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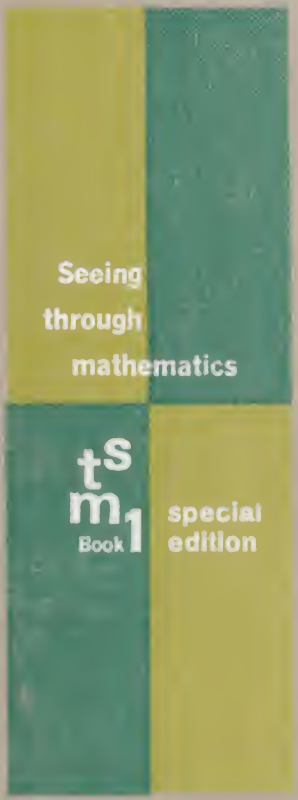

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Seeing through mathematics
special edition

Book 1

W. J. Gage Limited
Toronto

This is the seventh book of
The Basic Mathematics Program, published
by Scott, Foresman and Company,
and edited for use in Canadian schools
by W. J. Gage Limited.

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We live in a world of change. Every day the newspapers and news commentators report new developments in this breath-taking space age. New products, new methods of communication, new ways of travelling, and new scientific discoveries are being developed daily. Many of these modern miracles have been made possible because of new applications for mathematics in electronic computation, nuclear energy, rocketry, and space travel.

Not all of mathematics, however, is immediately concerned with such applications. Many mathematicians are interested in the way mathematics is constructed; they find beauty in the patterns they discover, much as the musician finds beauty in the patterns of music and the artist finds beauty in the patterns of art.

To give you the skills you need for applied mathematics is, of course, one of the purposes of this book. But an equally important purpose is to introduce you to the world of mathematical ideas and patterns.

In *Seeing Through Mathematics, Book 1*, you will learn about special words and symbols that are used to communicate in mathematics; about new uses for numbers; about points, lines, and planes and the ways in which they are related; about patterns and reasoning in mathematics. The ideas presented in many of the lessons will be new and different to you, but they are exciting and they are not difficult. We have planned the book so that you can discover many of

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these ideas for yourself.

In the first six units of this book, you will use only the natural numbers—that is, 0, 1, 2, and so on—as you examine ideas and solve problems. The geometry in unit 1 gives you an introduction to some of the properties of geometric figures. In the last unit of the book, you will use rational numbers like $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{4}$ in learning new concepts and in problem solving.

We enjoyed planning this book. Often we argued for hours about one idea. Sometimes, when we ended a day that had been filled with discussion, we still were not sure of the best way to present the ideas we had been talking about.

We wanted to pass along some of the ideas of mathematicians to you. And so we were tempted just to tell you about mathematics. But then we remembered that it is more fun to do mathematics than to read about it or listen while someone else explains it. Since we enjoyed the arguments we had while we were planning this book, we felt that we should make the book in such a way that you could have the same fun with mathematics as we had. That is why we decided to ask you questions rather than to tell you answers. We think that by answering questions for yourself, you will learn more mathematics—and learn it better—than by studying answers to questions.

We hope that you will find the lessons in this book stimulating and thought-provoking. We know that your understanding of the mathematical ideas will give you a sound introduction to the entire field of mathematics. For we have planned this part of the book as the first stage in your career as a student of mathematics. It is our wish that this career will be a long and fruitful one.

THE AUTHORS

Unit 1

Sets, conditions, and variables

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1 1 Exploring ideas

Sentences and placeholders

Throughout your work in mathematics, you will find that certain important ideas are used over and over again. Some of these ideas are explained in this lesson.

Examine the information card shown in D1. ("D1" means "display 1.") Each semester the mathematics teachers in Logan School ask each of their students to fill out a card like the one shown in D1.

A Each blank on the card calls for a special kind of information. Would it make sense for Steve Jones to write "Steve Jones" on the blank following "Age"?

B On which blanks is it correct to write the names of persons?

C The names of persons in one group are written on the blank following "Name." The names of persons in another group are written on the blank following "Mathematics teacher." How are these groups of persons different? Which group has more persons in it?

D Steve wrote "7" on his card after "Grade." Joe wrote "VII." How do you know that Steve and Joe are in the same grade? "7" and "VII" are names of the same number. Do you know any other names for the number seven?

The names of numbers are *numerals*.

nu mer al (nü′mər əl or nū′mər əl). A name of a number. The names 1, 15, 265, three, and V are examples of numerals.

In mathematics it is always important to know exactly what is being talked about. For example, it is important to know whether a number or a numeral is being talked about. Remember that a numeral is a name of a number. A number is an idea. A numeral is a name.

E Logan School has seven class periods a day. Joe wrote the numeral 8 on the blank after "Study period number" on his card. Why did his teacher, Miss Howe, ask him to correct his card?

F Suppose your teacher were to ask the students in your class to fill out a card like the card in D1. What numerals might they write on the blank following "Study period number"?

G Logan School has grades seven, eight, and nine only. What numerals would you expect to see on the blank following "Grade"?

H When you see "Name" on the card, you think of the sentence "My name is ~~~." What sentence do you think of when you see "Age" on the card?

I What sentence do you think of when you see "Favorite subject" on the card? What words might be written on the blank following "Favorite subject"?

Information card

LOGAN SCHOOL MATHEMATICS DEPARTMENT

Name.....Age.....

Birthday.....

Mathematics teacher.....

Mathematics period number.....

Study period number.....

Grade.....

Favorite subject.....

D1

Now look at D2. This information came from twelve of the cards that Miss Howe's class turned in. You will need this information to explore some of the ideas in the exercises that follow.

J Read the sentence in D3. If you replace the letter x by the name Richard from D2, will you have sentence A in D4?

K Does sentence A in D4 express a true idea? Could you answer this question if you were unable to look at D2?

L When you replace the letter x by the name of each student listed in D2, you get the sentences in D4. Which sentences express true ideas? Which sentences express false ideas?

M In the sentence "x has a birthday in September," replace the letter x by your name. Does the sentence you obtain express a true idea? Does it express a false idea?

N Imagine that each student in your class has replaced the letter x in "x has a birthday in September" by his name. Will some sentences express true ideas? Will other sentences express false ideas?

Certain exercises in this book are labeled with a large question mark. Exercise O below is such an exercise. Try to answer these exercises without help. When you think you know the answer (or answers), turn to the answer section in the back of the book. The answers in this section will show you whether or not your thinking is correct. Do not turn to this section until you have tried to answer the question without help.

O In exercises K through N you have been ? deciding whether or not sentences express true ideas. Look again at the sentence in D3. Is it possible to decide whether the idea expressed by this sentence is true or false? How does the sentence in D3 differ from sentence A in D4?

Birthday Months
of
Some Students in Miss Howe's
Class

Richard	June
Mary	September
Louise	January
David	November
Susan	September
Joanne	July
Bill	August
Joe	May
Dale	February
Ellen	September
Sandra	April
Debra	December

D2

---

x has a birthday in September.

D3

A Richard has a birthday in September.
B Mary has a birthday in September.
C Louise has a birthday in September.
D David has a birthday in September.
E Susan has a birthday in September.
F Joanne has a birthday in September.
G Bill has a birthday in September.
H Joe has a birthday in September.
I Dale has a birthday in September.
J Ellen has a birthday in September.
K Sandra has a birthday in September.
L Debra has a birthday in September.

D4

P Look at D5. Is it possible to decide whether the idea expressed by this sentence is true or false? Explain your answer.

Q Suppose you replace the star symbol by each letter of the alphabet, in turn. Which of these replacements make sentences that express true ideas?

R How many replacements make sentences that express false ideas?

S Is it sensible to make any other replacements in the sentence besides the letters of the alphabet?

Now you will begin to use numerals as replacements in sentences. In this lesson and in many of the lessons that follow, the numerals that you use will be the names of the numbers 0, 1, 2, 3, 4, and so on.

A Read the sentence in D6. Is it possible to decide whether the idea expressed by this sentence is true or false? Explain your answer.

B Write the sentences you obtain when you replace the letter *n* in the sentence by a name for each of the first nine numbers. Be sure to include zero.

When you replace the letter *n* in the sentence "*n* is a number greater than 6," you replace it by a numeral. For example, you may use the numeral 5 as a replacement for the letter *n*.

C Examine each sentence you wrote for exercise B. Label it with "T" if it expresses a true idea. Label it with "F" if it expresses a false idea.

D Look at D7. Is it possible to decide whether the idea expressed by the sentence is true or false?

E In the sentence in D7, replace the letter *y* by the numeral 8. Does the sentence you obtain express a true idea? Now replace the letter *y* by the numeral VIII. Does the sentence you obtain express a true idea?

☆ is a letter in the word "mathematics."

D5

n is a number greater than 6.

D6

y is less than 1.

D7

s is greater than 6 and less than 10.

D8

F Does using a different name for the number ? change the idea expressed by the sentence from a false idea to a true idea?

G Suppose you replace the letter *y* by the numerals 1, 2, 3, and so on. Will you obtain any sentences that express true ideas?

H Now replace the letter *y* by the numeral 0. Does the sentence express a true idea?

I Do the replacements you use make a difference in the number of sentences you can make that express true ideas?

J Read the sentence in D8. Then make ten sentences by replacing the letter *s* by the numerals 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. Which of the numerals that you used made sentences that express true ideas?

K Suppose you replace the letter *s* by fraction numerals like $\frac{1}{2}$, $\frac{1}{3}$, $\frac{9}{8}$, $\frac{5}{4}$, and so on. Are there any fraction numerals that will make sentences that express true ideas?

L The sentence in D8 uses the letter *s* as a symbol in a special way. What other symbols have been used in this way in the sentences you have

n is a number between 15 and 65.

D9

place hold er (plās′hōl′dər). A symbol that holds the place for numerals or the names of things that are being talked about. ☆, *x*, *n*, and many other symbols may be used as placeholders.

been working with? Symbols used in this way in sentences are *placeholders*. What is the placeholder in the sentence "*w* is less than 14"?

M Read the sentence in D9. Is it possible to decide whether the idea expressed by the sentence is true or false?

N What is the placeholder in the sentence?

O Replace the placeholder in the sentence "*n* is a number between 15 and 65" by each of the following numerals: 10, 20, 30, 40, 50, 60, 70, 80, 90, and 100.

P Which of the numerals that you used in exercise O made sentences that express true ideas?

Q Which of the numerals that you used in exercise O made sentences that express false ideas?

Now you know that some sentences contain placeholders. You also know that when placeholders are replaced by names, you obtain sentences that express ideas that are either true or false, but not both true and false.

On your own

1 What symbol is used as a placeholder in the sentence below?

∗ writes with the left hand.

2 Choose the names of six students in your class. Then use each of these names in turn as a replacement for the placeholder in the sentence "∗ writes with the left hand." Write the six sentences you obtain.

3 Mark with a "T" each of your sentences that expresses a true idea.

4 Mark with an "F" each of your sentences that expresses a false idea.

5 If any of your sentences express true ideas, list the names of the students you used in these sentences.

For each sentence in exercises 6 through 9, follow the instructions you used for exercises 1 through 5.

6 *z* is a boy who wears glasses.

7 *n* is less than 17 years old.

8 ☆ plays an instrument in the school band.

9 *x* is more than 5 years old.

10 In the sentence "*n* is less than 5," replace the placeholder by each of the following numerals: 1, 2, 3, 4, 5, 6, 7, and 8.

11 Mark with a "T" each of your sentences that expresses a true idea.

12 Mark with an "F" each of your sentences that expresses a false idea.

13 List the numerals you used in the sentences that express true ideas.

For each sentence in exercises 14 through 19, follow the instructions you used in exercises 10 through 13.

14 *x* is an even number.

15 *y* is the number of days in the week.

16 *w* is greater than 7.

17 ⁓ is less than 12.

18 *y* is a number less than 8 or equal to 8.

19 *z* is an odd number.

20 In the sentence "*x* is the number of months in a year," replace the placeholder by each of the numerals 0, 1, 2, 3, 4, 5, 6, 7, and 8.

21 Do any of the sentences you made for exercise 20 express true ideas?

22 Suppose you replace the placeholder in the sentence "*x* is the number of months in a year" by the numeral 12. Does the sentence you obtain express a true idea?

Sets and subsets

The idea of a collection is very important in mathematics. In this lesson you will begin to see how this idea is used.

The picture shows several collections of objects. The plants make up one collection of objects. The books between book ends make up another collection. A collection of books, a collection of persons, or even a collection of numbers is a *set* of objects.

A What sets of objects besides the sets of plants and books do you see in the picture?

B How many pencils are in the set of writing instruments? How many pens are in the set of writing instruments?

C Are the books between book ends the only books in the picture? What other books belong to the set of books?

Think for a minute about the Logan School information card. Collections, or sets of objects, were discussed in connection with this card. The teachers in Logan School, the students in a mathematics class, the subjects taught, and the numerals used to identify the class periods make up sets of objects.

D D1 shows a sentence that you have worked with before. What set of objects did you use to get replacements for the placeholder in this sentence? If you do not remember, look again at page 6.

D1 x has a birthday in September.

E When you replaced the letter x by each of the names Mary, Susan, and Ellen, you made sentences that express true ideas. Do the names Mary, Susan, and Ellen form a set of names?

The set of objects whose names are used as replacements for the letter x in "x has a birthday in September" is a set of persons. In mathematics you will frequently use sets of numbers, sets of lines, sets of triangles, and so on. Such sets are sets of mathematical objects.

F What sets of objects do you see below?

G What objects belong to the set of baseball equipment? When an object belongs to a set, it is a *member* of the set.

H Is the football a member of the set of balls? Is the bowling ball a member of this set?

set (set). A definite collection of objects. The students in your school, the numbers from 0 through 100, and the books in your classroom are examples of sets.

I How many members are in the set of objects that are used as a kind of club in games?

J How many members are in the set of gloves? How many members are in the set of boxing gloves? Is each member of the set of boxing gloves also a member of the set of gloves?

K How many members are in the set of objects used on ice or on snow? Are the ice skates members of this set? Is each member of the set of ice skates also a member of the set of

sub set (sub′set′). If each member of one set is also a member of a second set, then the first set is a subset of the second set. The set of cats is a subset of the set of animals; the set of numbers from 2 through 5 is a subset of the set of numbers from 0 through 10.

objects used on ice or on snow? The set of ice skates is a *subset* of the set of objects used on ice or on snow.

L Is each member of the set of numbers from 2 through 4 also a member of the set of numbers from 0 through 6? Is the set of numbers from 2 through 4 a subset of the set of numbers from 0 through 6?

M Is the set whose members are 2, 4, 6, and 8 a subset of the set of numbers from 1 through 10? Is the set whose members are 3, 5, 7, 9, and 11 a subset of the set of numbers from 1 through 10?

N Use the numerals from 1 through 10 as replacements for the letter *n* in the sentence "*n* is greater than 5 and less than 9." Which numerals make sentences that express true ideas?

O What are the members of the set of numbers that are greater than 5 and less than 9?

Suppose you want to talk about a set. You will need some way of making clear exactly what set you are talking about. One way would be to *describe* the set. You could describe one set by saying, "The set of students who have birthdays in September." You could describe another set by saying, "The set of numbers used in counting from 1 through 10." A description of a set must be so clear and definite that the listener or reader can tell exactly what objects belong to the set.

A Think about the set whose members are Mars, Venus, Earth, Jupiter, Saturn, Uranus, Pluto, Mercury, and Neptune. You can describe this set as "The set of known planets in our solar system." Now describe the set whose members are Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, and Saturday.

B How would you describe the set whose members are 1, 3, 5, 7, and 9?

Another way to tell about a set is to list the names of its members. The members of the set described as "The set of Great Lakes" can be listed as shown in D2.

Notice that the names of the members of the set are listed between braces like those shown at the right. Listing the names of the members of a set between braces is called *tabulating* the set. The tabulation of a set can be spoken or written. From a tabulation you can decide what objects are members of the set.

C D2 shows how to read the tabulation of a set. Now read the tabulation that follows: {2, 4, 6, 8, 10}.

D Look at the tabulation of a set in D3. How would you describe the set tabulated in this display?

"The set whose members are

{ Michigan, Huron, Superior, Erie, Ontario }

Michigan, Huron, Superior, Erie, Ontario"

D2

{Huron, Superior, Michigan, Ontario, Erie}

D3

E Look at D4. Is the same set tabulated in three different ways? Explain your answer.

F Does changing the order in which you list the names of the members of a set change the membership of the set?

G D5 shows two tabulations. Are they tabulations of the same set? Read each tabulation.

H Now read each tabulation in exercises I through P.

I {June, January, July}

J {Tom, Bill, Jack, Ed, Ken}

K {w, x, y, z}

L {book, chair, desk}

M {1, 3, 5, 7, 9}

N {10, 20, 30, 40, 50}

O {99, 75, 16, 102, 115, 176}

P {11, 101, 1001, 10001}

Q Tabulate the set of students who sit in your row, or at your table.

R Tabulate the set of numbers from 50 through 60.

S Tabulate the set of even numbers from 50 through 60.

T Use the numerals from 1 through 12 as replacements for the letter *n* in the sentence "*n* is greater than 3 and less than 12." Tabulate the set of numbers greater than 3 and less than 12.

Now you know what is meant by a set, a member of a set, and a subset. You also know how to describe sets, tabulate sets, and read tabulations.

On your own

A set is described in each of exercises 1 through 5. Each description is followed by a list. Which objects named in the list are members of the set described?

1 The set of months of the year.

January	July	Halloween
Sunday	December	October
Autumn	May	Thursday

{John, Mary, Joe}

{Mary, Joe, John}

{Joe, John, Mary}

D4

{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

{3, 5, 6, 9, 10, 7, 8, 1, 2, 4}

D5

2 The set of Prime Ministers of Canada.

Sir John A. Macdonald	Mackenzie King
Thomas Edison	Maurice Richard
Louis St. Laurent	John Diefenbaker

3 The set of Canadian coins, each of which has a value that is less than one dollar.

nickel	franc	silver dollar
half dollar	dime	cent

4 The set of letters of the alphabet.

a b ☆ q 7 r

5 The set of even numbers from 0 through 10.

8 5 12 3 4 15

Describe each of the sets tabulated in exercises 6 through 13.

6 {North America, South America, Asia, Africa, Europe, Antarctica, Australia}

7 {September, April, June, November}

8 {British Columbia, Alberta, Saskatchewan, Manitoba}

9 {Halifax, Fredericton, Charlottetown}

10 {0, 1, 2, 3}

11 {1, 3, 5, 7, 9}

12 {21, 22, 23, 24, 25, 26, 27, 28, 29}

13 {52, 54, 56, 58}

14 Tabulate the set of measuring instruments shown in the picture on page 9.

15 Tabulate the set of football equipment shown in the picture on page 10.

16 Tabulate the set of fishing equipment shown in the picture on page 10.

1 3 Exploring ideas

Sentences, statements, and conditions

In this lesson you are going to learn about two kinds of sentences. You will learn that these sentences express ideas. You will see that there is a difference between ideas and the sentences that express ideas.

A Read each sentence in D1.

B Each sentence in D1 is made up of symbols. Do all the sentences use the same symbols? How do the four sentences differ?

C These four sentences are alike in one way. Each sentence expresses an idea. Sentence A expresses the idea that the number three is less than the number four. What idea does each of the other sentences express?

D Do all the sentences in D1 express the same idea? Does each of these sentences express a true idea?

E Write a sentence different from those in D1 but expressing the same idea as each of the sentences in D1.

F Now read each sentence in D2.

G Do all the sentences in D2 express the same idea? Does each sentence express a true idea? Does each sentence express a false idea?

Look again at D1. This display shows different sentences, all of which express the same true idea. Now look again at D2. This display shows different sentences, all of which express the same false idea. Sentences that express ideas, whether true or false, are *closed sentences*. Ideas that are expressed by closed sentences are *statements*.

A 3 is less than 4.
B 3 is less than four.
C Three is less than four.
D III is less than IV.

D1

E 8 is greater than 12.
F Eight is greater than 12.
G Eight is greater than twelve.

D2

H 15 is less than 20.
I Fifteen is less than twenty.

D3

J 5 is greater than 9.
K V is greater than IX.

D4

H Look at D3. Are two different closed sentences shown in D3? Explain your answer.

I Do both sentences in D3 express the same idea? Is the idea true?

A true idea that is expressed by a closed sentence is a *true statement*.

J Write a sentence different from those in D3 but expressing the same true statement as each of the sentences in D3.

K Now look at D4. Are two different closed sentences shown in the display? Explain your answer.

L Do both sentences in D4 express the same idea? Is the idea false?

A false idea that is expressed by a closed sentence is a *false statement*.

M Write a sentence different from those in D4 but expressing the same false statement as each of the sentences in D4.

Distinction between closed sentence and statement; between open sentence and condition; between placeholder and variable

N Now look at D5. Which of the sentences in D5 express the following statement: "The number nine is less than the number twenty"? Does each of these sentences express a true statement?

O What statement is expressed by each of the other sentences? Do these sentences express the same statement? Does each of these sentences express a true statement?

You have learned that closed sentences express either true ideas or false ideas. The ideas expressed by such sentences are statements. Now you will learn about other sentences and the ideas they express.

A Read sentence A in D6. Is it possible to decide whether the idea expressed by sentence A is a true idea or a false idea? Is sentence A a closed sentence? Explain your answer.

B Is sentence B a closed sentence? Explain your answer.

C Are there two different sentences in D6? What is the placeholder in sentence A? In sentence B?

L 9 is less than 1.
M 9 is less than 20.
N IX is less than 20.
O Nine is less than one.

D5

A n is less than 8.
B x is less than eight.

D6

The sum of 3 and n is less than 20.

D7

D Each sentence in D6 expresses an idea about numbers. In each sentence the idea expressed is a requirement. Sentence A expresses the requirement that each number you are to find must be less than eight. What requirement does sentence B express?

E Do both sentences in D6 express the same requirement? Does the sentence "y is less than $5 + 3$" also express this requirement? Sentences that express requirements about objects are *open sentences.*

F Write a sentence that expresses the same requirement as each of the sentences in D6. Is the sentence that you wrote an open sentence?

G You have learned about closed sentences and open sentences. Each kind of sentence expresses an idea. The idea expressed by a closed sentence is different from the idea expressed by an open sentence. Explain how these ideas differ.

H Now look at D7. Is the sentence in D7 an open sentence, or is it a closed sentence? The requirement expressed is that the sum of 3 and each replacement for n must be less than what number?

I Does the open sentence "$3 + n$ is less than twenty" express the same requirement as the sentence in D7?

When you see an open sentence like the one in D7, think of the requirement that the symbols express, and not of the symbols themselves. The requirement expressed by an open sentence is a *condition.*

J Now look at D8. Which sentences express the condition that each number you are to find must be less than eighteen?

K What condition does each of the other sentences in D8 express?

L Write an open sentence that expresses this condition: The number you are to find must be less than the sum of five and eleven.

C x is less than 18.
D y is greater than seventeen.
E n is greater than 17.
F r is less than eighteen.

D8

When you think about the condition expressed by an open sentence or the statement expressed by a closed sentence, you are not thinking about symbols on a page. You are thinking about ideas. When you read the sentence "3 is less than x," think about the condition that the sentence expresses and not about the symbols on the page. When you read the sentence "3 is less than 4," think about the statement about numbers and not about the symbols on the page.

In the condition "3 is less than x," the x is a *variable*. When a variable is mentioned in this book, remember that you are to think about an idea and not about the symbol that expresses the idea.

M The condition "y is less than 3" requires that each replacement for y be less than three. Should you think of y as a placeholder or as a variable?

In this lesson you have learned that the ideas expressed by closed sentences are statements. Statements may be true or they may be false. You also have learned that the ideas expressed by open sentences are conditions. Open sentences contain placeholders. Conditions involve variables.

On your own

In exercises 1 through 10, tell whether each closed sentence expresses a true statement or a false statement.

1 Lake Erie is one of the Great Lakes.
2 April is the first month of the year.
3 32 is less than 23.
4 $26 = 17 + 9$.
5 3×6 is greater than 5×3.
6 $45 \div 9 = 5$.
7 $4 + 7$ is less than $14 - 5$.
8 8×5 is greater than $42 \div 6$.
9 $7 \times 1 = 7$.
10 $14 + 0$ is greater than 7×2.

In exercises 11 through 25, tell whether each sentence is an open sentence or a closed sentence.

11 $n = 3 + 17$.
12 9 is less than fifteen.
13 3×20 is greater than 3×19.
14 $4 = 3 + 2$.
15 $5 \times m$ is greater than 24.
16 $28 \div 14 = w$.
17 $2 = 28 \div 14$.
18 $4 + 25$ is less than $35 - 12$.
19 19×32 equals x.
20 n is greater than $45 \div 9$.
21 Fourteen equals $13 + 1$.
22 $9 + 12 = 12 + 9$.
23 4×6 is less than $7 \times m$.
24 The sum of 19 and 8 equals 8×3.
25 $x + x$ is greater than 2.
26 Which sentences in exercises 11 through 25 express true statements? Which express false statements?
27 Which sentences in exercises 11 through 25 express conditions?

In exercises 28 through 35, tell what condition is expressed by each open sentence.

28 $4 + x$ equals 6.
29 n is less than 36.
30 $y - 17$ is greater than 5.
31 $10 + 5$ is less than y.
32 $3 + x$ equals 3.
33 25 is less than $x + 7$.
34 $8 \times 7 = 7 \times m$.
35 16 is greater than $18 - y$.

1 4 Exploring ideas

Sets, conditions, and equations

You have already learned about sets, conditions, and statements. In this lesson you will learn how members of sets are used to obtain statements from conditions.

A Look at the open sentence in D1. This display also shows how to read the sentence.

Now you will use the condition $x + 6 = 9$ to obtain statements. The condition is about numbers. Before you can obtain statements from the condition, you must have a set of numbers to use.

B A set of numbers is tabulated in D2. Read the tabulation. Describe this set.

$x + 6 = 9.$ "x plus six equals nine."

D1

{1, 3, 5, 7, 9}

D2

$x + 6 = 9.$

A $1 + 6 = 9.$
B $3 + 6 = 9.$
C $5 + 6 = 9.$
D $7 + 6 = 9.$
E $9 + 6 = 9.$

D3

C Use each number in the set tabulated in D2 as a replacement for x in $x + 6 = 9$. How many statements do you obtain? Look at D3. Do the closed sentences in the display express these statements?

D Which statements expressed in D3 are true statements?

In the open sentence $x + 6 = 9$, the letter x is a placeholder. In lessons 1 and 2 you learned that placeholders in mathematical sentences are replaced by numerals. However, in the condition $x + 6 = 9$, you think of x as a variable. Variables are mentally replaced by numbers.

When a number is used as a replacement for a variable in a condition and a true statement is obtained, we say that the number *satisfies* the condition. If a false statement is obtained, the number does not satisfy the condition.

E Does the number 3 satisfy $x + 6 = 9$? How do you know?

F How do you know that each of the numbers 1, 5, 7, and 9 does not satisfy $x + 6 = 9$?

G Now think about what you have learned so far. What set did you use when you made replacements for the variable x in $x + 6 = 9$? The set whose members are used as replacements for the variable is the *universe* for the variable.

H Did you use {1, 3, 5, 7, 9} as the universe for x in $x + 6 = 9$? How do you know?

I After you replaced x by each member of the universe, statements were obtained. How did you use the statements to find the number that satisfies the condition $x + 6 = 9$?

universe (ū′nə vėrs). The set consisting of all the objects that can be used as replacements for the variable in a condition. {1, 3, 5, 7, 9} and {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10} are sets that may be used as universes.

J Read the tabulation and the open sentence in D4.

K The set tabulated in D4 is the universe for n. Think of the statements expressed by the closed sentences in the display. How were these statements obtained from $12 - n = 8$? Which statements are true? Which are false?

L How many members of the universe satisfy $12 - n = 8$? How do you know?

M Read the tabulation of the universe and the open sentence in D5.

N Replace y by each member of the universe. Write the sentences that express the statements you obtain from $3 + y = 14$.

O How many members of the universe satisfy the condition? How do you know?

P Suppose that 11 is also a member of the universe for y. Does 11 satisfy the condition? How many members of the universe satisfy the condition now?

The symbols you use in mathematics are a short way of expressing ideas. To express the idea of equality, you write the symbol for equality (=). To express the number 3, you write the numeral 3.

To express the idea of "y multiplied by 3," or "3 times y," you write the symbol "$3y$." The symbol $3y$ is read "three y." $3y$ means that the number 3 and the number that replaces y have been multiplied.

When 5 is used to replace y, you can express the product by writing the symbol 3(5). The parentheses help to prevent confusion between the symbol 3(5) and the numeral 35. The symbol 3(5) expresses 3 times 5, or 15. The numeral 35 expresses the number 35. You can also use the times sign and write the symbol 3×5 instead of 3(5).

A Read each sentence in D6. Which sentences express conditions? How do you know?

$\{0, 1, 2, 3, 4\}$

$12 - n = 8.$

A $12 - 0 = 8.$
B $12 - 1 = 8.$
C $12 - 2 = 8.$
D $12 - 3 = 8.$
E $12 - 4 = 8.$

D4

$\{0, 2, 4, 6, 8\}$

$3 + y = 14.$

D5

A $3y = 12.$
B $6 - 2 = 4.$
C $x - 20 = 90.$
D $5(6) = 30.$
E $0 + 5 = 0.$
F $3 + n = 15.$

D6

equation (i kwā′zhən). A true statement that includes the idea of equality. The statement $5(6) = 30$ is an equation. The statement $7 - 4 = 2$ is not an equation.

B Which sentences in D6 express statements?

C Does each of the statements include the idea of equality?

D The statement $5(6) = 30$ means that $5(6)$ is the same number as 30. Is $5(6) = 30$ a true statement?

E The statement $0 + 5 = 0$ means that $0 + 5$ is the same number as 0. Is $0 + 5 = 0$ a true statement?

True statements such as $5(6) = 30$ and $6 - 2 = 4$ are *equations*. False statements such as $0 + 5 = 0$ and $6 \div 3 = 3$ are not equations.

F Is $9 + 6 = 15$ an equation? Is $3(9) = 39$ an equation?

G Look again at D4. How many of the statements expressed in D4 are equations?

H Is $3y = 12$ an equation? Is $3(4) = 12$ an equation? Explain your answers.

I Read the tabulation of the universe and the open sentence in D7. Replace x by each member of the universe. Write the sentences that express the statements you obtain.

J Which statements are equations?

K Is $x + 5 = 8$ an equation? Did you use $x + 5 = 8$ in obtaining an equation?

A condition like $x + 5 = 8$ is a *condition for equality*. A condition that includes the idea of equality is a condition for equality. You obtain a statement from a condition for equality by replacing the variable by a member of the universe. If the statement is true, it is an equation.

L Tabulate the set of numbers that satisfy $x + 5 = 8$. Is the set you tabulated a subset of the universe?

M The universe is a set. The members of the universe that satisfy a condition also form a set. This set is the *solution set* of the condition. Is {3} the solution set of $x + 5 = 8$? How do you know?

N Now look at D8. Which sentence expresses a condition for equality? Which sentences express statements obtained from the condition?

O Tabulate the solution set of $2x + 1 = 7$. Is the set you tabulated a subset of the universe?

P Is the solution set of a condition always a subset of the universe? How do you know?

Now you know how to use the members of a universe to obtain statements from conditions for equality. You know that the members of the solution set are the replacements that make equations from conditions for equality.

solution set of a condition (sə lü′shən). The set consisting of all members of the universe that satisfy the condition. If the universe is {0, 1, 2, 3, 4, 5}, then the solution set of $x + 2 = 2$ is {0}.

D7

{0, 1, 2, 3, 4, 5}

$x + 5 = 8.$

D8

{0, 1, 2, 3, 4, 5}

$2x + 1 = 7.$

A $2(0) + 1 = 7.$
B $2(1) + 1 = 7.$
C $2(2) + 1 = 7.$
D $2(3) + 1 = 7.$
E $2(4) + 1 = 7.$
F $2(5) + 1 = 7.$

On your own

Which of the sentences in exercises 1 through 14 express equations? Which express conditions for equality?

1 $0 + 3 = 3.$
2 $14 - x = 9.$
3 $y - 7 = 8.$
4 $2(9) = 18.$
5 $4n = 8.$
6 $13 + x = 24.$
7 $5 = 9 - 4.$
8 $25 = 5n.$
9 $2n = 6.$
10 $3(7) = 20 + 1.$
11 $z - 4 = 13.$
12 $n + 2 = 9.$
13 $7(11) = 70 + 7.$
14 $54 = 9x.$

15 Use {1, 3, 5, 7, 9} as the universe for x in $3 + x = 10$. Write the sentences that express the statements you obtain. Write "T" after each sentence that expresses an equation.

16 Is $3 + x = 10$ a condition for equality? How do you know?

17 How do you know that {7} is the solution set of $3 + x = 10$?

For each condition expressed below, use {0, 5, 10, 15, 20} as the universe for the variable. Tabulate the solution set of each condition.

18 $2 + x = 12.$
19 $16 = y + 6.$
20 $3n = 15.$
21 $31 - x = 16.$
22 $5(4) = x.$
23 $y + 9 = 9.$
24 $20 - 10 = x.$
25 $60 = 4n.$

1 5 Exploring ideas

Sets, conditions, and inequalities

In this lesson you will learn about conditions that include ideas different from the idea of equality. You will obtain solution sets of this new kind of condition.

A The tabulation in D1 is one way to name a set. Think of the braces and the numerals together as a symbol for a set. Read the symbol in D1. Describe the set tabulated in the display.

After a set has been described or tabulated, it can be identified as set A, or set B, or set K, or set Y. This makes it unnecessary to use the description or the tabulation when you talk about the set. You may use any capital letter as a short name for a set. When a short name like A or B is used, it is important, of course, to keep the description or tabulation of the set in mind.

B The set tabulated in D2 is set K. Study D2.

C Now look at D3. Can the set tabulated here be called set A? The symbol for equality tells you that A and {10, 20, 30, 40, 50} are the same set. How do you read the sentence in D3?

D If M = {1, 2, 3}, and N = {1, 2, 3}, are M and N the same set?

We use the symbol for equality between names for the same object or idea. If M and N are the same set, we say M = N. Notice that equal sets have the same members.

E R = {7, 9, 11}. T = {9, 11, 7}. Is {7, 9, 11} the same set as {9, 11, 7}? Does R = T?

F S = {13, 14, 15}, and T = {13, 14, 15, 13}. Does S = T? Explain your answer.

G Look at D4. The set tabulated has been chosen as the universe for x in the condition "3 + x is greater than 6." What letter is used as a name of this universe?

The letter U is often used as a name of the universe.

H What numbers will you use as replacements for the variable in "3 + x is greater than 6"? Can you use any other numbers? Explain your answer.

I Does 3 satisfy the condition expressed in D4? Is 4 the only member of the universe that satisfies the condition? Explain your answers.

J Tabulate the solution set of the condition 3 + x is greater than 6. The solution set is a subset of what set?

You will learn many new symbols as you study mathematics. You have often used the symbol for equality. In this lesson you will use

D1

{5, 6, 7, 8, 9, 10}

D2

"K equals the set whose members are five, six, seven, eight, nine, ten."

K = { 5, 6, 7, 8, 9, 10 }.

D3

A = {10, 20, 30, 40, 50}.

D4

U = {0, 1, 2, 3, 4, 5}.
3 + x is greater than 6.

Using capital letters to name sets; conditions for inequality; inequalities; solution sets of conditions for inequality

three new symbols. These symbols express ideas different from the idea of equality.

Mathematicians use the symbol at the right to express the idea of "greater than." Study D5.

A Look at D6. Use the members of U as replacements for x in $x+2>6$. How many statements can you obtain from this condition? Do the sentences in D6 express all of these statements?

B How can you use the statements expressed in D6 to find the members of the universe that satisfy $x+2>6$?

C What is the name of the set of numbers that satisfy a condition? Tabulate this set for $x+2>6$.

D Can you use the condition $x+2>6$ to obtain an equation? Is $x+2>6$ a condition for equality?

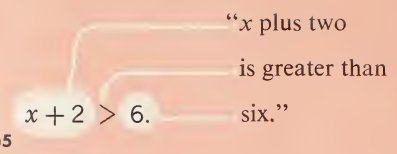
The symbol at the right below resembles the symbol for "greater than," but it points in the opposite direction. This symbol is used to express the idea of "less than." Study D7.

E Look at D8. The universe tabulated in D8 is the universe you will use for n in $2n<9$. Read the tabulation.

F If each member of the universe is used as a replacement for n in $2n<9$, how many statements are obtained? Write the sentences that express these statements. Will you write the sentence $2(10)<9$? Will you write the sentence $2(0)<9$?

G How many of the sentences that you wrote for exercise F express true statements? How can you use the statements expressed by the sentences you wrote to find the members of the universe that satisfy $2n<9$?

H Tabulate the solution set of the condition $2n<9$. How is the solution set related to the universe?



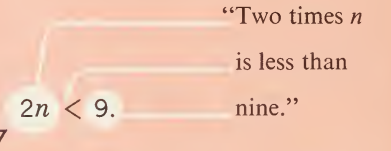
D5

$U = \{0, 1, 2, 3, 4, 5, 6\}.$

$x+2>6.$

A $0+2>6.$ **E** $4+2>6.$
B $1+2>6.$ **F** $5+2>6.$
C $2+2>6.$ **G** $6+2>6.$
D $3+2>6.$

D6



D7

$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$

$2n<9.$

D8

$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}.$

$y \neq 5.$

D9

I Can you use the condition $2n<9$ to obtain an equation? Is $2n<9$ a condition for equality?

J Now look at the open sentence in D9. Notice the symbol between the placeholder y and the numeral 5. How is this symbol different from the symbol for equality?

K The symbol used in D9 and shown at the right is used to express the idea of "not equal to." The condition expressed in D9 requires that each replacement for y be not equal to what number?

A $11 + 5 = 16$.
B $3 + 8 < 12$.
C $19 + 6 > 16$.
D $12 \neq 9$.
E $4 + 3 < 9$.
F $2(4) \neq 7$.
G $9 - 2 = 7$.
H $9(3) > 20$.

D10

$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

$n + n < 15$.

D11

$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

$n + n < 15$.

A $0 + 0 < 15$.
B $1 + 1 < 15$.
C $2 + 2 < 15$.
D $3 + 3 < 15$.
E $4 + 4 < 15$.
F $5 + 5 < 15$.
G $6 + 6 < 15$.
H $7 + 7 < 15$.
I $8 + 8 < 15$.
J $9 + 9 < 15$.

D12

L The universe that you are to use for y is tabulated in D9. What letter is used as a name of the universe?

M Using this universe, how many statements can you obtain from the condition $y \neq 5$? Are any of the statements equations? Is $y \neq 5$ a condition for equality?

N Is 5 a member of the solution set of $y \neq 5$? Explain your answer.

O Tabulate the solution set of $y \neq 5$.

P Now look at D10. Read each sentence. Does each sentence express a statement? How do you know? Which statements are true?

Q You have learned that an equation is a true statement that includes the idea of equality. Which sentences in D10 express equations?

R Look at sentence B in D10. Does the statement expressed by the sentence include the idea of "greater than," or "less than," or "not equal to," or equality? Is the statement true?

S Which of the ideas named in exercise R is used in the statement expressed by sentence C? Is the statement true?

A true statement that includes the idea of "greater than," or "less than," or "not equal to" is an *inequality*. A false statement that includes one of these three ideas is not an inequality.

T Which of the sentences in D10 express inequalities?

U Now you know that equations and inequalities are true statements. Is $3 + 5 = 8$ an inequality? Is $2 < 5$ an equation? Explain your answers.

V Look at D11. The universe for the variable n is tabulated in the display.

W How many times does the variable n occur in $n + n < 15$? When the same variable is used more than once in a condition, the same number must be used to replace the variable each time that the variable occurs.

X Look at D12. Do the sentences express the statements that result when the variable in $n + n < 15$ is replaced by each member of the universe?

Y Tabulate the solution set of $n + n < 15$.

Z Which sentences in D12 express inequalities?

Conditions like $n + n < 15$ and $n + 2 > 9$ are *conditions for inequality*. Conditions for inequality usually include the idea of "greater than," or "less than," or "not equal to." A

in e qual i ty (in′i kwol′ə ti). A true statement that includes the idea of "greater than," "less than," or "not equal to." The statements $3 < 5$, $5 > 3$, and $3 \neq 5$ are inequalities. The statements $3 > 5$, $5 + 2 = 7$, $9 < 6$, and $5 \neq 5$ are not inequalities.

statement is obtained from a condition for inequality by replacing the variable by a member of the universe. The statement that results, if true, is an inequality.

Now you know about the symbols that express inequalities. You know how to find the solution set of a condition for inequality. You know that the members of the solution set make inequalities from a condition for inequality.

On your own

Which of the following sentences express inequalities?

1 $3 < 7$.
2 $4(2) = 8$.
3 $3n < 10$.
4 $9 > 7$.
5 $10 \neq 11$.
6 $17 < 15$.

Which of the following sentences express equations? Which express inequalities?

7 $20 + 5 = 25$.
8 $7 < 11 + 2$.
9 $14 \neq 2(8)$.
10 $14 = 16 - 2$.
11 $2(10) > 15$.
12 $10 - 9 < 3$.

In exercises 13 through 18, replace each wavy line by the symbol $=$, $>$, or $<$ to make a sentence that expresses a true statement.

13 $5 + 2 \sim 7$.
14 $4 \sim 8 - 6$.
15 $16 + 3 \sim 20$.
16 $3 \sim 8 - 5$.
17 $3 \times 5 \sim 14$.
18 $12 \sim 4(6)$.

For each condition expressed in exercises 19 through 36, use {10, 9, 0, 6} as the universe for the variable. Tabulate the solution set of the condition.

19 $x + 3 > 5$.
20 $4 + n = 10$.
21 $5y < 7$.
22 $z \neq 6$.
23 $x < 16$.
24 $z + 1 = 11$.
25 $17 - y < 8$.
26 $3n > 19$.
27 $y + 2 \neq 12$.
28 $x + 6 > 1$.
29 $4n = 40$.
30 $16 - x = 10$.
31 $x < 6$.
32 $15 - x = 15$.
33 $7z + 5 < 30$.
34 $21 - n = 21$.
35 $n + 8 = 8$.
36 $x > 5 + 2$.

16 **Exploring ideas**

Finite sets and infinite sets

In this lesson you will learn how to tabulate sets that have many members. You will learn about two kinds of sets. You will learn how to use a condition for equality to help find the solution set of a condition for inequality.

You are already familiar with the numbers 0, 1, 2, 3, 4, 5, and so on. In this book we will call these numbers the *natural numbers*.

A The natural numbers from 0 through 1000 make up a set of numbers. Which is easier, to describe this set or to tabulate it?

There is an easy way to tabulate sets that have many members. This easy way of tabulating the set of natural numbers from 0 through 1000 is shown in D1. First write the numerals 0, 1, 2. Then make three dots. Finally, write the numeral 1000 for the last number in the set. The next display, D2, shows how to read the tabulation of this set.

B In the symbol {0, 1, 2, . . ., 1000}, the three dots are used instead of what numerals?

D1 {0, 1, 2, . . ., 1000}

D2 { 0, 1, 2, . . ., 1000 } "The set whose members are the natural numbers from 0 through 1000"

C How many members are in the set of natural numbers from 0 through 1000? Remember to include 0.

D How do you read the symbol in D3?

E In D3 the three dots are used instead of what numerals?

F How many members are in the set tabulated in D3?

So far you have worked only with sets in which you can determine the number of members. When the number of members in a set is a natural number, the set is a *finite* set.

G Can you count the members in {5, 10, 15, 20}? Can you count the members in {5, 10, 15, . . ., 10,000,000}? Explain your answers.

H Why are the three dots used in the symbol {5, 10, 15, . . ., 10,000,000}?

I How many members are there in this set: {5, 10, 15, . . ., 10,000,000}?

J Is {5, 10, 15, . . ., 10,000,000} a finite set? How do you know?

K Is {0, 1, 2, . . ., 100,000,000} a finite set? How do you know?

Now you will learn about sets that are not finite. The set of natural numbers includes all the numbers from 0 through 1000. It also includes 1001, 1002, 10,000, 3,999,999, and many, many more.

L Is there a natural number that is greater than every other natural number? Why is it impossible to list the names of all the members in the set of natural numbers?

Since it is impossible to list the names of all the members in the set of natural numbers, it is necessary to find some other way to express this set. D4 shows how to do this. D5 shows how to read the new symbol.

M What do the three dots tell you? Why is no last member named in the tabulation?

N What natural number immediately follows 36? How do you find this number?

{50, 51, 52, . . ., 100}

D3

{0, 1, 2, . . .}

D4

{0, 1, 2, . . .}

"The set whose members are the natural numbers"

D5

O For each natural number, is there a next greater natural number? How do you find this number?

P Can you use a natural number to tell how many members are in the set of natural numbers? Explain your answer.

Q Is the set of natural numbers a finite set? Explain your answer.

A set that is not finite is an *infinite* set. You cannot tell how many members are in an infinite set by using a natural number. There is no end to the members in an infinite set.

R Is the set of even numbers an infinite set? Use braces, numerals, and three dots to express the set of even numbers.

S Is the set of odd numbers an infinite set? Use braces, numerals, and three dots to express the set of odd numbers.

T Is the set of natural numbers less than 5 a finite set? An infinite set?

U Is the set of natural numbers greater than 5 a finite set? An infinite set?

V Is there a natural number that is less than every other natural number?

Because the set of natural numbers is used very often, it is convenient to name this set by a letter. The letter N is often used as a name of the set of natural numbers.

A Look at D6. What is the universe for x? Is the universe an infinite set?

B Is 5 a member of the solution set of $x+4>9$? Is any number less than 5 a member of the solution set?

C What are the three least numbers in the solution set of $x+4>9$?

D Is there a greatest number in the solution set?

Look again at D4 (repeated on this page), which shows how to indicate the tabulation of an infinite set. When you tabulate an infinite set, list the names of the first few members and then use three dots to show that the list goes on and on. You must list at least three names so that it is clear how the list continues.

E Tabulate the solution set of $x+4>9$. If necessary, use three dots in your tabulation.

F Describe the solution set of $x+4>9$. Is the solution set of $x+4>9$ an infinite set?

G Look again at D6. Is 5 a member of the solution set of $x+4<9$? Is any number greater than 5 a member of the solution set?

H Tabulate the solution set of $x+4<9$. Is the solution set an infinite set? A finite set?

You know that 0, 1, 2, 3, and 4 are the members of the solution set of $x+4<9$. The numbers 0, 1, 2, 3, and 4 are *solutions* of $x+4<9$. The members of the solution set of a condition are solutions of the condition.

I Look again at D6. Is any number less than 5 a solution of $x+4=9$? Is any number greater than 5 a solution?

J Tabulate the solution set of $x+4=9$. How many members does this solution set have? Is the solution set a finite set?

$$\{0,\ 1,\ 2,\ \ldots\}$$

D4

$$U = N.$$
$$x+4>9.$$
$$x+4<9.$$
$$x+4=9.$$

D6

$$U = N.$$

$x+4>9.$	$\{6,\ 7,\ 8,\ \ldots\}$
$x+4=9.$	$\{5\}$
$x+4<9.$	$\{0,\ 1,\ 2,\ 3,\ 4\}$

D7

$$U = N.$$
$$7+m=10.$$
$$7+m>10.$$
$$7+m<10.$$

D8

solution of a condition. Each member of a solution set of a condition. If the universe is the set of natural numbers, then each member of $\{0,\ 1,\ 2,\ 3,\ 4\}$ is a solution of $x<5$.

K Which sentences in D7 express conditions for inequality? Which express conditions for equality?

L Look at the tabulation of the solution set of $x+4>9$. Is each solution greater than the solution of $x+4=9$?

M Look at the tabulation of the solution set of $x+4<9$. Is each solution less than the solution of $x+4=9$?

When you know the solution of a condition like $x+4=9$, you can use this information to

help you find the solutions of conditions like $x+4>9$ and $x+4<9$. In the exercises that follow, notice how you can use a condition for equality to help find the solution set of a condition for inequality.

N Look at D8. What is the solution of $7+m=10$?

O Is each solution of $7+m>10$ greater than the solution of $7+m=10$?

P Is each solution of $7+m<10$ less than the solution of $7+m=10$?

Q Tabulate the solution set of $7+m>10$. Tabulate the solution set of $7+m<10$.

You know now that some solution sets are finite sets and some are infinite sets. Some finite sets have many members, some have a few members, and some have only one member.

A Look at D9. Read the sentence that expresses a condition. Now replace s in the condition by each member of the universe. Tabulate the solution set of the condition.

B $11+2<15$ and $12+2<15$ are true statements. Why are 11 and 12 not solutions of the condition expressed in D9?

C Look again at D9. How many members are in the universe for the variable? How many members are in the solution set of the condition?

D Is each member of the solution set also a member of the universe? Is the solution set a subset of the universe?

E Is each member of the universe a member of the solution set? Is the universe a subset of the solution set? Is the solution set of $s+2<15$ the same set as the universe?

F Look at D10. Replace x by 2. Do you obtain a true statement?

G Now replace x by 5,999,999. Do you obtain a true statement?

H What is the solution set of $x+3=3+x$?

$U=\{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\}.$
$s+2<15.$

D9

$U=N.$
$x+3=3+x.$

D10

$U=\{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\}.$
$x+5=110.$

D11

empty set (emp′ti). A set that has no members. Two descriptions of the empty set are: the set of women who have been Prime Minister of Canada and the set of natural numbers less than 0.

I Is the solution set of $x+3=3+x$ an infinite set?

J Are all the members of the universe solutions of $x+3=3+x$?

K Now look at D11. Replace x in the condition by each member of the universe. How many true statements do you obtain? How many members of the universe are in the solution set?

A set that has no members is the *empty set*.

The symbol at the right shows how to express the set that has no members. This symbol is read "the empty set." $\{\ \}$

Since it is possible to use a natural number to tell how many members there are in the empty set, the empty set is a finite set.

L Is the solution set of the condition expressed in D11 the empty set?

M Think of the set consisting of students in your class who are younger than five years of age. Tabulate this set.

$U = N.$
$1 + x = x.$

D12

N Look at D12. What is the universe for x? Replace x by 2. Do you obtain a true statement? Now replace x by 10,000. Do you obtain a true statement?

O Is there any member of the universe that satisfies the condition? How can you express the solution set? How do you know that the solution set is a finite set?

Now you know what finite sets and infinite sets are. You know that solution sets may be either finite or infinite sets. You also know how to use a condition for equality to help find the solutions of a condition for inequality.

On your own

Tabulate each set described in exercises 1, 2, and 3.

1 The set of odd numbers from 1 through 40
2 The set of natural numbers greater than 10
3 The set of odd numbers from 3 through 7

Describe each set tabulated below.

4 {0, 1, 2, . . ., 34}
5 {3, 6, 9, . . ., 30}
6 {11, 13, 15, . . .}

In each condition expressed in exercises 7 through 18, the universe for x is N. Tabulate the solution set of each condition. If a solution set has no members, use the symbol for the empty set. Tell whether each solution set is a finite or an infinite set.

7 $x + 7 > 21.$
8 $x + 3 = 8.$
9 $15 + x = 20.$
10 $20 < 15 + x.$
11 $2x + x = 3x.$
12 $x + 4 = x.$
13 $x + 4 < x.$
14 $x + 4 > x.$
15 $x + 3 < 3.$
16 $x + 4 \neq 4.$
17 $24 < x + 2x.$
18 $15 > 3x + 1.$

1 7 Exploring ideas

Standard description of sets

For several lessons now you have been finding solution sets of conditions. You have expressed these solution sets by tabulation. Now you will learn to express solution sets in a different way.

A You have learned to name sets in two ways, by tabulation and by description. Read this tabulation: {0, 1, 2, 3, 4, 5}. Can you describe the set just tabulated by writing "the set whose members are the natural numbers less than six"?

B When you tabulate a set, you use braces instead of what words? You also may use braces when you describe a set. For example, you can write {the natural numbers less than six} for the set that is described in exercise A.

C Now read the open sentence in D1. Also read the description of the solution set of the condition $x < 6$. The universe for x is N. The braces are used instead of what words?

There is a way of writing the description of a solution set that is shorter and more convenient than the description given in D1. We will work out this new description step by step.

First you use the symbol x instead of the words "natural numbers." You use the symbol x again instead of the word "number."

D Read the description of the solution set given in D2.

E Next you use the symbol $<$ for the words "is less than" and use the numeral 6 instead of the word "six." Notice how these symbols are

$x < 6$.
{all natural numbers that satisfy the condition that each number is less than six}

D1

{all x that satisfy the condition that each x is less than six}

D2

{all x that satisfy the condition that each $x < 6$}

D3

$\{\text{all } x | \text{each } x < 6\}$

D4

$\{x | x < 6\}$

D5

"The set whose members are all x that satisfy the condition that x is less than six"

$\{x \mid x < 6\}$

D6

$U = N$.
$x + 4 < 20$.
$\{0, 1, 2, \ldots, 15\}$

D7

used in D3. Read the description of the solution set given in D3.

F Finally you use a symbol instead of the words "that satisfy the condition that." This symbol is a picture of a vertical line. Notice how this new symbol is used in D4. Read the description of the solution set given in D4.

It usually is not necessary to write the words "all" and "each." You may write the description of the solution set as shown in D5. D6 shows how to read the description of the solution set given in D5.

When braces and an open sentence are used to describe a solution set, such a description is a *standard description.*

G If you want to, you may use the words "such that" instead of the words "that satisfy the condition that." Reread the standard description of the solution set given in D5. This time use the words "such that."

Write a standard description of each of the solution sets described in exercises H, I, and J. $U = N$.

H {all x such that each $x + 4 < 7$}

I {all numbers that satisfy the condition that each number is less than ten}

J The set whose members are all numbers that satisfy the condition that each number is greater than 100

A standard description is a convenient way of describing a solution set that has many members or that is an infinite set.

K Look at D7. Is each member of the tabulated set a solution of the condition? Is this set the solution set of $x + 4 < 20$?

L The solution set may be described in this way: $\{x | x + 4 < 20\}$. Is $\{x | x + 4 < 20\}$ the same set as $\{0, 1, 2, \ldots, 15\}$?

M Does the sentence below express a true statement? Explain your answer. $U = N$.

$\{x | x + 4 < 20\} = \{0, 1, 2, \ldots, 15\}$.

$U = N.$
$5 + x > 7.$
$\{3, 4, 5, \ldots\}$

D8

N Look at D8. Is $\{x|5 + x > 7\}$ a standard description of the solution set of $5 + x > 7$? Is $\{x|5 + x > 7\}$ the same set as $\{3, 4, 5, \ldots\}$?

Now you know that you can express a solution set by using a standard description.

On your own

Write a standard description of each set described in exercises 1 through 9. U = N.

1 {all x such that each $x < 50$}

2 {all n that satisfy the condition that each n is not equal to three}

3 {all y such that each $y + 3 > 9$}

4 {all numbers that satisfy the condition that two times each number is less than twenty}

5 {all z such that each $z + 1 \neq 4$}

6 {all numbers such that each number < 7}

7 {all r that satisfy the condition that each r is equal to nine}

8 {all x|each $x + 3 > 8$}

9 The set whose members are all numbers that satisfy the condition that each number added to itself is greater than ten

Write in words each standard description given in exercises 10 through 13. U = N.

10 $\{x|x > 2\}$

11 $\{n|n + 3 \neq 10\}$

12 $\{x|x + x < 17\}$

13 $\{z|6z = 18\}$

For each condition expressed in exercises 14 through 17, write a standard description of the solution set. U = N.

14 $x + 3 > 20.$

15 $y + 6 = 14.$

16 $m + 3 < 10.$

17 $r + 2 \neq 2.$

18 For each condition expressed in exercises 14 through 17 above, tabulate the solution set. Remember that U = N.

Read each sentence below. Tell which sentences express true statements and which express false statements. U = N.

19 $\{0, 1, 2, 3\} = \{x|x < 3\}.$

20 $\{2, 0, 3, 1\} = \{x|x < 4\}.$

21 $\{x|x > 10\} = \{11, 12, 13, \ldots, 50\}.$

22 $\{x|x + 2 = x\} = \{\ \}.$

Tabulate each set described below. U = N.

23 $\{x|x < 7\}$

24 $\{m|m > 9\}$

25 $\{z|z = 10\}$

26 $\{x|x + 1 \neq 1 + x\}$

27 $\{y|2y > 50\}$

28 $\{w|2w = w + w\}$

CHECKING UP

Use lessons 1 through 7 if you need to restudy the ideas expressed in these tests.

Test 1

For each condition expressed in exercises 1 through 4, write the sentences that express all the statements that can be obtained by using $\{0, 1, 5, 10\}$ as the universe for x. Tell which of these sentences express true statements.

1 $10 > 3x.$

2 $5 \neq x.$

3 $8 + x = 17.$

4 $12 - x < 3.$

Tabulate the solution set of each condition expressed in exercises 5 through 12.
$U = \{0, 1, 2, 3, 4, 5\}.$

5 $x > 4.$

6 $2 < x.$

7 $3 \neq x.$

8 $6 - x < 3.$

9 $x + 1 > 4.$

10 $2 \neq x + x.$

11 $3 + 3 = x.$

12 $x + 2 < 5.$

Test 2

For each set tabulated in exercises 13, 14, and 15, what numerals would you write instead of the three dots?

13 $\{5, 6, 7, \ldots, 12\}$

14 $\{0, 2, 4, \ldots, 14\}$

15 $\{7, 9, 11, \ldots, 19\}$

Which of the sets tabulated in exercises 16 through 19 are subsets of $\{1, 9, 7, 12, 5\}$?

16 $\{1, 7, 12, 5\}$

17 $\{5, 7, 9, \ldots\}$

18 $\{1, 3, 5, \ldots\}$

19 $\{12\}$

Sets of points

You know that mathematics is concerned with numbers and with sets of numbers. Mathematics also deals with such objects as points, lines, planes, and space. The part of mathematics that deals with these objects is called *geometry*. Some of these geometric objects are represented in D1 and D2.

A Look at D1. The dot in picture A represents a *point*. The dots in picture B represent several points. These points form a set of points. Picture C was obtained by first reproducing the dots in picture B and then placing another dot between each two dots. Do the points that are represented in picture C also form a set of points?

B Explain how picture D was obtained from picture C. Do the points represented in picture D also form a set of points?

C Suppose that you have a pencil with the sharpest tip possible. Imagine that with this pencil you place a dot between each two dots in picture D and that you continue placing dots in this way. Do you think there are many points between any two points represented by dots?

D Picture E represents the set of points indicated in picture D and all points between these points. Do you think that picture E represents an infinite set of points?

Look at the pictures of geometric objects in D2. The triangle is made up of three sides. Each side is a set of points like the set represented by picture E in D1. The triangle itself is the set of points that contains all the points in its three sides.

D1

D2

ge o met ric fig ure (jē′ə met′rik fig′yər). A set of points. Triangles, rectangles, circles, and spirals are geometric figures.

E The rectangle is made up of four sides. Is each side of the rectangle a set of points? Is the rectangle a set of points?

F Do you think that the circle is a set of points? Do you think that the spiral is a set of points?

Each geometric object represented in D2 is a set of points. A set of points is a *geometric figure*. As you continue to study geometry, you will learn more about some of these geometric figures.

G Think of two geometric figures that are different from those represented in D2. Make sketches of these two figures.

H Now look at D3. Notice that several dots have been drawn to show how a dot would look if it were shrinking. If the dot continues to shrink, will it finally disappear?

Even when there is no dot, there is a point. You can only imagine the point. A point is a mathematical idea.

I Look at D4. How many dots are there in the display? How many points are represented by dots?

Notice the letter A beside the dot farthest to the left. This letter names the point represented by the dot. You may call the point "point A" or, simply, "A."

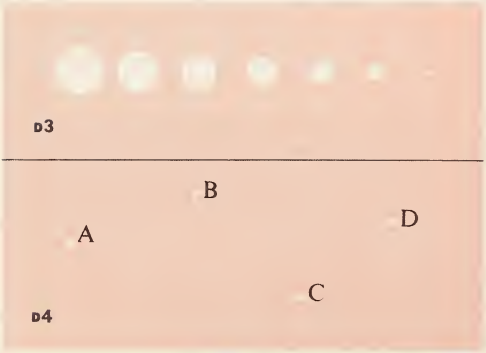
J Is {A, B, C, D} the set of points represented by dots in D4?

Now you will use the idea of a point as you study other geometric figures.

K From your study of science, you know that the earth's atmosphere contains particles of dust. Are there many particles of dust in the earth's atmosphere?

L Each dust particle suggests a point. Do all the dust particles in the earth's atmosphere suggest many points?

M Imagine that more and more particles of dust are released into the atmosphere. Do these dust particles suggest more and more points?



N Imagine the earth's atmosphere completely filled with dust particles. These particles still do not suggest all possible points. The set of all points is called *space*. Do you think that space is an infinite set of points?

O Look at the photograph above. Can you think of the tip of the pencil as suggesting a point in space? For each of the other objects pictured, what part best suggests a point?

Now you will learn about some special subsets of space.

A Imagine an unlighted room filled with dust particles. Light is being directed into the room through an extremely small pinhole in the door. The dust particles illuminated by the light form a set of dust particles. Is this set a subset of the set of dust particles in the room?

> REMINDER
> If each member of one set is also a member of a second set, then the first set is a subset of the second set.
> *See lesson 2, page 11.*

B Since each dust particle suggests a point, the set of illuminated particles suggests a set of points. Is this set of points a subset of space?

The set of points suggested by the set of illuminated particles is a *line*. Since a line is a set of points, a line is a subset of space. You can only imagine a line. A line is a mathematical idea.

C Is a line a geometric figure? How do you know?

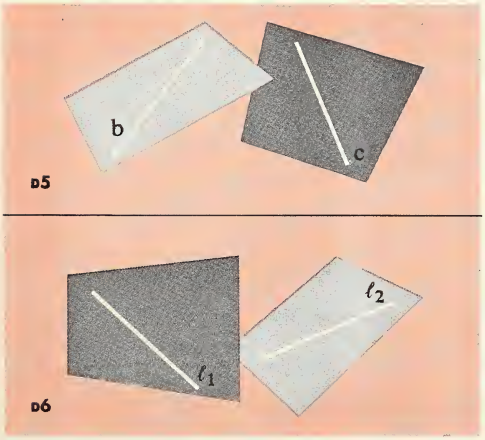
D Can you think of a tightly stretched wire as suggesting a line? Name two objects in your classroom that suggest lines.

In geometry, a line is understood to be a "straight line." In this unit, when we say "line," we will mean "straight line."

E Look at D5. The display shows two ways to represent a line. One picture has arrows to remind you that a line goes on and on in opposite directions. The other picture does not have arrows. When you look at a picture of a line without arrows, remember that the line goes on endlessly in opposite directions. Each picture in D5 is an incomplete picture of a line.

F Is each line represented in D5 an infinite set of points? Is any line an infinite set of points?

G Notice the letter b beside one picture in D5. One way to name a line is to use a letter. You may call this line "line b" or, simply, "b."



What is the name of the other line represented in D5?

H Is it sensible to ask if line b is longer than line c? Explain your answer. How does line b differ from line c?

I D6 shows another way to name lines. The picture of one line is labelled with the letter ℓ and a small numeral 1. This numeral 1 is called a *subscript*. It is written at the right and below the letter. This name of the line is read "ℓ subscript one," "ℓ sub one," or, simply, "ℓ one." What is the name of the other line represented in D6?

J Does ℓ_1 extend endlessly in opposite directions? Does ℓ_2 extend endlessly in opposite directions?

Now you will learn about another subset of space.

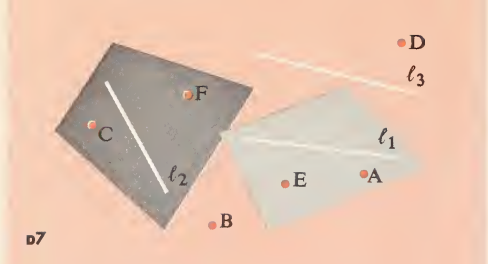
A Again imagine an unlighted room filled with dust particles. Light is being directed into the room through a door that is opened slightly. The dust particles illuminated by the light form a set of dust particles. Is this set a subset of the set of dust particles in the room?

B Since each dust particle suggests a point, the set of illuminated dust particles suggests a set of points. Is this set of points a subset of space?

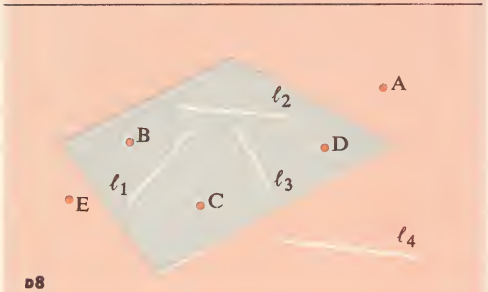
The set of points suggested by the set of illuminated particles is a *plane*. Since a plane is a set of points, it is a subset of space. A plane is unlimited in extent; that is, it has no boundaries. A plane, like a point or a line, can only be imagined. A plane is a mathematical idea.

C Is a plane a geometric figure? How do you know?

D Can you think of the surface of a mirror as suggesting a plane? Name two other objects that suggest a plane.



D7



D8

E Look at D7. It includes pictures of points, lines, and planes. When you look at the pictures, remember that planes extend on and on.

F Is {A, B, C, D, E, F} the set of points that are represented by dots in D7?

G Point A is in a plane represented in D7. What other point in this plane is represented by a dot?

H For each plane represented in D7, tabulate the set of points in the plane that are represented by dots.

I Do the planes contain any points besides the points represented by dots? Is a plane an infinite set of points? Explain your answers.

J Point B is a member of space. Is point B in a plane that is represented in D7?

K Is D a member of space? Is D in a plane that is represented in D7?

L Is $\{\ell_1, \ell_2, \ell_3\}$ the set of lines represented in D7? For each plane represented, you can find a line that is a subset of the plane. For example, ℓ_1 is a subset of one plane represented in D7. What other line named in D7 is a subset of a plane represented in the display?

When we say that a line is "*included* in a plane," we mean that the line is a subset of the plane.

M Are any lines other than ℓ_1 and ℓ_2 included in the planes represented in D7? Explain your answer.

N Is the set of lines that are included in a plane an infinite set?

Now you know that points, lines, planes, and space are mathematical ideas. Although no one has ever seen a point, a line, a plane, or space, you can use drawings and objects to represent or suggest these mathematical ideas.

On your own

Use D8 to help you with exercises 1 through 7. The plane referred to is the plane represented in D8.

1 Tabulate the set of points in the plane that are represented by dots.

2 Tabulate the set of points not in the plane that are represented by dots.

3 Is ℓ_1 a subset of the plane? What other lines named in D8 are included in the plane?

4 Tabulate the set of lines that are not included in the plane.

5 Tabulate the set of all points represented by dots in D8.

6 Is the set that you tabulated for exercise 5 a subset of space?

7 Is the set that you tabulated for exercise 1 a subset of the set that you tabulated for exercise 5?

Make a sketch of each geometric figure described below. Label each sketch.

8 A plane including line b and containing point F

9 A plane and lines ℓ_1 and ℓ_2 not included in the plane

10 A plane including line c and point E not in the plane

Relating points, lines, and planes

In this lesson you will learn how points, lines, and planes are related.

A Look at D1. What points are represented by dots? Line ℓ_1 contains A and B.

B Is ℓ_1 a set of points? Is {A, B} a subset of ℓ_1? How do you know?

C What lines are represented in D2? Is C contained in each of these lines?

D Is the set of lines, each of which contains C, an infinite set? Explain your answer.

E Are all lines that contain C included in the same plane?

F How many lines represented in D2 contain both C and D?

G How many lines are represented in D3? Does more than one line contain both E and F?

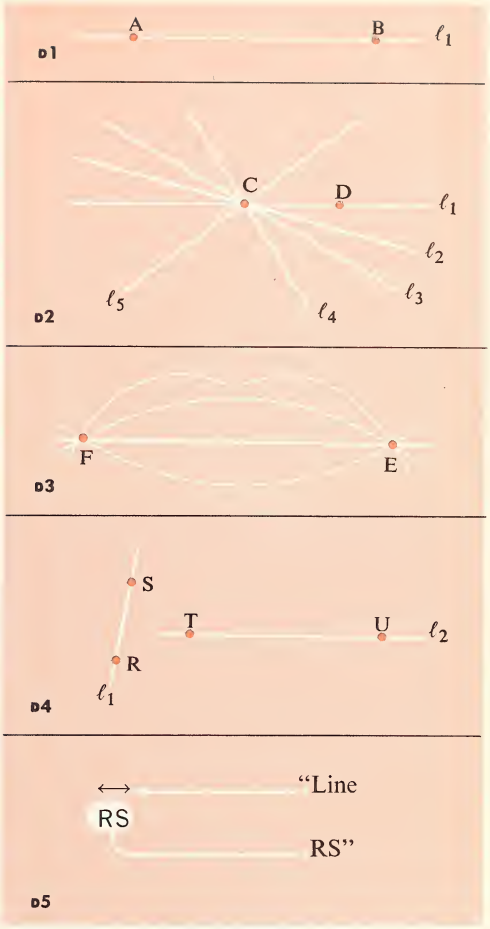
H Is the set of geometric figures, each of which contains both E and F, an infinite set?

Exercises F, G, and H have helped you discover that there is only one line that contains two different points. This idea is common to, or characteristic of, any two different points. This idea is a *property* that relates points and lines. The property is expressed below.

For any two different points there is exactly one line that contains both of these points.

From now on, when we say "two points," we will mean "two *different* points." In the same way, when we refer to two lines or to two planes, we mean two different figures.

I Look at D4. What line contains R and S? What line contains T and U?



J Since there is exactly one line that contains both R and S, points R and S *determine* ℓ_1. What line is determined by T and U? Do U and T determine the same line as T and U?

K How many lines are determined by any two points?

Since any two points determine exactly one line, you can use the names of the two points in naming the line. D5 shows another way to name ℓ_1, represented in D4. The words in D5 tell how to read this name.

L Notice the double-headed arrow in the symbol $\overleftrightarrow{RS}$. Why is an arrow like this a good symbol to use in naming a line?

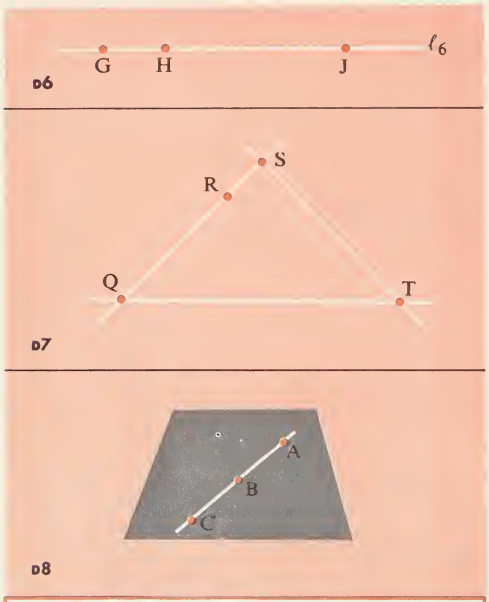
M Now look at D6. Is ℓ_6 determined by G and H? By H and J? What other pair of points represented by dots determines ℓ_6?

N Is $\overleftrightarrow{GH}$ the same line as ℓ_6? Is $\overleftrightarrow{HG}$ the same line as $\overleftrightarrow{GH}$? What are two other names for ℓ_6?

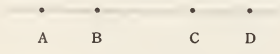
Points G, H, and J are contained in the same line. Points that are contained in the same line are *collinear points*.

O Look at D7. Are the members of {Q, R, S} collinear?

P Are any two points always collinear? Explain your answer.



col lin e ar points (kə lin′ē ər). Points that are contained in the same line. Points A, B, C, and D, represented below, are collinear points.



Q Look again at D7. How do you know that Q, R, and T are not collinear?

Points Q, R, and T are *not* contained in the same line. They are *noncollinear points*.

Next you will study a property that relates lines and planes.

A Look at the photographs. Does the surface of the chalkboard in each photograph suggest a plane? Does each stick suggest a line?

B Think of the stick in the photograph at the left as touching the board at dot A. Does this stick touch the board at any other place?

C Think of the other stick as touching the board at dots X and Y. Does this stick touch the board at any other place?

D If a stick touches a board at two places, does it touch the board at more than two places?

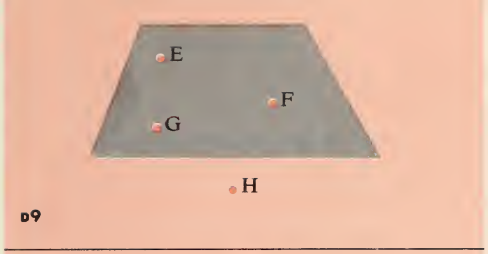
E Look at the picture of a plane in D8. A, B, and C are collinear points. A and B are in the plane. Is C in the plane?

F Is $\overleftrightarrow{AC}$ included in the plane? Is $\overleftrightarrow{CB}$ included in the plane?

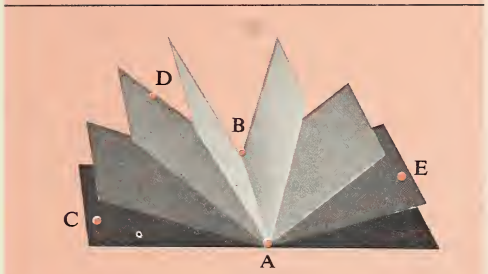
Now you have learned about a property that relates lines and planes. This property is expressed below.

If a line contains two points of a plane, the line is included in the plane.

G Points E, F, and G are in the plane represented in D9. Point H is not in the plane. Is $\overleftrightarrow{EF}$ included in the plane? Name two other



D9



D10

lines that are included in the plane. Name three lines that are not included in the plane.

In the following exercises you will learn more about the properties that relate points, lines, and planes.

A Look at D10. Point A and point B determine how many lines? How do you know?

B Is $\overleftrightarrow{AB}$ included in the plane suggested by the surface of the book cover? How do you know?

C The surface of the book cover suggests a plane. Each leaf of the book also suggests a plane. Is $\overleftrightarrow{AB}$ included in each of these planes?

D Is every plane that includes $\overleftrightarrow{AB}$ represented in D10? Explain your answer.

E Is the set of planes, each of which includes $\overleftrightarrow{AB}$, an infinite set? Explain your answer.

When we say that the set of planes, each of which includes $\overleftrightarrow{AB}$, is an infinite set, we mean that infinitely many planes include $\overleftrightarrow{AB}$.

You have learned about another property that relates lines and planes. This property is expressed as follows:

A given line is included in infinitely many planes.

F Look again at D10. Name the points that are represented by dots. Which of these points are not contained in $\overleftrightarrow{AB}$?

G Point C is in the plane suggested by the surface of the book cover. How many planes that contain C also include $\overleftrightarrow{AB}$?

H How many planes that contain D also include $\overleftrightarrow{AB}$? How many planes that contain E also include $\overleftrightarrow{AB}$?

In exercises F, G, and H you have seen how a line and a point not in the line are related to a plane. The following sentence expresses this property.

There is exactly one plane that includes a given line and contains a point not in the line.

We say that point C and $\overleftrightarrow{AB}$ determine the plane that contains C and includes $\overleftrightarrow{AB}$.

I Do point D and $\overleftrightarrow{AB}$ determine a plane? Explain your answer.

J Look again at D10. You know that point A and point B determine $\overleftrightarrow{AB}$. You also know that $\overleftrightarrow{AB}$ and point D determine the plane suggested by a leaf of the book. Are A, B, and D noncollinear points? How do you know?

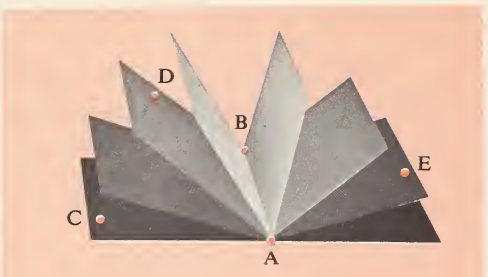
K Are A, B, and D contained in just one plane? Are A, B, and E contained in just one plane? Explain your answers.

L Are A, D, and E contained in just one plane? How do you know? Is the plane that contains A, D, and E suggested by a leaf of the book?

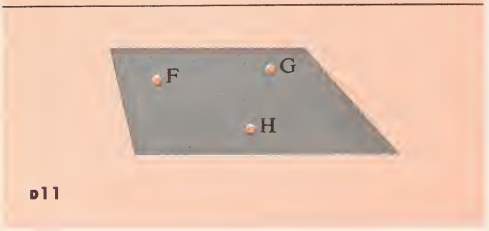
Now you know how three noncollinear points are related to a plane. This property is expressed by the sentence below.

There is exactly one plane that contains three noncollinear points.

M How many planes represented in D10 are determined by A, B, and C? Explain your answer.



D10



D11

Since three noncollinear points are contained in exactly one plane, you can use the names of these points to name the plane.

N Look at D11. The dots represent three noncollinear points. Are F, G, and H in the same plane? You may call the plane represented in D11 "plane FGH."

O Look again at D10. Is plane ADB represented? Name two other planes represented in D10.

P Look again at D11. Imagine that D is in plane FGH but is not in $\overleftrightarrow{GH}$. Is plane DGH the same plane as FGH? Explain your answer.

Q Can you name a plane by using the names of any three noncollinear points in that plane?

Now you know certain properties that relate points, lines, and planes.

On your own

For some of the following exercises, you will use a *straightedge* to draw pictures of lines. A ruler can be used as a straightedge. Make only one drawing for exercises 1 through 4.

1 Draw a picture of $\overleftrightarrow{AB}$. To do this, first place two dots on your paper. Label one dot with the letter A and the other dot with the letter B. Use a straightedge to draw a picture of $\overleftrightarrow{AB}$.

2 Make a dot to represent point C so that C is not contained in $\overleftrightarrow{AB}$. Draw a picture of $\overleftrightarrow{BC}$.

3 Make a dot to represent point D so that $\overleftrightarrow{AB}$ contains D.

4 Draw a picture of $\overleftrightarrow{DC}$ and of $\overleftrightarrow{AC}$.

For each of exercises 5 through 8, make dots to represent the points described. Label each dot. Draw pictures of the lines determined by the points and write a name for each line.

5 Points A, B, and C are collinear. Points A, B, and D are noncollinear.

6 Points A, B, and C are noncollinear. Points B, C, and D are noncollinear. Points A, B, and D are noncollinear. Points A, D, and C are collinear.

7 Points A, B, and C are collinear. Points B, D, and E are collinear. Points A, B, and D are noncollinear.

8 No three of points A, B, C, and D are collinear.

9 How many lines are determined by four points, no three of which are collinear?

10 Represent five points, no three of which are collinear. How many lines are determined by these points? Draw a picture of each line.

11 How many lines can contain a given point?

12 How many lines can contain two points?

13 How many planes can contain a given point?

14 How many planes can include a given line?

15 If $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are two lines, what do you know about points A, B, and C?

16 How many planes are determined by four points that are not all in the same plane, and no three of which are collinear?

17 If A, B, and C are in more than one plane, what do you know about A, B, and C?

Segments and congruent segments

You have studied some of the properties that relate points, lines, and planes. Now you will study subsets of lines and learn how these subsets are related to each other.

A Look at D1. Point E is between point D and point F. Point E is also between D and G. Is E between D and H?

B Is F between D and H? Is G between D and H? Are points D, E, F, G, and H collinear points?

C Look again at D1. Are points D, J, and H collinear?

Since D, J, and H are not collinear, it is not sensible to discuss whether or not J is between D and H. *Betweenness* applies only to collinear points. You have been studying one of the properties of betweenness. This property is expressed below.

If one point is between two other points, then the three points are collinear.

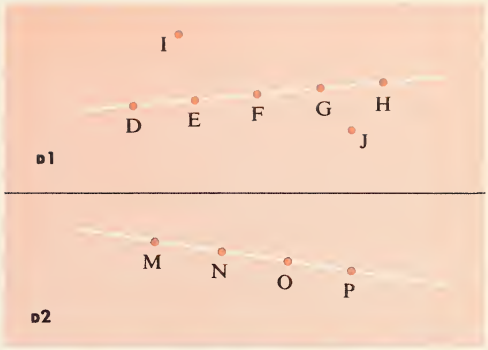
D Think of the three points A, B, and D. Point D is between B and A. Are A, B, and D collinear? Explain your answer.

E Tabulate the set of points named in D1 that are between D and H. Is I a member of this set? Explain your answer.

F Since E is between D and H, can you also say that E is between H and D?

G Now look at D2. Four collinear points are named in the display.

H Is N between O and P? Is P between O and N? Is O between P and N?



I When three points are collinear, is one of the points between the other two?

J When three points are collinear, is it possible for more than one of these points to be between the other two?

Now you know another property of betweenness. This property is expressed below.

For every three collinear points, just one of the points is between the other two.

K If you know that A is between B and C, then is it possible for B to be between A and C? For C to be between A and B? Explain your answers.

L S, V, and W are three collinear points. S is not between V and W. V is not between S and W. What can you decide about the position of the points S, V, and W?

M Look again at D2. Could you locate a point between M and N? Between N and O? Between O and P?

N Imagine that M and N are represented as "closer together" than they are in D2. Could you locate a point between M and N? Explain your answer. Remember that a point is an idea. A point does not have size.

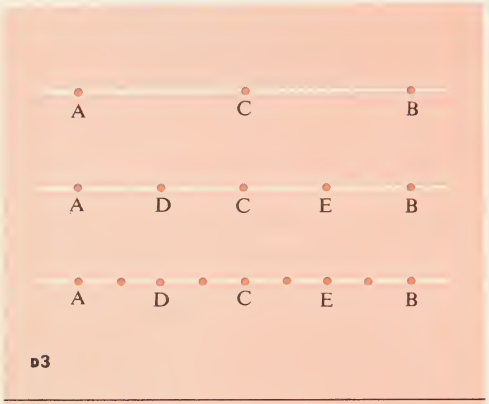
You have learned another property of betweenness. This property is expressed below.

For any two points, there is at least one point between them.

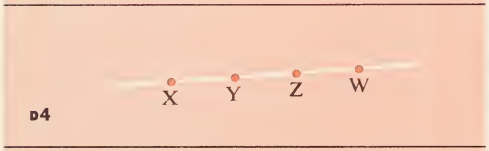
O Look at the first picture in D3. Is point C between A and B? Is there at least one point between A and C? Between C and B? How do you know?

P Look at the second picture in D3. Is D between A and C? Is E between C and B? Are C, D, and E between A and B?

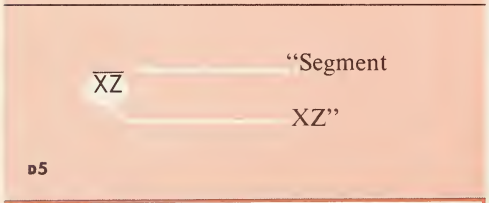
Q Now look at the third picture in D3. Four more points are represented by dots. Is each of these points also between point A and point B?



D3



D4



D5

seg ment (seg′mənt). A set of points whose members are two given points and all points between these two. The set whose members are A and B and all points between A and B is a segment. Points A and B are the **endpoints** of the segment.

R Could 10 points be located between A and B? Could 100 points be located between A and B?

S Are there infinitely many points between A and B?

T Which property of betweenness can you use to explain that there are infinitely many points between two given points?

You know now what it means to say that a point is between two other points. Next you will learn about a set of points whose members are two given points and all points between these two.

A Look at D4. You will work with the set of points whose members are X and Z and all points between X and Z. Is Y between X and Z? Is Y a member of the set you are working with?

B Is W between X and Z? Is W a member of the set you are working with?

The set of points whose members are X and Z and all points between X and Z is a *segment*. Points X and Z are the *endpoints* of the segment. Point Y is a member of the segment.

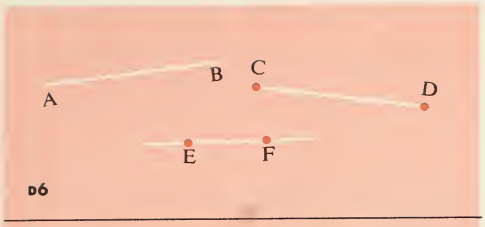
D5 shows how to write a name of the segment whose endpoints are X and Z. It also shows how to read the name of the segment.

C Is $\overline{ZX}$ the same segment as $\overline{XZ}$? Explain your answer.

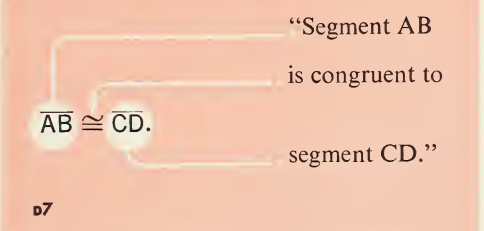
Sometimes a segment is called a "line segment." If you wanted to, you could say "line segment XZ" instead of "segment XZ." You would be naming the same set of points in a slightly different way. In this book we will use the name "segment."

D Three segments are represented in D6. Name each segment.

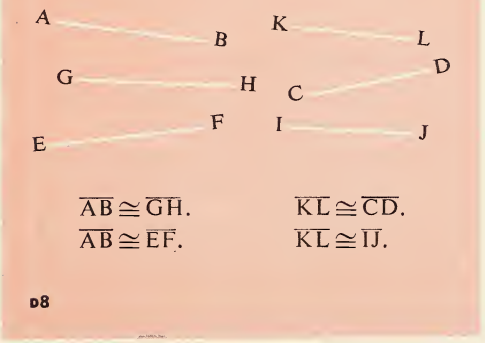
Notice the three different ways in which we have represented segments in D6. The endpoints of $\overline{AB}$ are not represented by dots. The



D6



D7



D8

endpoints of $\overline{CD}$ and $\overline{EF}$ are represented by dots. In this book we will use all three ways to represent segments.

Since two given points are the endpoints of exactly one segment, we say that two points determine a segment.

E Look again at D4. What segment is determined by X and Z? Is Y a member of this segment? Is W a member of this segment? Is Z a member of this segment?

F Can you draw a complete picture of a line? Can you draw a complete picture of a segment? Explain your answers.

G Look again at D4. Is $\overline{XW}$ a subset of $\overleftrightarrow{XW}$? Explain your answer.

Now you will learn about segments that are the same size. Such segments are *congruent* segments. When you are comparing segments, congruent means "the same size." Mathematicians use the symbol below to express the idea of congruence. Study D7, which shows how to read a sentence that includes this new symbol. $\overline{AB} \cong \overline{CD}$ is a true statement about two line segments. You will learn later that, for other geometric figures, congruence means "the same size *and* shape."

A Six segments are represented in D8. Name each segment.

B Now read each sentence in D8. Each sentence expresses a true statement about two of the segments represented.

Notice that the statements expressed in D8 use the idea of congruence and not the idea of equality. Remember that, in this book, equality means "the same as" or "identical."

C Is $\overline{AB}$ the same segment as $\overline{GH}$? Is $\overline{AB} = \overline{GH}$ a true statement? Explain your answer.

D Think about the statements expressed in D8. Since $\overline{KL} \cong \overline{CD}$ is a true statement, do you think that $\overline{CD} \cong \overline{KL}$ is also a true statement?

E Is $\overline{IJ} \cong \overline{KL}$ a true statement? Is $\overline{EF} \cong \overline{AB}$ a true statement? Is $\overline{GH} \cong \overline{AB}$ a true statement?

F In exercise E you accepted $\overline{GH} \cong \overline{AB}$ as a true statement. In other words, you *assumed* that this statement is true. You know that $\overline{AB} \cong \overline{EF}$ is a true statement. Do you think that $\overline{GH} \cong \overline{EF}$ is also a true statement?

$\overline{GH} \cong \overline{EF}$ is a true statement because of a property of congruence. This property, expressed below, applies to any three segments.

If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$ are true statements, then $\overline{AB} \cong \overline{EF}$ is a true statement.

This property can also be expressed in the shorter way shown below:

If $\overline{AB} \cong \overline{CD}$ *and* $\overline{CD} \cong \overline{EF}$, *then* $\overline{AB} \cong \overline{EF}$.

Reread the two ways of expressing the property. Notice that in the shorter way the phrases "are true statements" and "is a true statement" are omitted. From now on, when you see a sentence like $\overline{AB} \cong \overline{CD}$, assume that the sentence expresses a true statement unless you are told otherwise.

G You know that $\overline{CD} \cong \overline{KL}$ and $\overline{KL} \cong \overline{IJ}$. Is $\overline{CD} \cong \overline{IJ}$?

H Do you think that $\overline{AB} \cong \overline{KL}$ is a true statement? Explain your answer.

Now you know that segments are sets of points. You know what it means to say that two segments are congruent. You also know a property of congruent segments.

On your own

1 Draw a picture of a line. Then make six dots to represent six points in the line. Name these points, in order, with the letters H, I, J, K, L, and M. Now make two dots to represent two points not in the line. Name these points "N" and "O." Use your drawing to help you answer the questions in exercises 2 through 5.

2 Tabulate the set of points named in your drawing that are between H and M. The set of points that are between I and L. The set of points that are between J and K.

3 Tabulate the set of points named in your drawing that are between H and L. That are between L and H.

4 Did you tabulate two different sets for exercise 3?

5 Explain why neither point N nor point O is a member of the sets you tabulated for exercises 2 and 3.

6 Now draw another picture of a line. Make two dots to represent two points in the line. Name these points "A" and "B." Between A and B make a dot to represent a point. Name this point "C." Between C and B make a dot to represent point D. Between D and B make a dot to represent point E.

7 Could you locate at least one other point between any two points named in exercise 6? How many points are there between any two points? Explain your answers.

8 Draw pictures of $\overline{MN}$ and $\overline{OP}$.

9 Does each picture you made for exercise 8 represent a set of points?

10 What are the endpoints of each of the segments named in exercise 8? What is another name for each segment?

11 In your picture of $\overline{MN}$, make a dot, different from the dots representing M and N, to represent a member of $\overline{MN}$. Name this point "Q." Is Q between M and N?

12 Is $\overline{QN}$ the same set of points as $\overline{MN}$? Is $\overline{QM}$ the same set of points as $\overline{MN}$?

13 Suppose that $\overline{UV} \cong \overline{WX}$ and that $\overline{UV} \cong \overline{YZ}$ are true statements about three segments. Is $\overline{WX} \cong \overline{YZ}$? Explain your answer.

Each of sentences 14 through 17 expresses a true statement about two segments. Write the words that you would say in reading each sentence.

14 $\overline{AB} \cong \overline{CD}$.

15 $\overline{AB} \cong \overline{EF}$.

16 $\overline{GH} \cong \overline{IJ}$.

17 $\overline{KL} \cong \overline{MN}$.

$\overline{AB} \cong \overline{GH}$ is also a true statement. $\overline{AB} \cong \overline{KL}$ is a false statement. Use this information and the statements expressed in exercises 14 through 17 to decide which of the following sentences express true statements and which express false statements. You may use pictures to help you decide.

18 $\overline{EF} \cong \overline{CD}$.

19 $\overline{AB} \cong \overline{IJ}$.

20 $\overline{KL} \cong \overline{CD}$.

21 $\overline{GH} \cong \overline{EF}$.

22 $\overline{AB} \cong \overline{MN}$.

23 $\overline{GH} \cong \overline{MN}$.

24 $\overline{CD} \cong \overline{GH}$.

25 $\overline{MN} \cong \overline{EF}$.

1 11 Exploring ideas

Comparing segments

In this lesson you will use the idea of betweenness to help you understand a new idea, the idea of *separation*. Then you will learn to compare line segments without measuring them.

A Look at the picture of a plane in D1. Since the plane extends endlessly, it separates space into two sets of points. The two sets are called *half-spaces*. Is each half-space an infinite set of points?

B Tabulate the set of points that are named in D1.

Now you will decide whether or not all the members of the set that you have just tabulated are in the same half-space.

C Look again at D1. How many points between A and B are in the plane?

Since no point between A and B is in the plane, point A and point B are in the same half-space.

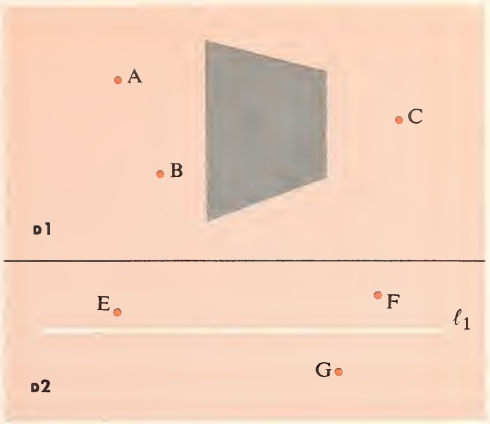
D How many points between B and C are in the plane?

Since there is one point between B and C that is in the plane, point B and point C are not in the same half-space.

E Are A and C in the same half-space? How do you know?

A plane that separates space into two half-spaces is the *boundary* of each half-space. The points in the boundary are not in either half-space.

F Does every plane separate space into two half-spaces?



Next you will learn that a line separates a plane.

G Look at D2. The picture represents a line included in plane EFG. Does ℓ_1 extend endlessly in opposite directions?

H Line ℓ_1, represented in D2, separates the plane into two sets of points. The two sets are called *half-planes*. Is each half-plane an infinite set of points?

I Tabulate the set of points in the plane that are named in D2.

Now you will decide whether or not all the members of the set that you have just tabulated are in the same half-plane.

J Look again at D2. How many points between E and F are in line ℓ_1?

Since no point between E and F is in ℓ_1, point E and point F are in the same half-plane.

K How many points between E and G are in line ℓ_1?

Since there is one point between E and G that is in ℓ_1, point E and point G are not in the same half-plane.

L Are F and G in the same half-plane?

A line that separates a plane is the boundary of each half-plane. The points in the boundary are not in either half-plane.

M Does every line included in a plane separate the plane into two half-planes?

Next you will learn that a point separates a line.

N Look at D3. Point R has been chosen as a point that separates the line into two sets of points. The two sets are called *half-lines*. Is each half-line an infinite set of points?

O Tabulate the set whose members are the points H, I, and J.

Now you will decide whether or not the members of the set that you have just tabulated are in the same half-line.

P Look again at D3. Remember that you are thinking about R as the point that separates the line. Is R between H and I?

Since R is not between H and I, point H and point I are in the same half-line.

Q Now think about H and J. Is point R between H and J?

Since R is between H and J, point H and point J are not in the same half-line.

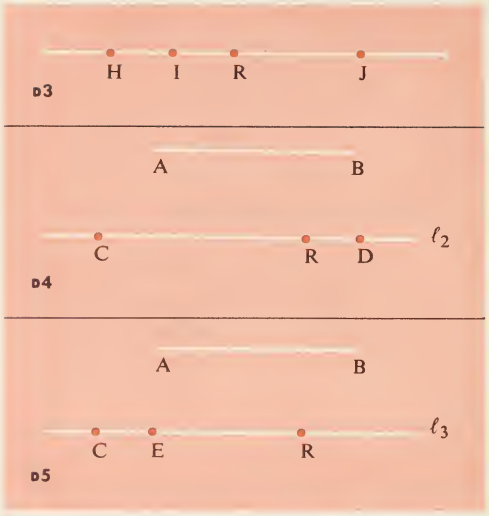
R Are points I and J in the same half-line?

A point that separates a line is the boundary of each half-line. The boundary is not in either half-line.

S Does H separate the line? Does J separate the line? Does I separate the line? Does every point in a line separate the line into two half-lines?

T Suppose that H is the boundary of two half-lines. Are points I and J in the same half-line? How do you know?

You know that a point separates a line into two half-lines. You will use this idea of separation and the idea of congruence to compare segments. You have already used the ideas of "less than" and "greater than" with numbers. You can also use these ideas to compare segments without measuring them.



A Look at D4. You are going to compare $\overline{AB}$ and $\overline{CD}$.

In the pictures of the two segments, $\overline{AB}$ appears to be less than $\overline{CD}$. Let us see if there is a way to decide whether or not $\overline{AB} < \overline{CD}$.

B Think about ℓ_2, represented in D4. Point C separates ℓ_2 into two half-lines. There is a point R, in the same half-line as point D, that makes $\overline{AB} \cong \overline{CR}$. In ℓ_2, is R between C and D?

Notice what you have done. You wanted to compare $\overline{AB}$ and $\overline{CD}$. You know that there is a point R, in the same half-line as D, so that $\overline{AB} \cong \overline{CR}$. Since it happens that R is between C and D, you are now sure that $\overline{AB} < \overline{CD}$.

C Now look at D5. You are going to compare $\overline{AB}$ and $\overline{CE}$.

In the pictures of the two segments, $\overline{AB}$ appears to be greater than $\overline{CE}$. Now you will decide whether or not $\overline{AB} > \overline{CE}$.

D Think about ℓ_3, represented in D5. Point C separates ℓ_3 into two half-lines. There is a point R, in the same half-line as point E, that makes $\overline{AB} \cong \overline{CR}$. In ℓ_3, is R between C and E? Is E between C and R?

Notice that you have compared $\overline{AB}$ and $\overline{CE}$ in the same way you compared $\overline{AB}$ and $\overline{CD}$. You know that there is a point R, in the same half-line as E, that makes $\overline{AB} \cong \overline{CR}$. This time it happens that R is *not* between C and E. Instead, E is between C and R. You are therefore sure that $\overline{AB} > \overline{CE}$.

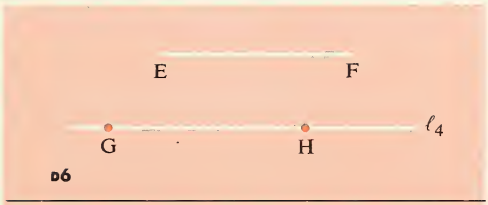
E Look at D6. You cannot tell by looking at the pictures of $\overline{EF}$ and $\overline{GH}$ whether $\overline{EF}$ is less than, congruent to, or greater than $\overline{GH}$. To compare $\overline{EF}$ and $\overline{GH}$, you must use what you have just learned about comparing segments.

F First think about line ℓ_4. Point G separates ℓ_4 into two half-lines. Imagine a point R, in the same half-line as H, so that $\overline{EF} \cong \overline{GR}$.

G If R is between G and H, is $\overline{EF} < \overline{GH}$?

H If H is between G and R, is $\overline{EF} > \overline{GH}$?

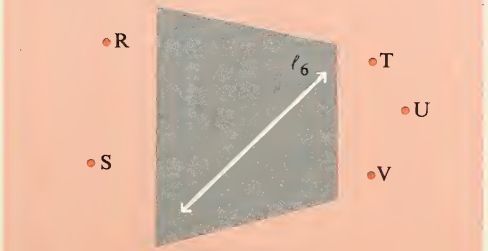
I If R is the same point as H, what do you know about $\overline{EF}$ and $\overline{GH}$?



D6

$\overline{IJ} \cong \overline{KL}$. $\overline{IJ} < \overline{OB}$. $\overline{OB} < \overline{RS}$.

D7



D8

Now you know what it means to compare segments. The exact meanings of "less than" and "greater than" are expressed below. You should make sketches to help you understand these meanings.

Let $\overline{UV}$ be any segment. Let $\overline{XY}$ be any segment included in line ℓ_5. Point X separates ℓ_5 into two half-lines. A point R in line ℓ_5 is in the same half-line as Y, so that $\overline{UV} \cong \overline{XR}$. If R is between X and Y, then $\overline{UV} < \overline{XY}$. If Y is between X and R, then $\overline{UV} > \overline{XY}$.

J Suppose that $\overline{JK} < \overline{LM}$ is a true statement. ? Is $\overline{JK} \cong \overline{LM}$? Is $\overline{JK} > \overline{LM}$? Explain your answers.

K Look at D7. Each sentence expresses a true statement. Do you think that $\overline{OB} < \overline{IJ}$? Do you think that $\overline{OB} > \overline{IJ}$? Explain your answers.

L Do you think that $\overline{KL} < \overline{OB}$? Explain your answer.

M Do you think that $\overline{IJ} < \overline{RS}$? Explain your answer.

$\overline{IJ} < \overline{RS}$ because of a property of "less than" for segments. This property is expressed below.

If $\overline{AB} < \overline{CD}$, and $\overline{CD} < \overline{EF}$, then $\overline{AB} < \overline{EF}$.

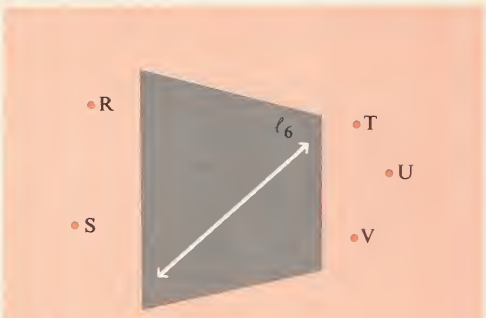
From this lesson you have learned that a plane separates space, that a line separates a plane, and that a point separates a line. You also know how the ideas of "less than" and "greater than" are used to compare segments.

On your own

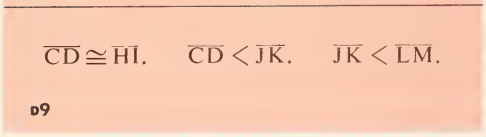
Use D8 as you answer the questions in exercises 1 through 13. Line ℓ_6 is included in the plane represented in D8. Points R, S, T, U, and V are not in this plane.

1 Does the plane separate space into two half-spaces?

2 Is there a point between R and T that is in the plane? Are R and T in the same half-space?



D8



D9

3 Is there a point between R and V that is in the plane? Are R and V in the same half-space?

4 Is there a point between R and S that is in the plane? Are R and S in the same half-space?

5 Is there a point in the plane between R and U? Are R and U in the same half-space?

6 Does line ℓ_6 separate the plane into two half-planes?

7 Points X, Y, and Z are not represented in D8. Suppose that they are in the plane represented, but are not in line ℓ_6. There is a point between X and Y that is in ℓ_6. Are X and Y in the same half-plane?

8 No point between Y and Z is in ℓ_6. Are Y and Z in the same half-plane?

9 There is a point between X and Z that is in line ℓ_6. Are X and Z in the same half-plane?

10 Now think about a point Q in line ℓ_6. Does Q separate ℓ_6 into two half-lines?

11 Point Q separates line ℓ_6. Suppose points M, N, and O are also in line ℓ_6. Point Q is between N and O. Are N and O in the same half-line?

12 Point Q is not between M and N. Are M and N in the same half-line?

13 Point Q is between M and O. Are M and O in the same half-line?

Make sketches to help you answer exercises 14, 15, and 16.

14 Let $\overline{AB}$ be any segment, and let ℓ_7 be any line containing points E and F. Point E separates ℓ_7 into two half-lines. There is a point R, in the same half-line as F, that makes $\overline{AB} \cong \overline{ER}$. Suppose that R is between E and F. Write a sentence that expresses a true statement comparing $\overline{AB}$ and $\overline{EF}$.

15 Suppose that F is between E and R. Write a sentence that expresses a true statement comparing $\overline{AB}$ and $\overline{EF}$.

16 Suppose that R and F are the same point. Write a sentence that expresses a true statement comparing $\overline{AB}$ and $\overline{EF}$.

Use the following information to answer the questions in exercises 17, 18, and 19: $\overline{WS}$ is any segment, and ℓ_8 is any line containing points T and U. Point T separates line ℓ_8 into two half-lines. There is a point R, in the same half-line as U, that makes $\overline{WS} \cong \overline{TR}$.

17 If $\overline{WS} < \overline{TU}$, is R between T and U?

18 If $\overline{WS} > \overline{TU}$, is R between T and U? Is U between T and R?

19 If $\overline{WS} \cong \overline{TU}$, what do you know about points U and R?

Each sentence in D9 expresses a true statement. Use the sentences to help you answer the questions in exercises 20 through 29.

20 Is $\overline{JK} > \overline{CD}$?

21 Is $\overline{CD} < \overline{LM}$?

22 Is $\overline{LM} > \overline{CD}$?

23 Is $\overline{JK} > \overline{HI}$?

24 Is $\overline{HI} < \overline{JK}$?

25 Is $\overline{LM} < \overline{CD}$?

26 Is $\overline{JK} < \overline{CD}$?

27 Is $\overline{HI} < \overline{LM}$?

28 Is $\overline{LM} > \overline{HI}$?

29 Is $\overline{JK} < \overline{HI}$?

30 Suppose that $\overline{MO} > \overline{RQ}$ and also that $\overline{RQ} > \overline{ST}$. Do you think that $\overline{MO} > \overline{ST}$ is a true statement?

The circle

You have studied universes, variables, conditions, and solution sets of conditions. Up to now, the universes and solution sets have been sets of numbers. Variables and conditions are also used with sets of points. Universes and solution sets can be sets of points as well as sets of numbers.

A Study D1. The sentence $\overline{AX} \cong \overline{AB}$ expresses a condition. The variable in the condition is X. What is the universe for X? How do you know that points can be used as replacements for X?

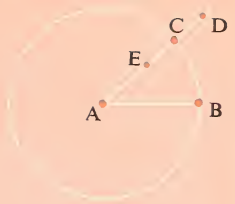
Notice that the letter X, used to express the variable, is a capital letter. This will help you remember that the replacements for X in the condition will be points, not numbers.

You remember that a condition is a requirement that applies to members of the universe. Usually, some members of the universe satisfy the requirement, and some members do not. The condition $\overline{AX} \cong \overline{AB}$ means that each point you are to find must, together with A, determine a segment that is congruent to $\overline{AB}$. Keep in mind that the universe is plane ABC and the points you find must be in the universe.

B Is D in this universe?

When you work with conditions about numbers, you are usually given certain numbers. In $2x = 8$, for example, you are given the numbers 2 and 8. When you work with conditions about points, you will be given certain points. In the condition $\overline{AX} \cong \overline{AB}$, points A and B are given points.

Universe for X is plane ABC.
$\overline{AX} \cong \overline{AB}$.



D1

Universe for X is plane ABC.
$\overline{AX} \cong \overline{AB}$.

D2

C Look again at D1. Suppose that $\overline{AC} \cong \overline{AB}$. Does C satisfy $\overline{AX} \cong \overline{AB}$? Is C a member of the solution set?

> REMINDER
> The solution set of a condition consists of all members of the universe that satisfy the condition.
> A standard description of the solution set of $\overline{AX} \cong \overline{AB}$ is $\{X \mid \overline{AX} \cong \overline{AB}\}$.
> This description is read, "The set of all points X such that segment AX is congruent to segment AB."
> *See lesson 4, page 18. See lesson 7, page 27.*

D Look again at D1. You know that C is a member of $\{X \mid \overline{AX} \cong \overline{AB}\}$. $\overline{AE} < \overline{AC}$, and $\overline{AD} > \overline{AC}$. Is $\overline{AE} < \overline{AB}$? Is $\overline{AD} > \overline{AB}$?

E Are D and E members of $\{X \mid \overline{AX} \cong \overline{AB}\}$?

F Look at D2. You know that B and C are members of $\{X \mid \overline{AX} \cong \overline{AB}\}$. From the display,

certain other points that are represented by dots appear to be members of $\{X \mid \overline{AX} \cong \overline{AB}\}$. Name these other points.

G H is a solution of $\overline{AX} \cong \overline{AB}$. $\overline{AN} < \overline{AH}$, and $\overline{AV} > \overline{AH}$. How do you know that neither N nor V is a solution of $\overline{AX} \cong \overline{AB}$?

H Do you think that K satisfies $\overline{AX} \cong \overline{AB}$? Do you think that R satisfies $\overline{AX} \cong \overline{AB}$?

I Now tabulate the set of points named in D2 that satisfy $\overline{AX} \cong \overline{AB}$. Do you think that these points are the only members of $\{X \mid \overline{AX} \cong \overline{AB}\}$?

J Are any solutions of $\overline{AX} < \overline{AB}$ also solutions of $\overline{AX} \cong \overline{AB}$? Are any solutions of $\overline{AX} > \overline{AB}$ also solutions of $\overline{AX} \cong \overline{AB}$?

K Look at D3. Are all the points in plane ABC that satisfy $\overline{AX} \cong \overline{AB}$ represented by dots?

L The geometric figure that is the solution set of $\overline{AX} \cong \overline{AB}$ is a *circle*. Is a circle an infinite set of points?

M Point A is the *centre* of the circle represented in D3. Is A in the solution set of $\overline{AX} \cong \overline{AB}$? Is the centre of the circle a member of the circle?

To name a circle, you may use the name of its centre. The circle represented in D3 is called "circle A."

N Is the centre of the circle an endpoint of $\overline{AB}$? What is the other endpoint of $\overline{AB}$? Is this endpoint in circle A?

A segment one of whose endpoints is the centre of a circle and whose other endpoint is in the circle is a *radius* of the circle.

O Look again at D3. Is $\overline{AB}$ a radius of circle A? Do A and C determine a segment? Is $\overline{AC}$ a radius of circle A? Explain your answers.

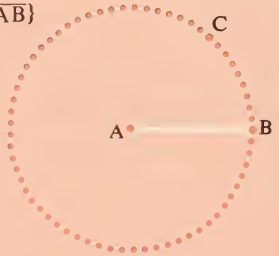
You see that a circle can have more than one radius. You use the word *radii* (rā′ dē ī) when you refer to more than one radius.

P Is each point in the circle an endpoint of a radius? For a given circle, how many radii are there?

Universe for X is plane ABC.
$\overline{AX} \cong \overline{AB}$.

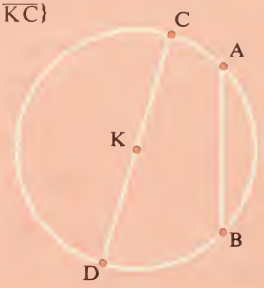
D2

Universe for X is plane ABC.
$\{X \mid \overline{AX} \cong \overline{AB}\}$



D3

Universe for X is plane AKB.
$\{X \mid \overline{KX} \cong \overline{KC}\}$



D4

cir cle (sėr′kəl). The set of all points in a plane that satisfy $\overline{RX} \cong \overline{RT}$. The centre of the circle is point R. A segment one of whose endpoints is the centre of the circle and whose other endpoint is in the circle is a **radius** (rā′di əs) of the circle.

Q For a given circle, is each radius congruent to each of the other radii?

R How many centres does a circle have?

If point A is the centre of a circle that has a radius AB, then the circle is the set of all points in a plane that satisfy $\overline{AX} \cong \overline{AB}$. We say that, in a given plane, circle A is determined by point A and radius AB. A circle is determined by a point called the centre and a segment called a radius.

A Name the circle represented in D4. What is the centre of this circle?

B Name the radii represented in D4.

C Is each point in circle K also in plane AKB?

D Is circle K the solution set of $\overline{KX} \cong \overline{KC}$? Is $\{X \mid \overline{KX} \cong \overline{KC}\}$ the same circle as K? Remember that the universe for X is plane AKB.

E Is $\{X \mid \overline{KX} \cong \overline{KD}\}$ also the same circle as K?

F To write a standard description of a circle, what must you know?

G Look again at D4. Are both endpoints of $\overline{AB}$ in circle K? What other segment represented in D4 has both endpoints in circle K?

A segment that has both of its endpoints in a circle is a *chord* of that circle.

H Is $\overline{CD}$ a chord of circle K? Does $\overline{CD}$ contain the centre of circle K?

A chord that contains the centre of a circle is a *diameter* of the circle.

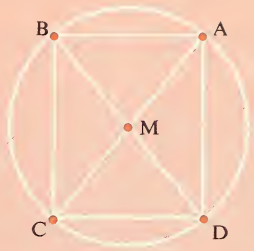
I Is $\overline{CD}$ a diameter of circle K? Is $\overline{AB}$ a diameter of circle K?

J Look at D5. How many radii of circle M are represented? Name them. Name the chords represented. Name the diameters represented.

K How do a radius of a circle and a chord of the same circle differ?

L Suppose that E and F are in circle M. Is $\overline{EF}$ a chord of circle M? Do any two points in a circle determine a chord of that circle? Does a circle have infinitely many chords?

Universe for X is plane MCD.
$\{X \mid \overline{MX} \cong \overline{MD}\}$



D5

chord of a circle (kôrd). A segment whose endpoints are in a circle. A chord of a circle that contains the centre is a **diameter** (dī am′ə tər) of the circle.

co pla nar points (cō plā′nər). Points that belong to the same plane. A circle is a set of coplanar points.

M Think of any point G in circle M. Is G an endpoint of a diameter of circle M? Is each point in circle M an endpoint of a diameter of circle M? Does a circle have infinitely many diameters?

N Look again at D5. Suppose the universe is changed from plane MCD to space. Then what geometric figure is the solution set of $\overline{MX} \cong \overline{MD}$?

You know that all the points in a circle are in the same plane. When all the points of a set belong to the same plane, they are *coplanar points*.

A Are any three given points coplanar? Explain your answer.

B Make a sketch to represent four points that are not coplanar.

You have been studying the circle. Now you will learn about the interior and the exterior of a circle.

C Look at D6. Is $\overline{RS}$ a radius of circle R? What other radius of circle R is represented in D6?

D Read sentence A in D6. What is the universe for X?

E The condition expressed by sentence A requires that each replacement for X, together with the centre of the circle, determine a segment less than a radius. What segment is determined by A and the centre of the circle? Is this segment less than a radius? Is A a solution of $\overline{RX} < \overline{RS}$?

F $\overline{RT} \cong \overline{RS}$, and $\overline{RG} > \overline{RS}$. Is T a solution of $\overline{RX} < \overline{RS}$? Is G a solution of $\overline{RX} < \overline{RS}$? How do you know?

The solution set of $\overline{RX} < \overline{RS}$ is the *interior* of circle R. The interior of circle R is $\{X \mid \overline{RX} < \overline{RS}\}$.

G What points named in D6 are in the interior of circle R?

H Are there any other points in the interior of circle R? Is the interior of a circle an infinite set of points?

I How do circle R and the interior of circle R differ?

J Now read sentence B in D6. Is G a solution of $\overline{RX} > \overline{RS}$? Explain your answer.

K $\overline{RT} \cong \overline{RS}$, and $\overline{RD} < \overline{RS}$. Is T a solution of $\overline{RX} > \overline{RS}$? Is D a solution of $\overline{RX} > \overline{RS}$? How do you know?

The solution set of $\overline{RX} > \overline{RS}$ is the *exterior* of circle R. The exterior of circle R is $\{X \mid \overline{RX} > \overline{RS}\}$.

L What points named in D6 are in the exterior of circle R? Is the exterior of a circle an infinite set of points?

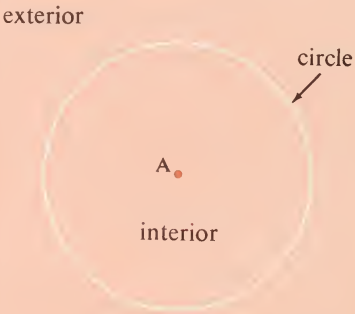
M How do circle R and the exterior of circle R differ?

You have studied three important geometric figures included in a plane. A geometric figure, all of whose points are coplanar, is called a *plane figure*. The plane figures that you have studied in this lesson are the circle, the interior of a circle, and the exterior of a circle.

Universe for X is plane RSN.

A $\overline{RX} < \overline{RS}$. **B** $\overline{RX} > \overline{RS}$.

D6



D7

in te ri or of a circle (in tēr′ē ər). The set of all points in a plane that satisfy $\overline{AX} < \overline{AB}$. If $\overline{AB}$ is a radius of circle A, then $\{X \mid \overline{AX} < \overline{AB}\}$ is the interior of circle A. The universe for X is the plane that includes circle A. The points in the interior of a circle and in the circle are coplanar.

ex te ri or of a circle (eks tēr′ē ər). The set of all points in a plane that satisfy $\overline{AX} > \overline{AB}$. If $\overline{AB}$ is a radius of circle A, then $\{X \mid \overline{AX} > \overline{AB}\}$ is the exterior of circle A. The universe for X is the plane that includes circle A. The points in the exterior of a circle and in the circle are coplanar.

N Study D7. Can any point that is in circle A also be in the interior of circle A? Can any point in the exterior of circle A also be in the interior of circle A?

O A circle separates a plane into two *regions*. The circle is the boundary of each region. One region is the interior of the circle. What is the other region? Can you say that each point in the plane is either in one of the regions or in the boundary?

P Suppose that G is in the interior of a circle and H is in the exterior of the same circle. Is there a point in the circle between G and H?

Q Suppose that G and H are in the interior of a circle. Is there a point in the circle between G and H?

In this lesson you have learned about geometric conditions. Now you know that the solution set of a geometric condition is a geometric figure.

On your own

1 Make a sketch of circle A. Now locate B in the exterior of circle A. Use your sketch for exercises 2, 3, and 4.

2 Locate C in the interior of circle A.

3 Locate D in circle A.

4 Make a sketch of radius AE of circle A. Be sure to label the dot that represents E.

5 Now make a sketch of $\overline{FG}$. Use your sketch for exercises 6 through 11.

6 Locate H between F and G.

7 Locate I between H and G.

8 Make a sketch of the circle that has $\overline{FG}$ as a radius and G as a point in the circle.

9 Make a sketch of the circle that has $\overline{FH}$ as a radius and H as a point in the circle.

10 Now make a sketch of the circle that has segment FI as a radius and I as a point in the circle.

11 How many circles can have the same centre?

A standard description of a circle is given in each of exercises 12 through 15. For each exercise, name the circle, the centre, and one radius. The universe is the plane that includes the circle described.

12 $\{X \mid \overline{RX} \cong \overline{RM}\}$

13 $\{X \mid \overline{BC} \cong \overline{CX}\}$

14 $\{X \mid \overline{XL} \cong \overline{KL}\}$

15 $\{X \mid \overline{ED} \cong \overline{XE}\}$

A circle is described in each of exercises 16 through 19. Write a standard description of each circle. The universe is the plane that includes the circle described.

16 Circle with centre at A and a radius AB

17 Circle C with a radius DC

18 Circle E with a radius congruent to $\overline{FG}$

19 Circle with radii HJ and HK

20 What is a standard description of the interior of circle L with a radius LM? The universe is the plane that includes circle L.

21 What is a standard description of the exterior of a circle that has $\overline{NP}$ and $\overline{NQ}$ as two radii? The universe is plane NPQ.

22 The universe for X is the plane that includes circle R. Circle R is $\{X \mid \overline{XR} \cong \overline{TR}\}$. What is a standard description of the interior of circle R?

Make a sketch of circle M. Now locate points E, F, and G in circle M. Use this sketch for exercises 23 through 30.

23 Make a sketch of radius MG.

24 Make a sketch of the radius that has F as an endpoint.

25 Is each radius of a circle congruent to each of the other radii of the circle?

26 Make a sketch of chord EF.

27 Make a sketch of chord EG.

28 Make a sketch of the diameter that has F as an endpoint.

29 Make a sketch of diameter EH. Label the dot that represents H.

30 Is each diameter of a circle congruent to each of the other diameters? Is a diameter congruent to a radius of the same circle?

CHECKING UP

If you have trouble with these two tests, you should restudy lessons 8 through 12.

Test 3

Use D1 for exercises 1 through 8.

1 How many planes include {B, G, D}?

2 How many segments are determined by C and E?

3 What are the endpoints of $\overline{EG}$?

4 What points named are collinear with C and F?

5 What points named are noncollinear with B and G?

6 What points named are between B and F?

7 E separates ℓ_1 into how many half-lines?

8 Think of the half-lines determined by E. What point named is in the same half-line as C? What points named are in ℓ_1 but are not in the same half-line as C?

Test 4

Use D2 for exercises 9 through 16.

9 Name the centre of circle K.

10 Name two radii of circle K.

11 Name two chords of circle K.

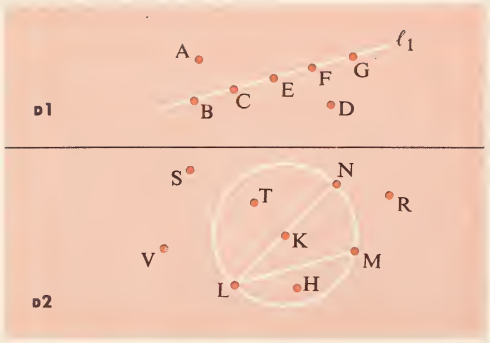
12 Name a diameter of circle K.

13 Name 3 points in the interior of circle K.

14 Name 3 points in the exterior of circle K.

15 Name 3 points that are in circle K.

16 Name a pair of congruent segments.



1 13 Exploring ideas

Number lines and graphs

So far in this book you have been working with sets of objects. First you worked with sets of numbers and with conditions about numbers. Then you worked with sets of points and with conditions about points. In this lesson you first will see how numbers and points are associated. Then you will see how you can use dots to represent a set of numbers.

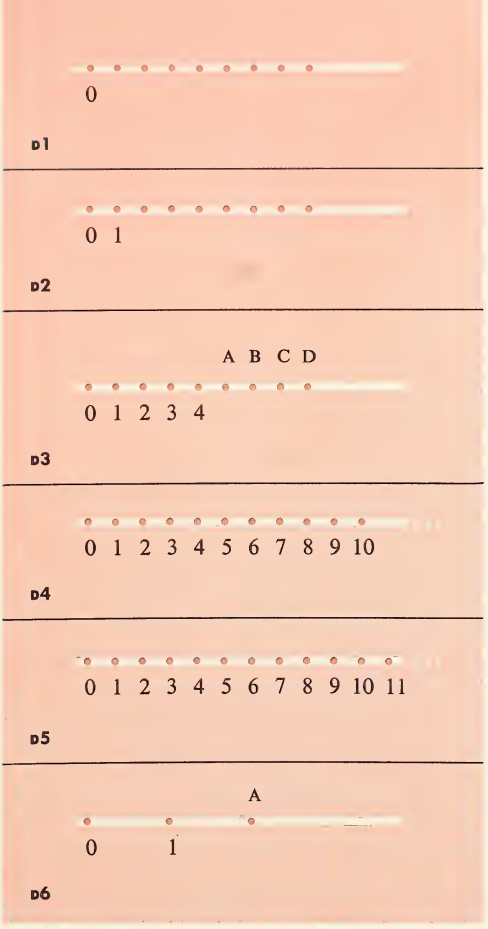
A Look at the picture of a line in D1 on page 51. How many points in the line are represented by dots?

A natural number can be matched with each point that is represented by a dot. Of course, a line is an infinite set of points, but only those points represented by dots are referred to here. Suppose you match, or assign, the number 0 to the point represented by the dot farthest to the left. You may call this point "point 0."

B What natural number is one more than 0? Assign this number to the point whose dot is immediately to the right of the dot for point 0. Remember that only those points represented by dots are being discussed. Call this point "point 1." D2 shows the numerals for 0 and 1 placed below the first two dots.

C What natural number is one more than 1? What number should be assigned to the point represented by the next dot in D2? What is the name of this point?

So far we have assigned numbers to points. We can also assign points to numbers. In either case, numbers and points are *associated*. When



we say "The number 1 is associated with point 1," we mean that the number 1 is assigned to point 1, and that point 1 is assigned to the number 1.

D Look at D3. How many numbers and points have been associated? Why is point 3 to the right of point 2? Why is point 4 to the right of point 3?

E Suppose that you want to locate points to which you can assign natural numbers that are greater than 4. Look again at D3. The points represented by successive dots determine segments. These segments are congruent.

F What numbers should be assigned to points A, B, C, and D represented in D3? Is the segment determined by point 7 and point 8 congruent to the segment determined by point 0 and point 1?

G How would you locate point 9? Point 10?

H Look at D4. You already know that the set of natural numbers is an infinite set. You can think, then, about associating more and more points with the natural numbers. The three small dots to the right of the picture of the line in D4 tell you that the points you can associate with the natural numbers go on and on. Do the points associated with the natural numbers form an infinite set of points?

I A line that has points associated with natural numbers is a *natural-number line*. Is a natural-number line represented in D5?

J In D5, what do the three small dots to the right of the picture of the line represent?

K If more points were located, you could assign a natural number to each of these points. Imagine that you have located all the points to point 50. What points are to the left of point 38?

L What point is represented by the dot immediately to the right of the dot for point 38?

M The universe is the set of natural numbers. ? Are the points assigned to the members of $\{x \mid x < 26\}$ to the right of point 26 in a natural-number line? To the left of point 26?

N Now look at D6. Is the segment determined by point 0 and point 1 greater than the segment determined by point 0 and point 1 as represented in D5?

O Can you use this new segment to locate points in a natural-number line?

P What number should be assigned to point A represented in D6? How would you locate point 3? Point 4?

Any segment may be selected as the segment determined by point 0 and point 1 in a number

line. But once the segment is chosen, it is convenient to use this segment in locating other points.

Point 1 can be located either to the right or to the left of point 0, but it is customary to locate point 1 to the right of point 0, and we have done so in this book.

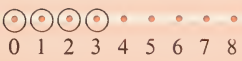
Now you will see how pictures of natural-number lines are used to help you understand subsets of the set of natural numbers.

Look at D7. The dots in D7 form a picture, or *graph*, of the set of natural numbers. Of course, it is impossible to make a dot for each natural number, but the three small dots to the right of the picture of a line indicate that the dots in the graph of the set of natural numbers form an infinite set.

A Describe the set tabulated in D7.

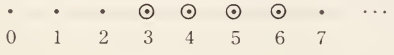
B In D7 the dot representing point 0 is encircled to show that the number 0 belongs to the set tabulated. Why are the dots representing point 1, point 2, and point 3 also encircled? The four dots that are encircled form a graph of {0, 1, 2, 3}. Is {0, 1, 2, 3} a subset of N?

{0, 1, 2, 3}

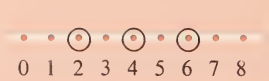


D7

graph (graf). The graph of a set of numbers is a set of dots. Each dot in the graph represents a point associated with a member of the set of numbers. Each member of the set of numbers is associated with a point represented by a dot in the graph. The dots below form a graph of the set of natural numbers. The encircled dots form a graph of {3, 4, 5, 6}, a subset of the set of natural numbers.



{2, 4, 6}



D8

C Look at the picture of a natural-number line in D8. What dots form the graph of N? Does each encircled dot represent a point associated with a member of {2, 4, 6}? What dots form the graph of {2, 4, 6}?

You have learned that a set of numbers and a set of points may be associated. You have also learned that in a graph of the set of natural numbers, you may indicate a particular set of dots by encircling them. The encircled dots form the graph of the particular set of numbers.

D Now look at D9. What is the universe for x in $x < 5$?

E How do you read the standard description of the solution set of the condition? Tabulate the solution set of $x < 5$.

F Look at D10. What dots form the graph of the universe? Why are there no small dots to the right of the picture of the line?

G What dots form the graph of $\{x | x < 5\}$? Why is the dot for point 2 encircled? Why is the dot for point 7 not encircled? Are the dots that form the graph of $\{x | x < 5\}$ a subset of the graph of the universe?

H What is the universe for x in the condition expressed in D11? How is the graph of this universe different from the graph of the universe in D10? What condition is expressed by the sentence $x + 1 < 8$?

I How do you read the standard description of the solution set of $x + 1 < 8$? Tabulate the solution set.

J What dots form the graph of $\{x | x + 1 < 8\}$? Why are certain points in the number line

represented by encircled dots? Why is the dot for point 7 not encircled? How is the graph of $\{x \mid x + 1 < 8\}$ related to the graph of the universe?

Remember that to make a graph of a solution set, you first use a picture of a natural-number line to make a graph of the universe. Then you encircle the dots that form the graph of the solution set.

K Look at D12. Tabulate the solution set described.

L Make a graph of the universe and of the solution set described in D12.

M The graph of the solution set of a condition is shown in D13. What is the condition?

Notice that the three small dots to the right of the picture of a line in D13 have been encircled. This tells you that all the dots representing points to the right of point 10 belong to the graph of $\{x \mid x + 3 > 9\}$. The solution set of $x + 3 > 9$ is an infinite set.

N What is the least solution of $x + 3 > 9$? What is the greatest solution?

O Tabulate the solution set of $x + 7 > 12$. $U = N$. Is $\{x \mid x + 7 > 12\}$ an infinite set? Make a graph of this set.

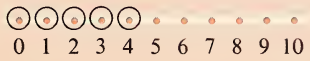
P Tabulate the set of numbers whose graph is shown in D14. Remember that the members of the set are the numbers that are assigned to the points represented by encircled dots.

Q The graph of the solution set of a condition is shown in D14. Give a condition whose solution is shown in the graph. $U = N$.

$U = \{0, 1, 2, \ldots, 10\}$.
$x < 5$.
$\{x \mid x < 5\}$

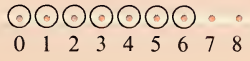
D9

$U = \{0, 1, 2, \ldots, 10\}$.
$\{x \mid x < 5\}$



D10

$U = N$.
$x + 1 < 8$.
$\{x \mid x + 1 < 8\}$

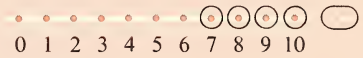


D11

$U = N$.
$\{x \mid x + 3 = 9\}$

D12

$U = N$.
$\{x \mid x + 3 > 9\}$



D13

D14

Now you know how to use a picture of a number line to draw a graph of a set of numbers. You also know that you can use a graph of a set of numbers to make a tabulation of the set.

On your own

For exercises 1 through 8, you are to make a graph of each of the sets tabulated. First

make a graph of the set of natural numbers. Extend your picture of the line to contain as many dots as you need for the particular exercise. Then encircle the dots that represent the points associated with the members of each set.

1 $\{3, 5, 9\}$
2 $\{0, 1\}$
3 $\{9\}$
4 $\{0, 1, 2, \ldots, 12\}$
5 $\{0, 1, 2, 3, 4\}$
6 $\{7, 8, 9, \ldots\}$
7 $\{5, 6, 4, 7, 0\}$
8 $\{\ \}$

Tabulate each of the sets of numbers whose graphs are shown in exercises 9 through 13.

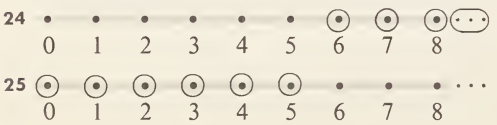
Tabulate each solution set described in exercises 14 through 19. Then make a graph of the universe and of the subset described. U = N.

14 $\{x \mid x < 6\}$
15 $\{x \mid x + 3 > 10\}$
16 $\{x \mid x \neq x\}$
17 $\{x \mid 2x < 20\}$
18 $\{x \mid x + 1 = 1 + x\}$
19 $\{x \mid 3x > 2x\}$

For each solution set described in exercises 20 through 23, make a graph of the universe and of the subset described. U = N.

20 $\{x \mid 7 < x\}$
21 $\{x \mid x + 6 > 11\}$
22 $\{x \mid 2x + 1 = 3\}$
23 $\{x \mid 3x + 1 = x\}$

The graphs of two solution sets are shown in exercises 24 and 25. Give a condition for each solution set. U = N.



CHECKING UP

These tests will help you find out how well you can compute.

Test 5
Add
1 3647, 8492, 6386.
2 7652, 5295, 9821, 1370.
3 4288, 9095, 2927, 4696.
4 1967, 3499, 2028, 356.
5 5026, 428, 7309, 819.
6 9273, 429, 738, 4976.
7 32167, 89450, 89629.
8 14023, 37259, 28435.
9 34276, 8075, 50392.
10 72943, 408, 6661.

Test 6
Subtract
11 54298 from 76032.
12 27292 from 36376.
13 43986 from 72010.
14 385749 from 942617.
15 293872 from 400307.
16 84729 from 620038.
17 96273 from 142968.
18 63672 from 70241.
19 5682 from 12013.
20 7683 from 303452.

Test 7
Multiply
21 48612 by 8.
22 368 by 40.
23 38 by 24.
24 109 by 60.
25 318 by 75.
26 849 by 93.
27 506 by 41.
28 180 by 86.
29 726 by 29.
30 570 by 59.

Test 8
Multiply
31 308 by 900.
32 639 by 796.
33 810 by 593.
34 168 by 321.
35 703 by 327.
36 514 by 548.
37 480 by 307.
38 630 by 180.
39 725 by 905.
40 608 by 305.

Test 9
Divide
41 5184 by 54.
42 1504 by 16.
43 1400 by 40.
44 4450 by 89.
45 2048 by 32.
46 4816 by 43.
47 23850 by 75.
48 13356 by 84.
49 6960 by 29.
50 23923 by 47.

Test 10
Divide
51 7616 by 136.
52 12992 by 224.
53 10791 by 109.
54 29718 by 127.
55 613470 by 726.
56 55480 by 380.
57 37761 by 123.
58 106797 by 367.
59 253495 by 419.
60 150700 by 275.

Multiplication

The computation you will use to find solutions of conditions includes multiplication. This lesson will help you review multiplication involving natural numbers.

When you compute, using multiplication, the number you multiply is the *multiplicand*. The number you multiply by is the *multiplier*. The answer you get is the *product*. In $7 \times 4 = 28$, the multiplier is 7; the multiplicand is 4; and the product is 28.

$56 = 50 + 6.$ To multiply 423 by 56, first think of 56 as $50 + 6$.

Imagine that 6 groups of 423 are combined. Multiply 423 by 6.

```
 423
   6
2538 <-------- 6 × 423 = 2538.
```

Now imagine that 50 groups of 423 are combined. Multiply 423 by 50.

```
 423
  50
21150 <------- 50 × 423 = 21150.
```

You have multiplied 423 by 6 and by 50. Imagine that the 50 groups of 423 are put with the 6 groups of 423. Add 2538 and 21150.

```
 2538
21150
23688 <------- 56 × 423 = 23688.
```

You should write all of the work in one place, as shown at the left.

```
  423
   56
 2538 <-------- 6 × 423 = 2538.
21150 <-------- 50 × 423 = 21150.
23688 <-------- 56 × 423 = 23688.
```

The following exercises will help you multiply 508 by 49.

$49 = 40 + 9.$ First think of 49 as the sum of what two numbers?

```
 508
  49
4572
```

You first imagine that how many groups of 508 are combined? You multiply 508 by what number? What product do you get?

```
 508
  49
4572
20320
```

You next imagine that how many groups of 508 are combined? You multiply 508 by what number? What product do you get?

```
 508
  49
4572
20320
24892
```

Why is the numeral 20320 written under the numeral 4572? Why do you add 4572 and 20320? What product do you obtain when you multiply 508 by 49?

The work that follows will help you multiply 276 by 394. First think of 394 as $300 + 90 + 4$.

```
 276
 394
1104
```

You first imagine that how many groups of 276 are combined? You multiply 276 by what number? What product do you get?

276 394 1104 24840	You next imagine that how many groups of 276 are combined? You multiply 276 by what number? What product do you get?
276 394 1104 24840 82800	You then imagine that how many groups of 276 are combined? You multiply 276 by what number? What product do you get?
276 394 1104 24840 82800 108744	What product is obtained when you multiply 276 by 394?

For each of exercises A through F, find the product. After you complete your work, compare it with the work given below the exercise.

A Multiply 47092 by 7.

47092
7
329644

B Multiply 469 by 80.

469
80
37520

C Multiply 568 by 38.

568
38
4544
17040
21584

D Multiply 196 by 408.

196
408
1568
78400
79968

E Multiply 8692 by 24.

8692
24
34768
173840
208608

F Multiply 716 by 490.

716
490
64440
286400
350840

Now you know how to multiply.

On your own

Multiply

1 95763 by 9.
2 98 by 76.
3 167 by 45.
4 4382 by 78.
5 659 by 52.
6 3759 by 800.
7 302 by 178.
8 412 by 645.
9 382 by 791.
10 107 by 570.
11 902 by 405.
12 780 by 208.

1 15 Learning computation

Preparation for division

This lesson will help you review certain ideas you need to understand when you divide.

In your work in arithmetic, you learned about rounding numbers. 7614 can be rounded to the next greater thousand, 8000, or to the next lesser thousand, 7000. It can be rounded to the next greater hundred, 7700, or to the next lesser hundred, 7600. 7614 can also be rounded to the next greater ten, 7620, or to the next lesser ten, 7610.

A If you round 42,304 to the next greater thousand, do you get 43,000? If you round 42,304 to the next lesser hundred, do you get 42,300?

B Round 217 to the next greater hundred. To the next lesser hundred. To the next lesser ten.

C Round 4568 to the next lesser thousand. To the next greater ten. To the next lesser hundred.

D Round 74 to the next lesser ten. Round 72,468 to the next lesser hundred.

E Round 592 to the next greater hundred. Round 17,009 to the next lesser thousand.

When you compute using division, the number you divide is the *dividend*. The number you divide by is the *divisor*. The answer you get is the *quotient*. In $28 \div 4 = 7$, the dividend is 28; the divisor is 4; and the quotient is 7.

When we say "the quotient of 30 and 5," we mean the number that is obtained when the first number, 30, is divided by the second number, 5. The quotient of 30 and 5 is 6.

F What is the quotient of 24 and 6? Divide both 24 and 6 by 2. What two numbers do you get? What is the quotient of these two numbers?

G When you divide 24 by 6, do you get the same quotient as when you divide 12 by 3?

H The quotient of 270 and 90 is 3. Divide both the dividend and the divisor by 10. What two numbers do you get? What is the quotient of these two numbers?

I Is the quotient of 270 and 90 the same as the quotient of 27 and 9?

J The quotient of 5600 and 700 is 8. Divide both the dividend and the divisor by 100. What two numbers do you get? What is the quotient of these two numbers?

K Is the quotient of 5600 and 700 the same as the quotient of 56 and 7?

To show that you have divided both 270 and 90 by 10, you can write the sentence $27\text{Ø} \div 9\text{Ø} = 3$. To show that you have divided both 5600 and 700 by 100, you can write the sentence $56\text{ØØ} \div 7\text{ØØ} = 8$.

L If you divide both the dividend and the divisor by the same number, is the quotient changed?

M What is the quotient of 15 and 3? Multiply both 15 and 3 by 3. What two numbers do you get? What is the quotient of these two numbers?

N When you divide 15 by 3, do you get the same quotient as when you divide 45 by 9?

O What is the quotient of 24 and 8? Multiply both the dividend and the divisor by 100. What numbers do you get?

P The quotient of 2400 and 800 is 3. When you divide 24 by 8, do you get the same quotient as when 2400 is divided by 800?

Q If you multiply both the dividend and the divisor by the same number (except 0), is the quotient changed?

Now you know how to round numbers. You also know that when you multiply or divide both the dividend and the divisor by the same number (except 0), the quotient remains the same.

On your own

For exercises 1 through 9, round each of the numbers named to the next greater hundred, to the next lesser hundred, and to the next greater ten. For exercise 1, you write the numerals 400, 300, 370.

1 362	**4** 9836	**7** 760
2 815	**5** 3178	**8** 4105
3 427	**6** 19,368	**9** 24,214

For exercises 10 through 18, round each of the numbers named to the next greater thousand, to the next lesser hundred, and to the next lesser ten.

10 2476	**13** 7638	**16** 17,803
11 6911	**14** 53,259	**17** 8910
12 4532	**15** 42,015	**18** 2596

For each of exercises 19 through 32, first tell how the second dividend or divisor was obtained from the first. Then find the missing number. For exercise 19, you could write: "Divide by 100. The missing divisor is 8."

	Dividend	Divisor		Dividend	Divisor
19	2400	800	**21**	360	40
	24	?		?	4
20	4	2	**22**	28	7
	?	8		280	?

	Dividend	Divisor		Dividend	Divisor
23	72	8	**28**	32	8
	?	1		?	2
24	3200	800	**29**	15	3
	32	?		1500	?
25	54	6	**30**	4800	80
	?	600		?	8
26	81,000	90	**31**	21	3
	?	9		?	9
27	24	3	**32**	28,000	400
	240	?		280	?

Division

When you find the solutions of some conditions, you will need to use division. This lesson will help you review division involving natural numbers.

To find the quotient of 3057 and 39, you can use the ideas you reviewed in the last lesson. It is helpful to round the divisor to a greater number and the dividend to a lesser number. When you round the numbers in this way, you will always get a quotient that you can use. First round the divisor, 39, to the next greater ten, 40. Then round the dividend, 3057, to 2800 because you know that the quotient of 2800 and 40 is 70. Would it be sensible to round the dividend to 2900? To 3000?

$280\text{Ø} \div 4\text{Ø} = 70.$

```
39)3057 | 70   <-- Use 70 groups of 39.
 2730   <-- 70 × 39 = 2730.
  327   <-- Subtract 2730 from 3057 to find how many are left to divide into groups of 39.
```

Next divide 327 by 39. Round the 39 to 40 and the 327 to 320.

$32\text{Ø} \div 4\text{Ø} = 8.$

```
39)3057 | 70
 2730
  327 |  8   <-- Use 8 groups of 39.
  312   <-- 8 × 39 = 312.
   15   <-- 327 − 312 = 15.
```

```
39)3057 | 70
 2730
  327 |  8   Add the groups of 39.
  312
   15 | 78   <-- There are 78 groups of 39.
      ↑
      15 is left over. 15 is the remainder.
```

Since 15 is less than 39, you cannot remove another group of 39.

$70 + 8 = 78.$

The work below will help you find the quotient of 5423 and 78. Will you round 78 to 80? Will you round 5423 to 4800? Explain your answers. Why will you first use 60 groups of 78?

```
78)5423 | 60
 4680
  743
```

What product do you get when you multiply 78 by 60? Why do you subtract 4680 from 5423?

```
78)5423 | 60
 4680
  743 |  9
  702
   41
```

Next you divide 743 by 78. Round 78 to what number?

Round 743 to what number?

What is the quotient of 720 and 80? Why do you now use 9 groups of 78?

What product do you get when you multiply 78 by 9?

Why do you subtract 702 from 743?

```
78)5423 | 60
  4680  |
   743  |  9
   702  |
    41  | 69
```

How do you know that you cannot remove another group of 78?

How do you find the total number of groups of 78? Is there a remainder?

There are how many groups of 78 in 5423? What is the remainder?

The exercises that follow will help you find the quotient of 38,793 and 193. First round the divisor, 193, to 200. Then round the dividend, 38,793, to 20,000. You use 20,000 because you can easily find the quotient of 20,000 and 200.

What is the quotient of 20000 and 200?

```
193)38793 | 100
  19300   |
  19493   |
```

Why will you first use 100 groups of 193?

What product do you get when you multiply 193 by 100?

Why do you subtract 19,300 from 38,793?

```
193)38793 | 100
  19300   |
  19493   |  90
  17370   |
   2123   |
```

Next you divide 19,493 by 193.

You round 193 to what number?

You round 19,493 to what number?

What is the quotient of 18,000 and 200?

Why do you now use 90 groups of 193?

What product do you get when you multiply 193 by 90?

Why do you subtract 17,370 from 19,493?

```
193)38793 | 100
  19300   |
  19493   |  90
  17370   |
   2123   |  10
   1930   |
    193   |
```

Now you divide 2123 by 193. Round 193 to what number? Round 2123 to what number?

What is the quotient of 2000 and 200? Why do you use 10 groups of 193?

What product do you get when you multiply 193 by 10?

Why do you subtract 1930 from 2123?

Can you remove another group of 193?

```
193)38793 | 100
  19300   |
  19493   |  90
  17370   |
   2123   |  10
   1930   |
    193   |   1
    193   |
      0   | 201
```

What is the quotient of 193 and 193?

How do you find the total number of groups of 193?

Is the remainder 0?

What is the quotient of 38,793 and 193?

The work below shows the first steps you used in finding the quotient of 38,793 and 193.

20000 ÷ 200 = 100.

```
193)38793 | 100 ← Use 100 groups of 193.
  19300   ←——— 100 × 193 = 19300.
  19493   ←——— 38793 − 19300 = 19493.
```

```
193)38793 | 100
  19300   |
  19493   | 100
  19300   |
    193   |   1
    193   |
      0   | 201
```

Next you divided 19,493 by 193. What product do you get when you multiply 193 by 100?

Can you remove another 100 groups of 193 instead of 90 groups of 193? Explain your answer.

Sometimes it is very helpful to round the divisor and dividend. At other times, it is not necessary to round them. You can decide for yourself when you should round the dividend and divisor.

For each of exercises A through D, find the quotient. Then compare your work with the work shown below the exercises. The groups you used may not be the same as those shown below, but the answer should be the same.

A Divide 3000 by 56.

$$\begin{array}{r|l} 56\overline{)3000} & 50 \\ 2800 & \\ \hline 200 & 3 \\ 168 & \\ \hline 32 & 53 \end{array}$$

B Divide 2323 by 101.

$$\begin{array}{r|l} 101\overline{)2323} & 20 \\ 2020 & \\ \hline 303 & 3 \\ 303 & \\ \hline 0 & 23 \end{array}$$

C Divide 15603 by 538.

$$\begin{array}{r|l} 538\overline{)15603} & 20 \\ 10760 & \\ \hline 4843 & 8 \\ 4304 & \\ \hline 539 & 1 \\ 538 & \\ \hline 1 & 29 \end{array}$$

D Divide 50025 by 72.

$$\begin{array}{r|l} 72\overline{)50025} & 600 \\ 43200 & \\ \hline 6825 & 80 \\ 5760 & \\ \hline 1065 & 10 \\ 720 & \\ \hline 345 & 4 \\ 288 & \\ \hline 57 & 694 \end{array}$$

Now you know how to divide.

On your own

Divide

1 2840 by 29.
2 6302 by 46.
3 11700 by 38.
4 33040 by 74.
5 49640 by 246.
6 36320 by 89.
7 84384 by 308.
8 96600 by 190.
9 5000 by 19.
10 104000 by 759.
11 343200 by 65.
12 132615 by 315.
13 70000 by 563.
14 29700 by 605.

KEEPING SKILFUL

Find the sum.

1 7985, 856, 114
2 294, 703, 5578
3 7216, 6532, 1187
4 5769, 3014, 83118
5 18568, 88923
6 95410, 588736

Find the difference.

7 5701, 843
8 8563, 8097
9 7275, 5086
10 11345, 7649
11 90357, 89756
12 74001, 63428

Multiply

13 68 by 34.
14 1357 by 7.
15 80 by 54.
16 706 by 89.
17 145 by 86.
18 758 by 900.
19 150 by 408.
20 426 by 534.

Divide

21 4641 by 53.
22 3342 by 68.
23 2809 by 24.
24 16068 by 52.
25 2088 by 124.
26 16929 by 213.
27 31416 by 154.
28 18373 by 123.

For each of exercises 29 through 38, tabulate the solution set. U = {0, 2, 4, 6, 8, 10, 12}.

29 $x + 5 = 11$.
30 $5(3) > m$.
31 $12 - d = 4$.
32 $14 > 8 + s$.
33 $2v = 8$.
34 $3 < b + 2$.
35 $n + 3 = 6$.
36 $d > 76 \div 19$.
37 $x + 9 = 21$.
38 $s < 15 - s$.

For each of exercises 39 through 48, tabulate the solution set. U = N.

39 $27 + n = 41$.
40 $x + 9 < 17$.
41 $m + 21 = 59$.
42 $s + s > 14$.
43 $26 + n = 11$.
44 $f + f = 16$.
45 $6n = 35$.
46 $13 + d > 12$.
47 $2x = x$.
48 $21(8) > v$.

For each solution set described in exercises 49 through 56, make a graph of the universe and of the solution set described. U = N.

49 $\{n \mid n < 9\}$
50 $\{s \mid s + 2 > 3\}$
51 $\{b \mid b + b = 8\}$
52 $\{d \mid d + 3 > 2\}$
53 $\{w \mid w + 3 < w + 4\}$
54 $\{f \mid f \neq f\}$
55 $\{x \mid 13 + x < 14\}$
56 $\{m \mid m + 4 > 4\}$

Problems involving conditions for equality

In this lesson you will learn to write sentences that express conditions for equality for problems. U = N.

When you read a problem, you actually read sentences that express a problem. Before you can answer the question asked, you must think about the condition for the problem. The work below shows how to develop the condition for the problem in D1. The markers represent coins.

You know how many quarters Sue had at first. Because Sue had 15 quarters to begin with, use 15 in the condition.

15

---

n

Sue spent some quarters, but you do not know how many. You can use *n* as a variable for the number of quarters she spent.

You can use $15 - n$ in the condition. $15 - n$ refers to the number of quarters Sue had left.

$15 - n$

You know that Sue had 9 quarters left. You can also use 9 for the number of quarters she had left.

9

$15 - n = 9.$

Now you have the condition for the problem.

Sue had 15 quarters. After she spent some of them, she had 9 quarters left. How many quarters did Sue spend?

D1

George collects coins. After Joe gave him 5 coins, he had 39 coins. How many coins did George have before Joe gave him the 5 coins?

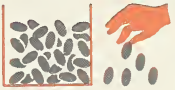
D2

Read the problem in D2. The work that follows shows how to develop the condition for the problem.

You do not know how many coins were in George's collection at first. You can use x as a variable for the number of coins he had at first.

x

You know that Joe gave 5 coins to George. You can use $x + 5$ in the condition. $x + 5$ refers to the number of coins George had then.



$x + 5$

You know how many coins George had after Joe had given him 5. You can also use 39 for the number of coins George had then.

39

$x + 5 = 39.$

Now you have the condition for the problem.

In later lessons you will find solution sets of conditions like $15 - n = 9$ and $x + 5 = 39$. The solutions of the conditions will be used to obtain the answers to the problems.

A Reread the problem in D1. Why is $15 - n$, and not $15 + n$, used in the condition for the problem?

B Why is the idea of equality used in the condition?

C Reread the problem in D2. Why is $x + 5$, and not $x - 5$, used in the condition for the problem?

D Why is the idea of equality used in the condition?

E What name is given to such conditions as $15 - n = 9$ and $x + 5 = 39$?

Read the problem in D3. First you will develop the condition for the problem.

F Is 17 the number of model cars Ernie had in May? 17

G Do you know how many model cars he built during the summer? For what is n a variable? n

H Why is $17 + n$ used in the condition instead of $17 - n$? $17 + n$

I Did Ernie have 32 model cars at the end of the summer? 32

J Why do you use the idea of equality in the condition? $17 + n = 32.$

Read the problem in D4. First it is necessary to develop the condition for the problem.

K Do you know how many stamps Ann bought? For what is y a variable? y

L Why is $y - 14$ used in the condition instead of $y + 14$? $y - 14$

M Does Ann have 18 stamps left? 18

N Why do you use the idea of equality in the condition? $y - 14 = 18.$

Read the problem in D5. Then read the sentence in D5 that expresses the condition for the problem.

Sue had 15 quarters. After she spent some of them, she had 9 quarters left. How many quarters did Sue spend?

D1

George collects coins. After Joe gave him 5 coins, he had 39 coins. How many coins did George have before Joe gave him the 5 coins?

D2

In May, Ernie had 17 model cars. During the summer, he built some more cars. Then he had 32 model cars. How many model cars did Ernie build during the summer?

D3

Ann bought some postage stamps. She has used 14 of them. Now she has 18 stamps left. How many stamps did Ann buy?

D4

The temperature was 38 degrees above zero at 6 A.M. By noon it had risen 7 degrees. What was the temperature at noon?

$$38 + 7 = x.$$

D5

Nancy had 25 tickets to sell for the school play. She has sold 14 of these tickets. How many tickets does Nancy have left to sell?

$$25 - 14 = n.$$

D6

O For what is x a variable?

P Why is $38 + 7$ used in the condition instead of $38 - 7$? Why is the idea of equality used in the condition?

Read the problem in D6. Then read the sentence in D6 that expresses the condition for the problem.

Q For what is n a variable?

R Why is $25 - 14$ used in the condition instead of $25 + 14$? Why is the idea of equality used in the condition?

In some problems, you must compare the number of objects in one group with the number of objects in another group. Now you will learn how to develop conditions for problems of this kind.

Read the problem in D7 and the sentence that expresses the condition for the problem. You will learn why you can use $37 - 26 = x$ as the condition for this problem.

A How many dimes are there? How many nickels are there?

You cannot, of course, remove the group of 26 nickels from the group of 37 dimes, but you can remove a group of dimes equal in number to the group of nickels. Imagine that a group of 26 dimes is removed from the group of 37 dimes. The number of dimes left tells you how many more dimes than nickels Joe has.

When you find how many more or how many fewer objects there are in one group than in another, you are finding the *difference* of the numbers of objects in the two groups. When we say "the difference of 100 and 74," we mean that the second number, 74, is subtracted from the first number, 100. The difference of 100 and 74 is 26.

In this kind of comparison problem, there are three things to think about: the number of objects in the larger group, the number of objects in the smaller group, and the difference of the numbers.

Joe has 37 dimes and 26 nickels. He has how many more dimes than nickels?

$$37 - 26 = x.$$

D7

Joe has 11 more dimes than nickels. He has 26 nickels. How many dimes does he have?

$$x - 26 = 11.$$

D8

Now read the problem in D8. You will learn why you can use $x - 26 = 11$ as the condition for this comparison problem.

B In this problem, what two groups are compared?

C Do you know the number of objects in the larger group? Do you know the number of objects in the smaller group?

D Which does the x in the condition refer to, the number of dimes or the number of nickels? The 26 refers to the number of dimes in a group that has the same number of objects as the group of nickels.

E How many more objects are there in the larger group than in the smaller group?

F Is 11 the difference of the numbers?

G Which does the problem in D8 ask you to find, the number of objects in the larger group, the number of objects in the smaller group, or the difference of the numbers? How do you know?

H Why do you imagine that a group of 26 dimes has been removed from the total group of dimes? Will the number of dimes left be the same as the difference of the numbers in the two groups?

Joe has 11 fewer nickels than dimes. He has 37 dimes. How many nickels does he have?

$$37 - x = 11.$$

D9

Read the problem in D9. You will learn why you can use $37 - x = 11$ as the condition for this comparison problem.

I What two groups have been compared in this problem?

J Do you know the number of objects in the larger group? In the smaller group?

K The 37 in the condition refers to the number of objects in which group? The x refers to the number of dimes in a group that has the same number of objects as what other group?

L Is 11 the difference of the numbers?

M Which does the problem in D9 ask you to find, the number of objects in the larger group, the number of objects in the smaller group, or the difference of the numbers?

N Why do you imagine that a group of dimes is removed from the 37 dimes? Does the group removed have the same number of objects as the group of nickels? Will the number of dimes left be the same as the difference of the numbers in the two groups?

Now you know how to write sentences that express conditions for equality for problems.

On your own

For each problem, write a sentence that expresses the condition for the problem. Use whatever letter you wish to express the variable. Do not find the solution set. U = N.

1 When Margaret went to camp, she weighed 98 pounds. She lost 3 pounds while she was at camp. How many pounds did Margaret weigh when she came home from camp?

2 Sam scored 6 points during the first quarter of a basketball game. By the end of the second quarter, he had scored a total of 14 points. How many points did Sam score during the second quarter?

3 Judy promised to make 12 posters for a school party. She has made 8 of the posters. How many more posters does she need to make?

4 The height of Mt. McKinley in the United States is 20,320 feet. The height of Mt. King in Canada is 17,130 feet. Mt. McKinley is how many feet higher than Mt. King?

5 In a federal election in 1957, 92,858 votes were cast in Newfoundland, and 394,130 votes were cast in Nova Scotia. How many votes in all were cast in these two provinces?

6 Jerry had 72 cents. He spent some of this money and had 63 cents left. How many cents did Jerry spend?

7 There were 3,638 Canadian post offices in 1868. By 1958 there were 12,500 post offices. How many more Canadian post offices were there in 1958 than in 1868?

8 Mr. Martin bought 40 lambs at a sale. Then he had a total of 117 lambs. How many lambs did Mr. Martin have before he bought the 40 lambs?

9 During a baseball season, Lou Gehrig of the New York Yankees hit 34 more doubles than triples. He hit 18 triples. How many doubles did he hit?

10 After Helen spent 34¢ of the money in her purse, she had 18¢ left. How many cents did Helen have in her purse before she spent the 34¢?

11 Alice collects pictures of famous singers. She has 63 pictures of folk singers. She has 25 fewer pictures of opera singers than she has of folk singers. How many pictures of opera singers does Alice have?

Problems involving conditions for inequality

Conditions help you to solve problems. You have made conditions for equality for some problems. You will now learn to make conditions for inequality for problems. U = N.

Read the problem in D1. First you will develop the condition for this problem.

You know how many airplanes Allan has and how many airplanes Pete has. The markers below represent the boys' airplanes.

Allan has 13 airplanes.

Pete has 10 airplanes.

Imagine that Pete's airplanes are put with Allan's airplanes. Because the 10 airplanes are put with the 13 airplanes, you use 13 + 10 in the condition. 13 + 10 refers to the total number of airplanes that Alian and Pete have.

13 + 10

You do not know how many airplanes Jim can have, but you do know that, together, Allan and Pete have more airplanes than Jim. Use x as a variable for the number of airplanes Jim can have. Together, Allan and Pete have more airplanes than Jim has. Now you have the condition for the problem.

x

$13 + 10 > x.$

Allan has 13 model airplanes, and Pete has 10. Together, Allan and Pete have more model airplanes than Jim. How many model airplanes can Jim have?

D1

Margaret bought 27 post cards. She mailed some of them and then had fewer than 8 post cards left. How many post cards could Margaret have mailed?

D2

Now read the problem in D2. The first thing to do is to develop the condition for the problem. You know the number of post cards Margaret bought. The markers at the left represent the 27 post cards that Margaret bought.

27

Margaret mailed some post cards, but you do not know how many. Use n as a variable for the number of post cards that Margaret could have mailed.

n

Because she removed some of the 27 post cards, use $27 - n$ in the condition. $27 - n$ refers to the number of cards she had left.

$27 - n$

Margaret had fewer than 8 post cards left. Now you have the condition for the problem.

$27 - n < 8.$

In later lessons you will find solution sets of conditions like $13 + 10 > x$ and $27 - n < 8$. The solutions of the conditions will be used to obtain answers to the problems.

A Read the problem in D1 again. Why is $13 + 10$, and not $13 - 10$, in the condition for this problem?

B Now reread the problem in D2. Why is $27 - n$, and not $27 + n$, in this condition?

C What name is given to such conditions as $13 + 10 > x$ and $27 - n < 8$? How are these conditions different from the conditions you made for problems in lesson 17?

Read the problem in D3. You will develop the condition for this problem.

D Do you know how many boards Ken could have bought? For what is x a variable? x

E Is 16 the number of boards that Ken used? 16

F Why is $x - 16$ used in the condition instead of $x + 16$? Why is $x - 16$ used to refer to the number of boards that Ken had left? $x - 16$

G Why is the idea of "less than" used in the condition? $x - 16 < 5.$

Read the problem in D4. The exercises that follow will help you to develop the condition for this problem.

H Do you know how many snapshots Judy could have had at first? For what is n a variable? n

I Is 9 the number of new snapshots that Judy put into her album? 9

J Why is $n + 9$, and not $n - 9$, used in the condition? For what do you use $n + 9$? $n + 9$

K Why is the idea of "greater than" used in the condition? $n + 9 > 50$

Read the problem in D5. A sentence that expresses the condition for the problem is also shown.

L For what is n a variable?

Allan has 13 model airplanes, and Pete has 10. Together, Allan and Pete have more model airplanes than Jim. How many model airplanes can Jim have?

D1

Margaret bought 27 post cards. She mailed some of them and then had fewer than 8 post cards left. How many post cards could Margaret have mailed?

D2

Ken bought some boards to build a bookcase and a shelf. He used 16 of the boards and had fewer than 5 boards left. How many boards could Ken have bought?

D3

After Judy put 9 new snapshots into her album, she had more than 50 snapshots in it. How many snapshots could Judy have had in her album before she put in the 9 new ones?

D4

Martha had 73 shells. She gave 18 of the shells to Rosemary. Then Martha had fewer shells than Ruth. How many shells could Ruth have had?

$$73 - 18 < n.$$

D5

M Why is $73 - 18$, and not $73 + 18$, used in the condition? What does $73 - 18$ refer to in the problem? Why is the idea of "less than" used in the condition?

Read the problem in D6. A sentence that expresses the condition is also shown.

N For what is y a variable?

O Why is $15 + y$, and not $15 - y$, used in the condition? For what do you use $15 + y$? Why is the idea of "greater than" used in the condition?

P Read the problem in D7. You are to write a sentence that expresses the condition for this problem. For what, in the problem, will you use a variable?

Q Why will you use $10 + 5$, and not $10 - 5$, in the condition?

R Why will you use the idea of "less than" in the condition?

S Write a sentence that expresses the condition for the problem.

Now you know how to write sentences that express conditions for inequality for problems.

On your own

For each problem write a sentence that expresses the condition. To express the variable, use whatever letter you wish. Do not find the solution set. U = N.

1 Dick had \$7. After he earned \$3, he had more dollars than John. How many dollars could John have had?

2 Ann bought 24 buttons. After she had sewed some of the buttons on a dress, she had more than 12 buttons left. How many buttons could Ann have sewed on the dress?

3 Virginia spent less than 75 cents for a notebook and pencils. She spent 49 cents for the notebook. How many cents could she have spent for the pencils?

4 Fred sold 29 papers, and Jim sold 33 papers. Together, Fred and Jim sold fewer papers than Don. How many papers could Don have sold?

5 Jane put 8 fish into her aquarium. Then she had more than 19 fish in the aquarium. How many fish could have been in the aquarium before Jane put in the 8 fish?

6 The total population of Saskatchewan and Manitoba is less than 1,810,000. The population of Manitoba is 885,000. What can the population of Saskatchewan be?

7 Jay had 17 feet of track for his miniature railroad. He bought more track and now has more than 31 feet of track. How many feet of track could Jay have bought?

8 A school auditorium can seat three hundred persons. Fewer than twelve seats were empty for the school play. How many persons could have attended the school play?

9 The area of Ontario is 412,582 square miles. The area of New Brunswick is 28,354 square miles. The difference in area of these provinces is greater than the area of British Columbia. What can be the area of British Columbia?

THINK What two groups are compared in problem 9?

10 During the year, Miss Green added \$300 to her savings account. By the end of the year, she had more than \$850 in her account. How many dollars could she have had in her account at the beginning of the year?

George had 15 arrowheads. After he found some more on his vacation, he had more than 40 arrowheads. How many arrowheads could George have found on his vacation?

$$15 + y > 40.$$

D6

Sarah had 10 phonograph records. She bought 5 more records, but she still has fewer records than Alice. How many phonograph records can Alice have?

D7

11 Ralph has 26 model cars. John has fewer model cars than Ralph. The difference in the number of cars they have is less than 9. How many model cars can John have?

12 Together, the students at Flint School and at Boone School collected less than $425 for playground equipment. The students at Boone collected $195. How many dollars could the students at Flint have collected?

13 Mr. Williams owns a ranch in Alberta. He sold 1280 acres of his land. Now he has more than 2480 acres of land left. How many acres of land could Mr. Williams have owned before he sold the 1280 acres?

KEEPING SKILFUL

Make one sketch for exercises 1 through 5.

1 Make a sketch of a plane.

2 Locate points A, B, and C in the plane. A, B, and C are noncollinear.

3 How many lines are determined by points A, B, and C? Make sketches of these lines.

4 Locate point D between A and C.

5 How many lines are determined by A, D, C, and B?

Make one sketch for exercises 6 through 12.

6 Make a sketch of circle K.

7 Make a sketch of diameter FG.

8 Make a sketch of chord FH. Locate J between F and H.

9 Locate L and M in the extérior of circle K.

10 Tabulate the subset of {F, G, H, J, K, L, M} that is included in circle K.

11 Tabulate the subset of {F, G, H, J, K, L, M} that is included in the interior of circle K.

12 Tabulate the subset of {F, G, H, J, K, L, M} that is included in the exterior of circle K.

13 Locate seven points, no three of which are collinear. How many lines are determined by these points? Make sketches of these lines.

Solution sets of conditions for equality

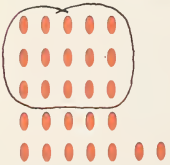
In this lesson you will use conditions for equality to help solve problems. You will compute to find the solutions of the conditions. $U = N$.

Read problems A and B in D1. Does the sentence in red express the condition for problem A? Does it also express the condition for problem B?

$15 + n = 27.$

Why is $15 + n$, and not $15 - n$, in the condition? For what, in problem A, is n a variable? For what, in problem B, is n a variable?

The 27 markers at the left represent the 27 place cards Mary needs in problem A. What do the 27 markers represent for problem B?



15 of the 27 markers represent the number of place cards that Mary has already made. What do the 15 markers represent for problem B?

Notice that 15 of the 27 markers are dimmed. Imagine that 15 of the place cards are removed from the 27 place cards. What does the dimming of the 15 markers suggest for problem B?

A Mary is making place cards for her birthday party. She has made 15 of them, but she needs 27 place cards in all. How many more place cards should she make?

B Jim had 15 pictures of old automobiles. He bought more pictures and now has 27 pictures of old automobiles. How many more pictures of old automobiles did Jim buy?

D1

C Bob did not have as many nails as he needed for building a shelf. After he found 8 more, he had 17 nails. How many nails did Bob have before he found the 8 nails?

D Joan made some paper flowers. She made 8 more and then had 17 flowers in all. How many flowers had Joan made before she made the 8 extra ones?

D2

A The removal of 15 markers from 27 markers suggests subtracting 15 from 27 to get the solution of $15 + n = 27$. Tabulate the solution set.

B When you replace n by 12 in $15 + n = 27$, do you obtain a true statement?

C Does 12 satisfy $15 + n = 27$?

Now you can answer the questions asked in problems A and B.

Mary should make 12 more place cards. Jim bought 12 more pictures.

D You can verify the answers by asking yourself these questions: If Mary has 15 place cards and makes 12 more, will she have the 27 she needs? If Jim had 15 pictures of old automobiles and bought 12 more, does he now have 27 pictures?

E Read the problem below. Can you also use the condition $15 + n = 27$ to solve this problem?

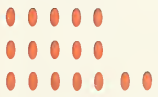
Jane had 15 pennies in her penny collection. She put in some new pennies. Then she had a total of 27 pennies. How many new pennies did she put in her collection?

F Make another problem that can be solved by using the condition $15 + n = 27$.

Read problems C and D in D2. Does the sentence in red express the condition for problem C? Does it express the condition for problem D?

$x + 8 = 17.$

Why is $x + 8$, and not $x - 8$, in the condition? For what, in problem C, is x a variable? For what, in problem D, is x a variable?



The 17 markers at the left represent the 17 nails Bob had. What do the 17 markers represent for problem D?

---

8 of the 17 markers represent the nails Bob found. What do the 8 markers represent for problem D?

---

8 of the 17 markers are dimmed. For problem C, imagine that 8 of the 17 nails are removed. What does the dimming of 8 markers suggest for problem D?

G The removal of 8 markers from 17 markers suggests subtracting 8 from 17 to get the solution of $x + 8 = 17$. Tabulate the solution set.

H When you replace x by 9 in $x + 8 = 17$, do you obtain a true statement?

I Does 9 satisfy $x + 8 = 17$?

Now you can answer the questions asked in problems C and D.

Bob had 9 nails before he found the 8 nails. Joan had made 9 paper flowers before she made the 8 extra flowers.

J You can verify these answers by asking yourself these questions: If Bob had 9 nails and found 8 more, did he then have 17 nails? If Joan had made 9 paper flowers and then made 8 more, had she then made 17 paper flowers in all?

K Make another problem that can be solved ? by using the condition $x + 8 = 17$.

Read the problems in D3. Does the sentence in red express the condition for problem E? Does it also express the condition for problem F?

$n - 9 = 15.$

Why is $n - 9$, and not $n + 9$, in the condition? For what, in problem E, is n a variable? For what, in problem F, is n a variable?

The 15 markers at the left represent the number of elms left. What do the 15 markers represent for problem F?

These 9 markers represent the number of elms cut down. What do the 9 markers represent for problem F?

The 9 markers are put with the 15 markers. Think of the 9 elms as put back with the 15 elms. What does putting 9 markers with 15 markers suggest for problem F?

L Putting 9 markers with 15 markers suggests adding 9 and 15 to get the solution of $n - 9 = 15$. Tabulate the solution set.

M Does 24 satisfy $n - 9 = 15$?

E One day 9 of the elms in the park were cut down. Then only 15 elms were left. How many elms were in the park before the 9 were cut down?

F Phyllis bought some ribbon. She used 9 yards of the ribbon and then had 15 yards left. How many yards of ribbon had she bought?

D3

G Tom had 32 tickets to sell for the school play. After one day of sales, he had 13 tickets left. How many tickets had he sold that day?

H Mrs. Ross had saved 32 dollars to buy a lamp. She bought a lamp on sale and had 13 dollars left. How many dollars had she spent for the lamp?

D4

N Use the solution of $n - 9 = 15$ to answer the question asked in problem E.

You have found that there were 24 elms in the park before 9 of them were cut down. To verify your answer, read the problem again and decide whether or not your answer is correct. If 9 of the 24 trees were cut down, you know that 15 trees were left. So 24 is the correct answer.

It is sensible to verify an answer because you may have the wrong condition for the problem or your computation may be incorrect. You can verify many answers mentally.

O Use the solution of $n - 9 = 15$ to answer the question asked in problem F.

P Verify your answer to problem F.

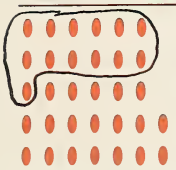
Q Make another problem that can be solved ? by using the condition $n - 9 = 15$.

Read problems G and H in D4. Does the sentence in red below express the condition for problem G? Does it also express the condition for problem H?

$32 - n = 13.$

Why is $32 - n$, and not $32 + n$, in the condition? For what, in problem G, is n a variable? For what, in problem H, is n a variable?

The 32 markers at the left represent the 32 tickets Tom had. What do the 32 markers represent for problem H?



13 of the 32 markers represent the 13 tickets Tom had left. What do the 13 markers represent for problem H?

13 of the 32 markers are dimmed. For problem G, imagine that 13 of the 32 tickets are removed. What does the dimming of 13 markers suggest for problem H?

R The removal of 13 markers from 32 markers suggests subtracting 13 from 32 to get the solution of $32 - n = 13$. Tabulate the solution set.

S Does 19 satisfy $32 - n = 13$?

T Answer the questions asked in problems G and H.

U Verify your answers to problems G and H.

V Make another problem that can be solved by using the condition $32 - n = 13$.

You can use what you have just learned about conditions and their solutions to help you think about familiar problems in a new way.

Don had 18 baseball cards. After he bought 24 more, he had how many baseball cards?

$18 + 24 = w.$

D5

Tim sold 49 of his 237 United States stamps to Ken. How many United States stamps did Tim have left?

$237 - 49 = n.$

D6

A Read the problem in D5. Why is $18 + 24$, and not $18 + w$, in the condition?

B For what, in the problem in D5, is w a variable?

C Suppose you were using markers to help you think about this problem. Why would you need a group of 18 markers and a group of 24 markers?

D Why would you put the 24 markers with the 18 markers? What computing should you do to find the solution of $18 + 24 = w$?

E Tabulate the solution set of $18 + 24 = w$.

F Does 42 satisfy $18 + 24 = w$?

G How many baseball cards did Don have after he bought 24?

H Verify your answer.

I Make another problem that can be solved by using the condition $18 + 24 = w$.

J Read the problem in D6. Why is $237 - 49$, and not $237 + 49$, in the condition?

K For what, in the problem in D6, is n a variable?

L Why must you think of 49 stamps as being removed from 237 stamps? What computing should you do to find the solution of $237 - 49 = n$?

M Tabulate the solution set of $237 - 49 = n$.

N Does 188 satisfy $237 - 49 = n$?

O Use the solution of $237 - 49 = n$ to answer the question asked in the problem. Verify your answer.

P Make another problem that can be solved by using the condition $237 - 49 = n$.

Now you know how to use computation to find the solutions of conditions for equality.

On your own

For each of the following problems, write a sentence that expresses the condition. Then tabulate the solution set of the condition and answer the question asked in the problem. Verify each answer. U = N.

1 At noon, the temperature was 28 degrees above zero. By seven o'clock in the evening, the temperature had dropped 15 degrees. What was the temperature at seven o'clock?

2 One week Mr. Brown travelled 798 miles. The following week he travelled 1092 miles. How many miles in all did Mr. Brown travel during these two weeks?

3 In 1959, 9942 vessels entered ports in Nova Scotia, and 6192 vessels entered ports in New Brunswick. How many fewer vessels entered in New Brunswick than in Nova Scotia?

4 Mrs. Reed picked 23 pints of strawberries. She sold some of these strawberries and then had 7 pints left. How many pints of strawberries had Mrs. Reed sold?

5 Elizabeth's weight now is 4 pounds less than her weight a month ago. She weighs 98 pounds now. Elizabeth weighed how many pounds a month ago?

THINK Why can $n - 98 = 4$ be thought of as the condition for problem 5?

6 John bought a bicycle with some of the money he had saved. He paid $34 for the bicycle and had $9 of his savings left. How many dollars had John saved?

7 Mr. Jones travelled by plane from Seattle to Chicago and from Chicago to Montreal. He flew a distance of 2489 air miles. The air distance from Seattle to Chicago is 1737 miles. The air distance from Chicago to Montreal is how many miles?

8 Carolyn was given $17 for her birthday. She earned some more money by baby-sitting. Then she had $23. How many dollars did she earn by baby-sitting?

9 After Mr. Hubert withdrew $75 from his savings account, he had $245 left in the account. How many dollars did Mr. Hubert have in his savings account before he withdrew the $75?

10 The Canadian Senate was established in 1867. The Supreme Court of Canada was established 8 years later. In what year must the Supreme Court of Canada have been established?

11 Mauvoisin Dam in Switzerland is 188 feet higher than Tignes Dam in France. The height of Tignes Dam is 592 feet. What is the height of Mauvoisin Dam?

12 The students at Rockville High School had 1513 tickets to sell for the spring music festival. They sold all but 195 of these tickets. How many tickets had the students sold?

13 After the Oakwood basketball team scored 43 points in a game with Clark, they had scored a total of 149 points against all their opponents. How many points had the Oakwood team scored before their game with Clark?

14 After Mr. Jameson bought 75 stamps, he had 423 stamps in his collection. How many stamps did he have before he bought the 75 new ones?

15 The Takakkaw Falls in British Columbia are 1038 feet higher than the Horseshoe Falls at Niagara. The height of the Takakkaw Falls is 1200 feet. How high are the Horseshoe Falls?

Solution sets of conditions for inequality

In this lesson, you will use conditions for inequality to help you solve problems. You will learn how a condition for equality can be used to help obtain the solutions of a condition for inequality.

So far in your work with conditions, you have used the set of natural numbers as the universe for the variable. However, other sets of numbers can be used. For example, the universe can be the set whose members are 1, 2, 3, and so on. This set is also an infinite set and is often called the set of *counting numbers*. In this book, we will use the letter C as a name of this set.

Notice that the only difference between set C and the set of natural numbers is that zero is not included in set C.

From now on, when you see the sentence "U = C," remember that the replacements for the variable are members of the set of counting numbers.

A Read problems A and B in D1. What is the universe for the variable in each problem?

$13 + x < 30.$

B Does the sentence at the left express the condition for problem A? For problem B?

C You can use a condition for equality to help you find the solutions of $13 + x < 30$. For problem A, imagine that Ann now has exactly 30 records. For problem B, imagine that the boys together have exactly 30 cars.

$13 + x = 30.$

U = C.

A Ann had 13 phonograph records. She bought more, but she still had fewer than 30 records. How many records could Ann have bought?

B Together, Stan and Gordon have fewer than 30 model cars. Stan has 13 cars. How many model cars can Gordon have?

D1

D How do the markers below at the left suggest that you subtract to find the solution of $13 + x = 30$? The solution set of $13 + x = 30$ is {17}. Is 17 a solution of $13 + x < 30$?

E Is any number greater than 17 a solution of $13 + x < 30$? Explain your answer.

F Now you can decide if any numbers less than 17 are solutions of $13 + x < 30$. Is 16 a solution of $13 + x < 30$? Is 15 a solution?

G Is each member of the universe that is less than 17 a solution of $13 + x < 30$?

H Look at the tabulation of the set at the left below. Is this set the solution set of $13 + x < 30$? Why is 0 not included in the solution set?

$\{1, 2, 3, \ldots, 16\}$

I Now you can write a standard description of the solution set of $13 + x < 30$ in the way shown at the left. Is $\{x \mid x < 17\}$ the same set as $\{x \mid 13 + x < 30\}$?

$\{x \mid x < 17\}$

You can give the answers to problems A and B as follows:

Ann could have bought from 1 through 16 phonograph records. Gordon can have from 1 through 16 model cars.

J Now verify these answers. If Ann had 13 phonograph records and bought from 1 through 16 more, would she now have fewer than 30 records?

K If Stan has 13 model cars and Gordon has from 1 through 16 cars, would the boys have fewer than 30 model cars in all?

Now you will develop conditions for two more problems. You will use conditions for equality to help obtain the solutions of the conditions for inequality. This time you will use a different universe.

A Read problems C and D in D2. What is the universe for the variable in each problem?

B Does the sentence at the left express the condition for problem C? For problem D?

$x + 12 > 40.$

C You can use a condition for equality to help you find the solutions of $x + 12 > 40$. Imagine that Mr. Blake had exactly 40 mathematics books; imagine that the temperature at two o'clock was exactly 40 degrees.

$x + 12 = 40.$

D What computation does the picture below suggest for finding the solution of $x + 12 = 40$?

The solution set of $x + 12 = 40$ is {28}. Is 28 a solution of $x + 12 > 40$?

E Is any number less than 28 a solution of $x + 12 > 40$? Explain your answer.

F Now you can decide if any numbers greater than 28 are solutions of $x + 12 > 40$. Is 29 a solution of $x + 12 > 40$?

G Is each member of the universe greater than 28 a solution of $x + 12 > 40$?

H Is the set tabulated below the solution set of $x + 12 > 40$?

{29, 30, 31, . . .}

I Is the solution set an infinite set?

U = N.

C Mr. Blake did not have enough mathematics textbooks to give one to each student in his third-period class. He got 12 more, and then had more than 40 textbooks. How many mathematics textbooks could he have had at first?

D At two o'clock in the afternoon the temperature in Elm Park was more than 40 degrees above zero. It had risen 12 degrees since noon. What could the temperature have been at noon?

D2

J How do you know that $\{x | x > 28\}$ is the same set as $\{x | x + 12 > 40\}$?

K Why is it convenient to use a standard description to describe the solution set of a condition for inequality like $x + 12 > 40$?

You know that solutions of the condition for the problem are used to obtain the answer to the problem. However, when the solution set is an infinite set, it is not sensible to use each solution in obtaining the answer to the problem. For example, 1000 is a member of $\{x | x > 28\}$, but it is unlikely that Mr. Blake would have had 1000 mathematics textbooks. Is it likely that the temperature could have been 1000 degrees at noon?

When the solution set of a condition for a problem is infinite, you usually need more information before you can find a sensible answer to the problem.

L Suppose that you are given the following information about problem C: "Mr. Blake had 36 students in his third-period class." Now you know that Mr. Blake had fewer than 36 textbooks at first. From the solution set of the

condition, you know that Mr. Blake had more than 28 textbooks at first. Each solution of $x + 12 > 40$ that satisfies all the requirements of the problem can be used to get an answer to the problem. Use the information just given to tabulate the set of numbers that can be used to answer problem C.

Now you can give a sensible answer to problem C.

Mr. Blake had from 29 through 35 textbooks at first.

M Verify this answer. If Mr. Blake had from 29 through 35 textbooks and got 12 more, would he have more than 40 textbooks?

N Why do you need more information before you can give a sensible answer to problem D?

O Suppose you know that the temperature was lower than 38 degrees at noon. What is the greatest number that could be used to get an answer to problem D?

P From the solution set of $x + 12 > 40$, you know that the temperature was higher than 28 degrees at noon. What is the least number of degrees the temperature could have been at noon?

Q With the new information, what set of numbers can you use to obtain the answer to problem D? Tabulate this set.

R Now you can give a sensible answer to problem D.

The temperature at noon could have been any number of degrees from 29 through 37.

S Why are the numbers that we used to answer the question in problem C different from those we used to answer the question in problem D?

You observed for problems C and D that you had to have more information before you could find sensible answers to the problems. Later on in this book, you will learn how to write sentences to express conditions that contain all the information needed for such problems. The solutions of these conditions will be used to obtain the answers to the problems.

$U = N$.

Mary made 24 popcorn balls for a party. After the party, she had fewer than 7 popcorn balls left. What is the least number of popcorn balls that could have been eaten at the party?

$24 - y < 7$.

D3

$U = N$.

Jim had a reading assignment. After he had read 9 pages of his assignment, he still had more than 15 pages to read. What is the least number of pages that could have been in the assignment?

$n - 9 > 15$.

D4

A Read the problem in D3. What is the universe for the variable?

B You can use $24 - y = 7$ to help find the solutions of $24 - y < 7$. Do you add, or do you subtract, to find the solution of $24 - y = 7$? Tabulate the solution set of $24 - y = 7$.

C Is 17 a solution of $24 - y < 7$?

D If you replace y in $24 - y < 7$ by any number less than 17, do you get a true statement?

E Does each solution of $24 - y < 7$ have to be a number greater than 17?

F Were more than 17 popcorn balls eaten? What is the least number of popcorn balls that could have been eaten?

G Verify your answer.

H Read the problem in D4. What is the universe for the variable? Why is $n - 9$ used in the condition?

I Use $n - 9 = 15$ to help find the solutions of $n - 9 > 15$. Do you add, or do you subtract, to find the solution of $n - 9 = 15$? Tabulate the solution set.

J Is 24 a solution of $n - 9 > 15$? Explain your answer.

K Is any number less than 24 a solution of $n - 9 > 15$?

L Tabulate the solution set of $n - 9 > 15$. What is the least solution?

M Is the least solution the least number of pages that Jim could have had to read?

N Verify your answer.

In this lesson you have learned to use conditions for equality to help find solutions of conditions for inequality.

On your own

For each problem, write a sentence that expresses the condition. Tabulate the solution set of the condition and give the answer to the problem. The universe is given for each problem.

1 After Betty deposited 7 dollars in her savings account, she had less than 100 dollars in the account. What is the greatest number of dollars Betty could have had in her account before she deposited the 7 dollars? U = N.

2 Ken answered all 60 questions in a test. He had more than 4 incorrect answers. How many correct answers could Ken have had? U = N.

3 Mr. King plowed fewer than 35 acres of land in two days. He plowed 19 acres the first day. How many acres could he have plowed the second day? U = C.

4 Henry had 128 stamps. He bought 36 more, and now he has more stamps than Terry. What is the greatest number of stamps that Terry can have? U = N.

5 Sally made 8 place mats for her mother. She said, "Now I have made fewer than 20 place mats." How many place mats could Sally have made before she made the 8 for her mother? U = N.

6 Evelyn had 82¢. She spent 35¢ and still had more money than Phyllis. How many cents could Phyllis have had? U = N.

7 Mr. Owen drove from his house to the Toronto airport, and then he flew 962 miles to Winnipeg. He travelled a total distance of less than 1000 miles. How many miles can his house be from the Toronto airport? U = C.

8 Dick has 45 arrowheads, and John has 29. The difference in the number of their arrowheads is greater than the number of Jack's arrowheads. How many arrowheads can Jack have? U = N.

9 The highest temperature in Danville one summer was 105 degrees. One afternoon that summer, the temperature dropped 25 degrees, but it was still higher than 60 degrees. What could the temperature have been before it started to drop? U = N.

10 Miss Jasper typed 590 words in ten minutes. In the first five minutes, she typed fewer than 325 words. What is the least number of words she could have typed in the last five minutes? U = N.

11 The highest peak in Canada is Mount Logan in the Yukon, some 19,850 feet above sea level. Mount Logan is more than 8,213 feet higher than Mount Temple in Alberta. What is the greatest height in feet Mount Temple can be? U = N.

12 During the first game of the football season, Ed scored more than 6 points. By the end of the season, he had scored 36 points. How many points could Ed have scored after the first game of the season? U = N.

13 Last spring Mr. Hansen planted 112 acres of his land in oats and corn. He planted more than 57 acres in corn. How many acres could he have planted in oats? U = C.

APPLYING MATHEMATICS

In this lesson you will solve problems by using conditions for equality and conditions for inequality. You will compute to find the solutions of the conditions. The universe is given for each problem.

After you have read each problem, write a sentence that expresses the condition for the problem. Then find the solution, or solutions, of the condition and answer the question asked in the problem.

1 Hayes School has a new auditorium. There are 823 seats on the main floor and 518 seats in the balcony. What is the total number of main-floor and balcony seats in the new auditorium? U = N.

2 David and John share a paper route. Last Sunday they delivered fewer than 105 papers. David delivered 67 of the papers. How many papers could John have delivered? U = C.

3 The Harrison Electronics Company bought 500 large envelopes. Some of these envelopes were used to mail the annual report. 245 envelopes were left. How many envelopes were used to mail the annual report? U = N.

4 Alice spent 8 hours doing homework during a two-week period. She spent more than 3 hours on homework the first week. How many hours could she have spent on homework the second week? U = N.

5 Mr. Johnson's weekly salary was increased by $19. Now his salary is $103 per week. What was Mr. Johnson's weekly salary before he received the increase? U = N.

6 The length of the Mackenzie River is 2635 miles. The length of the Peace River is 1195 miles. The Mackenzie is how many miles longer than the Peace? U = N

7 Ted had 201 foreign stamps. He gave 84 of them away and still had more foreign stamps than Phil. What is the greatest number of foreign stamps that Phil could have had? U = N.

8 Alex had $14. He was given some money for his birthday and now has $21. How many dollars was he given for his birthday? U = N.

9 After Mrs. Jones wrote a check for $145, she had $278 in her checking account. How many dollars did she have in her checking account before she wrote the check for $145? U = N.

10 A nursery owner had 531 pine trees to sell. He sold all but 76 of these trees. How many pine trees did he sell? U = N.

11 Last weekend Ann read the first 179 pages of a new book. Now she has fewer than 165 pages to read. What is the greatest number of pages that the book can contain? U = N.

12 Mr. Butler borrowed $1350 from a bank. At the end of the year, he repaid this amount plus $81 for the use of the money. How many dollars did he pay the bank at the end of the year? U = N.

13 Charles spent $51 for a camera and flash attachment. He spent $44 for the camera. How many dollars did he spend for the flash attachment? U = N.

14 The Richardson family drove from their home to Lake Louise in two days. Lake Louise is 421 miles from their home. The first day they drove more than 295 miles. How many miles could they have driven the second day? U = C.

15 It took Mr. Williams 3 hours to set out 140 tomato plants. He set out 46 plants the first hour and at least one plant during each of the remaining two hours. What is the greatest number of plants he could have set out the second hour? U = C.

THINK Is the number of tomato plants he set out in the first two hours less than 140?

Conditions for abstract problems

In lessons 19 and 20, you used solutions of conditions to get answers to problems about persons and things. Now you will learn to use solutions of conditions as answers to problems that concern numbers only. U = N.

A Read problems A and B in D1.

B Does the sentence in D2 express the condition for problem A? Does it also express the condition for problem B?

C For what, in problem A, is x a variable? For what, in problem B, is x a variable?

D Why is $16 + x$, and not $16 - x$, in the condition?

E Why is the idea of equality used in the condition?

F What computation should you use to find the solution of $16 + x = 33$?

G Tabulate $\{x \mid 16 + x = 33\}$.

H If Frank had 16 model cars and Jerry gave him 17 more, does Frank now have 33 cars?

I If 17 is added to 16, is the sum 33?

J Read problems C and D in D3. Does the sentence in D4 express the condition for problem C? Does it also express the condition for problem D?

K Use $13 + x = 20$ to help find the solution set of $13 + x < 20$. What computation should you use to find the solution of $13 + x = 20$?

L Tabulate $\{x \mid 13 + x = 20\}$.

M Is each solution of $13 + x < 20$ greater than 7? Less than 7?

N Tabulate $\{x \mid 13 + x < 20\}$.

A Frank had 16 model cars. Jerry gave him some more. Now Frank has 33 model cars. How many model cars did Jerry give to Frank?

B If a number is added to 16, the sum is 33. What is the number?

D1

$$16 + x = 33.$$

D2

C Dick scored 13 points during the first half of a basketball game. He scored fewer than 20 points during the entire game. How many points could he have scored during the second half?

D What numbers can be added to 13 so that each sum is less than 20?

D3

$$13 + x < 20.$$

D4

A farmer had 85 steers. He shipped some of them to market and then had 47 steers left. How many steers did he ship to market?

D5

O Does each member of the set that you tabulated in exercise N satisfy the condition for problem C? For problem D?

P Read the problem in D5. Write the sentence that expresses the condition for this problem.

Q Write a problem whose condition is the same as the condition for the problem in D5. Your problem should concern numbers only.

Now you know how to write sentences that express conditions for problems that are about

numbers only. You also know how to find the solutions of these conditions.

On your own

For each problem, first write a sentence that expresses the condition. Then tabulate the solution set of the condition. U = N.

1 The difference of 42 and 18 is what number?

2 What numbers can be added to 29 so that each sum is less than 35?

3 If 6 is added to a number, the sum is 25. What is the number?

4 The product of 16 and 12 is less than what numbers?

5 You can subtract 37 from what number to get a difference of 14?

6 The quotient of 1134 and 42 is greater than what numbers?

7 What number can you subtract from 120 so that the difference is 57?

8 To what numbers can you add 8 so that each sum is greater than 17?

9 When you add 139 to 376, what is the sum?

10 To what number can 16 be added so that the sum is 12?

THINK Can a solution set be the empty set?

11 What numbers can be subtracted from 26 so that each difference is greater than 12?

12 The difference of 147 and 15 is less than what numbers?

13 What number do you get when you multiply 108 by 350?

14 To what numbers can you add 13 so that each sum is greater than 15?

15 What number do you get when you divide 95,676 by 476?

16 What is the greatest number from which 6 can be subtracted so that the difference is less than 9?

17 What is the least number that can be subtracted from 28 so that the difference is less than 15?

KEEPING SKILFUL

Find the sum.

1 6851, 2334, 1815

2 8230, 694, 19

3 24768, 34619, 52116

4 86347, 49818, 60004

5 72683, 14931, 12386

6 29318, 3647, 7802

7 5655, 656, 19290

8 63147, 82598, 14006

Find the difference.

9 74160, 39890

10 41934, 67

11 62346, 12547

12 61842, 3457

13 36809, 6909

14 42321, 35312

15 35700, 24894

16 19258, 18369

17 27125, 18864

18 97325, 58320

Multiply

19 67 by 39.

20 125 by 80.

21 409 by 53.

22 348 by 46.

23 275 by 128.

24 106 by 349.

25 297 by 502.

Divide

26 4502 by 92.

27 5764 by 68.

28 10606 by 52.

29 9225 by 123.

30 10005 by 279.

31 26780 by 103.

32 77558 by 214.

Tabulate the solution set of each condition expressed below. For exercises 33 through 44, U = {1, 2, 3, 4, 5}. For exercises 45 through 56, U = N.

33 $y + 12 > 12$.

34 $8 - d > d$.

35 $3z + z = 4$.

36 $9 - x < 6$.

37 $97 - z = 95$.

38 $144 \div 48 < c$.

39 $41 - c < 32$.

40 $17 + n > 47$.

41 $14 - d = 9$.

42 $x + 37 < 41$.

43 $96 \div 16 > m$.

44 $7 - b > 2$.

45 $97(102) = z$.

46 $13 - y > 6$.

47 $3x + 1 < 3x$.

48 $65 - x = 60$.

49 $34(76) > m$.

50 $z + 2z = 6$.

51 $n + 7 > 59$.

52 $24 + n < 24$.

53 $n + 100 > 300$.

54 $5y = 4y + y$.

55 $x - 5 > 0$.

56 $43 - m > 42$.

CHECKING UP

The small numeral within parentheses tells you what page to turn to if you need help with an exercise.

Test 11

For each of exercises 1 through 4, choose the word from the following list that most clearly expresses the idea described by the exercise.

collinear	geometry
congruent	line
diameter	plane

1 The study of sets of points (29)

2 A chord of a circle (47)

3 Separates a plane into two half-planes (41)

4 Is contained in the same line (34)

For each of exercises 5 through 8, choose the word from the following list that most clearly expresses the idea described by the exercise.

equation	numeral
finite	tabulation
graph	universe

5 A way to name a set (11)

6 A true statement that includes the idea of equality (17)

7 A set consisting of all the objects that can be used as replacements for the variable (16)

8 A name of a number (5)

Test 12

Which of the following sentences express true statements? Which of the following sentences express false statements?

9 The number of members in an infinite set is a natural number. (23)

10 A circle has infinitely many radii. (46)

11 A variable may be used only once in a condition. (21)

12 The solution set of a condition must have fewer members than the universe. (25)

13 If U = N, then $\{x \mid 2 + x < 5\}$ contains exactly two members. (27)

14 If a line is included in a plane, then each point in the line is in the plane. (32)

15 Three noncollinear points determine two planes. (35)

16 A point separates space into two half-spaces. (41)

17 If three points are collinear, then one of the points is between the other two. (37)

18 Assume that $\overline{AB} \cong \overline{CD}$ and $\overline{CD} < \overline{EF}$. Then $\overline{EF} < \overline{AB}$. (43)

Test 13

What words or symbols best complete exercises 19 through 27?

19 The set that has no members is called the ——. (25)

20 An object that belongs to a set is a —— of the set. (10)

21 A name of the line that contains points A and B is ——. (33)

22 {1, 3} is a —— of {0, 1, 2, 3, 4}. (11)

23 If U = N, then {0, 1, 2} is the —— of $x + 4 < 7$. (18)

24 The sentence $13 - 2 < 11$ expresses a —— statement. (13)

25 {3, 6, 9, . . ., 30} has exactly —— members. (23)

26 The sentence $0(5) < 1$ expresses a —— statement. (13)

27 {1, 2, 3, . . .} is the set of —— numbers. (73)

Test 14

For each of the conditions expressed in exercises 28 through 31, first tabulate the solution set. Then make a graph of the universe and encircle each dot that represents a point associated with a member of the solution set.

U = {0, 1, 2, 3, 4, 5}.

28 $x + 3 < 5$. (53) **30** $2 + x = x + 2$. (53)

29 $4 \neq 7 - x$. (53) **31** $x + 9 > 8$. (53)

Test 15

From the list given below each exercise, choose the expression that correctly completes the exercise.

32 The sentence ~~~ expresses a false statement. (13)

a $2+3>4$. c $1+4<5$.
b $7-3=4$. d $5+2\neq 6$.

33 The set whose members are ~~~ is an infinite set. (23)

a all the known planets in our solar system
b the natural numbers from 2 through 10,000
c all the people in the world
d the counting numbers greater than 5000

34 The symbol ~~~ expresses the idea of "greater than." (20)

a $<$ b $>$ c $\cong$ d $\neq$

35 The symbol ~~~ expresses the idea of "not equal to." (20)

a $>$ b $\cong$ c $\neq$ d $\{\ \}$

36 Sentence ~~~ expresses an inequality. (21)

a $x+4\neq 5$. c $7-3<6$.
b $2+3=8$. d $1-0>1$.

37 If the universe is $\{2, 3, 4\}$, then ~~~ is a solution of $5<2+x$. (24)

a 2 b 3 c 4 d 5

38 Segments of the same size are ~~~. (39)

a collinear c coplanar
b congruent d infinite

Test 16

Use the display above for exercises 39 through 44.

39 What three radii of circle K are represented? (46)

40 What two chords of circle K are represented? (47)

41 Name the diameter of circle K that is represented. (47)

42 What points in circle K are named? (46)

43 What points in the exterior of circle K are named? (48)

44 What points in the interior of circle K are named? (48)

Test 17

For each of the following problems, first write a sentence that expresses the condition. Next tabulate the solution set of the condition. Then give the answer to the problem.

45 Ann bought 15 post cards and then had fewer than 31 post cards. How many post cards could she have had before she bought the 15? U = N. (73)

46 In the 1867 Canadian House of Commons, there were 181 members. In 1961 there were 265 members. The Canadian House of Commons had increased by how many members during this period of time? U = N. (68)

47 Sue had 57 magazines. She threw away the oldest ones, and then had more than 39 left. How many magazines could she have thrown away? U = C. (73)

48 Kay's weight now is 4 pounds more than it was a month ago. She weighs 101 pounds now. How many pounds did she weigh a month ago? U = N. (68)

49 What numbers can be subtracted from 71 so that each difference is greater than 13? U = N. (78)

50 To what number can 482 be added to get a sum of 620? U = N. (78)

Test 18

Multiply

51 359 by 9. (55)
52 105 by 46. (55)
53 77 by 77. (55)
54 812 by 238. (55)
55 768 by 552. (55)

Test 19

Divide

56 5280 by 6. (58)
57 6636 by 84. (58)
58 15542 by 38. (58)
59 14484 by 213. (59)
60 76000 by 125. (59)

Unit 2

Intersection and union of sets

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2 22 Exploring ideas

Intersection of sets

In this lesson you will learn how to obtain a new set from two given sets. In later lessons you will use this knowledge to help you solve problems.

A Look at the sets in D1. Set A is the set of South American countries that border on Bolivia. Set B is the set of South American countries that border on the Pacific Ocean.

B Is Peru a country that borders on Bolivia? Is Peru a country that borders on the Pacific Ocean? Name another country that borders on Bolivia and also on the Pacific Ocean.

C Read the tabulation in D2. Is each member of the set a member of set A? Of set B?

D Does the set tabulated in D2 contain any objects that are not members of both set A and set B?

E {Peru, Chile} is the set of countries that border on Bolivia and also on the Pacific Ocean. Why is Brazil not a member of this set? Why is Paraguay not a member of this set?

F How were A and B used to obtain {Peru, Chile}?

{Peru, Chile} is the set that contains those objects, and only those objects, that belong to both set A and set B. {Peru, Chile} is the *intersection* of set A and set B.

The intersection of two sets is the set that contains all the objects that belong to both sets. It does not contain any other objects.

G Look at D3. Is 6 a member of the intersection of C and D? Is 3 a member of the

Forming the intersection of two sets; using Venn diagrams to represent intersections of sets

intersection? Is 4 a member of the intersection? Explain your answers.

H How do you know that {1, 3, 6} is not the intersection of C and D?

I Is {2, 4} the intersection of C and D? How do you know?

The symbol below is used to express the idea of intersection. The words in D4 show how to write and read a name of the intersection of {1, 2, 3, 4} and {2, 4, 6}.

J Read the sentence in D5. How do you know that {1, 2, 3, 4} ∩ {2, 4, 6} and {2, 4} are the same set? Does the sentence in D5 express a true statement?

K D6 shows you how to read a sentence about the intersection of sets C and D. Does the sentence in D6 express the same statement as the sentence in D5? Explain your answer.

L Is the intersection of D and C the same set as the intersection of C and D? Is $C \cap D = D \cap C$ a true statement? Explain your answers.

M Look at D7. Tabulate set E. Tabulate set F. Which of these sets is an infinite set?

N Is 5 a member of the intersection of E and F? Is 10 a member of the intersection of E and F? Explain your answers.

O Tabulate E ∩ F. Is E ∩ F a finite set?

D1

A = {Brazil, Paraguay, Chile, Argentina, Peru}.
B = {Colombia, Peru, Ecuador, Chile}.

D2

{Peru, Chile}

D3

C = {1, 2, 3, 4}.
D = {2, 4, 6}.

D4

"The intersection of {1, 2, 3, 4} ∩ {2, 4, 6}
the set whose members are one, two, three, four and
the set whose members are two, four, six"

D5

{1, 2, 3, 4} ∩ {2, 4, 6} = {2, 4}.

D6

"The intersection of C and D equals C ∩ D = {2, 4}. the set whose members are two, four."

D7

U = N.
E = {x | x > 5}.
F = {x | x < 10}.

in ter sec tion (in′tər sek′shən). The intersection of two sets is the set that contains those objects, and only those objects, that belong to both sets. The intersection of {1, 2, 3} and {2, 3, 4, 5} is {2, 3}.

P Look at D8. Tabulate set K. Tabulate set L.
Q Tabulate $K \cap L$. Is $K \cap L$ an infinite set?
R Tabulate set M. What members of $K \cap L$ are also members of M?

{7, 8} is the intersection of set K, set L, and set M. Notice that the intersection of these three sets is the set that contains only those objects that are members of both $K \cap L$ and set M. In other words, the intersection of K, L, and M ($K \cap L \cap M$) contains only those objects that belong to all three sets.
S Is $K \cap L \cap M = \{7, 8\}$ a true statement?
T Sets I and J are tabulated in D9. Is any member of I also a member of J? Is any member of J also a member of I?

No member of I is a member of J, and no member of J is a member of I. No object in either set is a member of both sets. Sets I and J have *no common members*. Two sets that have no common members, are *disjoint sets*.
U Why are set I and set J disjoint sets?
V Look again at D9. Is the intersection of I and J the empty set? Explain your answer.

You can use diagrams to learn more about the intersection of sets. Such diagrams are called *Venn* diagrams.
A Two sets are represented by the Venn diagram in D10. The picture of the rectangle and of its interior represents a universe. The picture of the circle and of its interior represents set A. How does the diagram show that A is a subset of the universe?
B Suppose that U is the set of all Canadian citizens and that A is the set of all Canadian citizens who are registered voters. Is A a subset of U?
C Suppose that U is the set of all automobiles. Describe three sets, each of which is a subset of U.

Now you will study a Venn diagram that represents disjoint sets.
D What three sets are represented by the Venn diagram in D11? Notice that the pictures of A and B are shaded in different ways. These different shadings will help you remember that A is different from B.
E U and A are the same sets described in exercise B. Suppose that set B is the set of all Canadian citizens who are younger than ten years of age. Is B a subset of U?
F Do the set of all registered voters and the set of all Canadian citizens younger than ten have common members? Are A and B disjoint sets? Explain your answer.

$U = N$.
$K = \{x \mid x > 6\}$.
$L = \{x \mid x > 2\}$.
$M = \{x \mid x < 9\}$.

D8

$I = \{5, 10, 15, 20\}$.
$J = \{7, 14, 21, 28\}$.

D9

D10

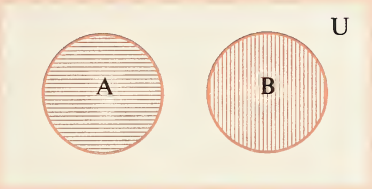
disjoint sets (dis joint′). Two sets are disjoint sets only when they have no common members. {0, 1, 2} and {3, 4, 5} are disjoint sets. {0, 1, 2} and {2, 3, 4} are not disjoint sets.

G For the sets described in exercises B and E, is $A \cap B$ the empty set? Explain your answer.

H Suppose that U is the set of natural numbers. Suppose also that A is the set of even numbers less than 10. Describe three subsets of U whose intersections with A are the empty set.

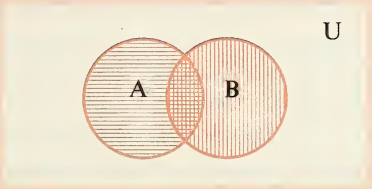
I If two sets are disjoint, what do you know about their intersection?

$A \cap B$ when A and B are disjoint sets



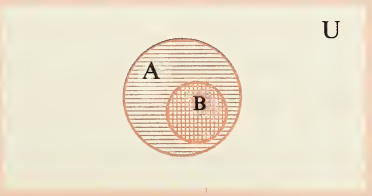
D11

$A \cap B$ when A and B meet



D12

$A \cap B$ when B is a subset of A



D13

J The Venn diagram in D12 represents sets that have some common members. Sets that have some common members are sets that *meet*. Again suppose that U is the set of all Canadian citizens and that A is the set of all Canadian citizens who are registered voters. Now suppose that B is the set of all Canadian citizens living in Ontario. How do you know that B is a subset of U?

K For the sets described in exercise J, is $A \cap B$ the set of all registered voters living in Ontario? How do you know?

L Look again at D12. How is $A \cap B$ represented by the Venn diagram?

M Suppose that C is the set of all citizens living in Quebec. Describe $A \cap C$.

N Suppose that U is the set of natural numbers and A is the set of odd numbers. Describe three subsets of U whose intersections with A are not the empty set.

O If two sets meet, what do you know about their intersection?

P Now look at D13. Again suppose that U is the set of all Canadian citizens and that A is the set of all Canadian citizens who are registered voters. Suppose that B is the set of all registered voters living in Manitoba. How do you know that B is a subset of A?

Q How does the diagram in D13 show that B is a subset of A?

R For the sets described in exercise P, is $A \cap B$ the set of all registered voters living in Manitoba? How is $A \cap B$ represented by the diagram in D13?

S For the sets described in exercise P, is $A \cap B$ the same set as B? Explain your answer.

T Suppose that U is the set of students in your school and that A is the set of all members of the school band. Describe three sets that are subsets of A.

U What is the intersection of A and each set that you described in exercise T?

V If one set is a subset of a second set, what do you know about the intersection of the two sets?

W What is the intersection of a set with itself? Explain your answer.

X Is the intersection of two sets a subset of each of the sets? Explain your answer.

Y What is the intersection of a set and the empty set? Explain your answer.

Now you know how to find the intersection of two sets. You also know that the intersection depends on how the two sets are related.

On your own

For each of exercises 1, 2, and 3, tabulate $A \cap B$.

1 $A = \{4, 8, 12, 16, 20, 24\}$.
$B = \{6, 12, 18, 24, 30\}$.

2 $A = \{0, 1, 2, \ldots\}$.
$B = \{0, 2, 4, \ldots\}$.

3 $A = \{1, 3, 5, \ldots\}$.
$B = \{0, 2, 4, \ldots\}$.

For each of exercises 4 through 7, tabulate set C and set D. Then tabulate $C \cap D$. $U = N$.

4 $C = \{x \mid x < 8\}$.
$D = \{x \mid x > 4\}$.

5 $C = \{x \mid x = 27\}$.
$D = \{x \mid x < 35\}$.

6 $C = \{x \mid x + 3 < 10\}$.
$D = \{x \mid x + 2 > 6\}$.

7 $C = \{x \mid 8 > x\}$.
$D = \{x \mid x + 2 > 13\}$.

Sets E, F, and G are described below. Use the descriptions to make a Venn diagram for each set named in exercises 8, 9, and 10. The universe is the set of students enrolled in Emerson High School.

E is the set of twelfth-grade students.

F is the set of tenth-grade students.

G is the set of students who are in the twelfth-grade chorus.

8 $F \cap E$ 9 $F \cap G$ 10 $E \cap G$

Union of sets

You have learned how to obtain the intersection of two sets. In this lesson you will learn how to obtain another set from two given sets.

A Look at D1. Set S is the set of all science teachers in Irving School. Set M is the set of all mathematics teachers in Irving School. Does Mr. Wilson teach science in Irving School? Does Mr. Wilson also teach mathematics?

B Name a mathematics teacher in the school who is not a science teacher.

C Does Mr. Adams teach science in the school? Does Mr. Adams also teach mathematics? Name another science teacher who is also a mathematics teacher.

D Read the tabulation in D2. Does the set contain all the teachers who are members of either S or M, or of both S and M? Does the set contain any teachers who are not members of either S or M, or of both S and M?

E The set tabulated in D2 is the set of all teachers in Irving School who are teachers of either science or mathematics, or of both science and mathematics. How were sets S and M used to obtain {Mr. Adams, Miss Evans, Mr. Wilson, Mr. Cox}?

The set tabulated in D2 is the set that contains those teachers, and only those teachers, who belong to either S or M, or to both S and M. The set tabulated in D2 is the *union* of set S and set M. The union of two sets is the set that contains all the objects that belong to either set or to both sets. It does not contain any other objects.

union (ūn′yən). The union of two sets is the set that contains those objects, and only those objects, that belong to either set or to both sets. The union of {1, 3, 5} and {0, 1, 2, 3} is {0, 1, 2, 3, 5}.

F Suppose that E is the set of all English teachers in Irving School. E = {Mrs. Robbins, Miss Schroeder, Mr. Johnson, Mr. Barnes, Miss Hansen}.

G Is any member of E also a member of S? Tabulate the union of E and S.

H Is any member of E also a member of M? Tabulate the union of E and M.

I Look at D3. Is 5 a member of the union of C and D? Is 6 a member of the union? Is 10 a member of the union? Is any other number a member of the union? If so, what is the number?

J How do you know that {5, 6, 7, 8, 9, 10} is not the union of C and D?

K Tabulate the union of C and D. Remember that, when you tabulate a set, you list the name of each member only once.

The symbol below is used to express the idea of union. Be careful not to confuse this symbol with the letter U that is often used as a name for a universe. The words in D4 show how to write and read a name of the union of {5, 10} and {6, 8, 10}.

L Now read the sentence in D5. Does the sentence express a true statement? How do you know?

M D6 shows how to read another sentence about the union of C and D. Does this sentence express the same statement as the sentence in D5?

N Is the union of D and C the same set as the union of C and D? Is $C \cup D = D \cup C$ a true statement? Explain your answers.

S = {Mr. Adams, Miss Evans, Mr. Wilson}.
M = {Mr. Adams, Miss Evans, Mr. Cox}.

D1

{Mr. Adams, Miss Evans, Mr. Wilson, Mr. Cox}

D2

C = {5, 10}.
D = {6, 8, 10}.

D3

"The union of {5, 10} ∪ {6, 8, 10}
the set whose members are five, ten and
the set whose members are six, eight, ten"

D4

$\{5, 10\} \cup \{6, 8, 10\} = \{5, 6, 8, 10\}$.

D5

"The union of C and D equals $C \cup D = \{5, 6, 8, 10\}$. the set whose members are five, six, eight, ten."

D6

$U = N$.
$E = \{x | x < 6\}$.
$F = \{x | x + 5 = 16\}$.
$G = \{x | x - 2 = 7\}$.

D7

$U = N$.
$K = \{x | x + 4 < 15\}$.
$L = \{x | x + 2 > 5\}$.

D8

O Look at D7. Tabulate set E. Tabulate set F. Tabulate set G.

P Do E and F meet? Explain why E and F are disjoint sets.

Q Is 6 a member of the union of E and F? Is ? 0 a member of the union? Is 11 a member of the union? Explain your answers.

R Tabulate $E \cup F$. Is $E \cup F$ an infinite set?

S Is each member of E a member of $E \cup F$? Is each member of F a member of $E \cup F$?

T Is each member of $E \cup F$ a member of either E or F?

U The union of $E \cup F$ and G is {0, 1, 2, 3, 4, 5, 9, 11}. Is the union of E, F, and G ($E \cup F \cup G$) the set that contains only those objects that belong to either $E \cup F$ or G, or to both $E \cup F$ and G?

Notice that the union of three sets contains all the objects that belong to the three sets. Some of the objects may belong to just one of the sets. Some objects may belong to two of the sets. Some objects may belong to all three sets.

V Look at D8. Tabulate K. Tabulate L.

W Is K a finite set? Is L a finite set? Tabulate $K \cup L$. Is $K \cup L$ a finite set?

X Is each member of K a member of $K \cup L$? Is each member of L a member of $K \cup L$?

Y Is each member of $K \cup L$ a member of either K or L? Which members of $K \cup L$ are members of both K and L?

Z Tabulate the intersection of K and L. Is ? $K \cap L$ a subset of $K \cup L$? How do you know?

In lesson 22 you used Venn diagrams to represent the intersection of sets. Venn diagrams can also be used to represent the union of sets.

A The Venn diagram in D9 represents disjoint sets. How is the universe represented? How are the disjoint sets represented?

B Suppose that U is the set of all cities in Canada. Suppose also that A is the set of all cities located in the Prairie Provinces, and B is the set of all cities in Ontario. Is A a subset of U? Is B a subset of U?

C Are A and B disjoint sets? Explain your answer.

D Is $A \cup B$ the set of all cities that are either in the Prairie Provinces or in Ontario? How do you know?

E How is $A \cup B$ represented by the diagram in D9?

F Suppose that U is the set of natural numbers and A is the set of odd numbers less than 20. Give descriptions of two other subsets of U so that A and each set that you describe will be disjoint sets.

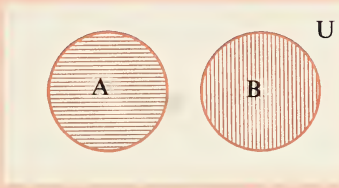
G What is the union of A and each set that you just described?

H The diagram in D10 represents sets that meet. Suppose that U and A are the sets described in exercise B. Suppose that set B is the set of all cities that are provincial capitals. Is B a subset of U?

I Describe $A \cup B$. How is $A \cup B$ represented by the diagram in D10?

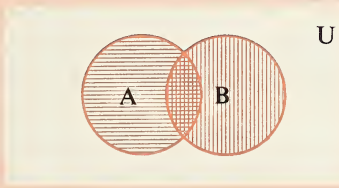
J Suppose that U is the set of natural numbers and A is the set of even numbers less

$A \cup B$ when A and B are disjoint sets



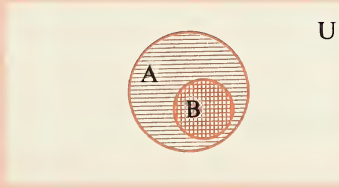
D9

$A \cup B$ when A and B meet



D10

$A \cup B$ when B is a subset of A



D11

than 10. Give descriptions of two other subsets of U so that A and each set that you describe will meet.

K For each set that you described in exercise J, what is the union of that set and A?

L Now look at D11. How is the statement that B is a subset of A represented by the diagram in D11?

M Again suppose that U is the set of all cities in Canada and that A is the set of all cities that are in the Prairie Provinces. Suppose that set B is the set of all cities in Alberta. Is B a subset of A? Explain your answer.

N For the sets described in exercise M, describe $A \cup B$. How is $A \cup B$ represented by the diagram in D11?

O Is $A \cup B$ the same set as A? Explain your answer.

P Suppose that U is the set of natural numbers and that A is the set of odd numbers. Give descriptions of two other sets so that each set is a subset of A.

Q What is the union of A and each set that you described in exercise P?

R If one set is a subset of a second set, what do you know about their union?

S Suppose A and B are two sets. Is A a subset of $A \cup B$? Is B a subset of $A \cup B$? Is $A \cap B$ a subset of $A \cup B$? Explain your answers.

T What is the union of any given set and the empty set? Explain your answer.

U What is the union of a set with itself? Explain your answer.

V If the union of two sets is the empty set, what do you know about the two sets?

W If the union of two sets is the same set as the intersection of the two sets, and the two sets are not empty, what do you know about the two sets?

Now you know how to find the union of two sets. You also know that the union depends on how the two sets are related.

On your own

For each of exercises 1 through 5, tabulate $A \cup B$.

1 A = {bluebird, crow, hawk, owl}.
B = {eagle, hawk, owl, vulture}.

2 A = {3, 6, 9, 12}.
B = {5, 10, 15}.

3 A = {0, 1, 2, . . .}.
B = {1, 3, 5, . . .}.

4 A = {1, 2, 5, 9}.
B = {2, 9, 5, 1}.

5 A = {0, 2, 4, . . .}.
B = {1, 3, 5, . . .}.

6 Sets E, F, and G are tabulated below. Tabulate E ∪ F ∪ G.

E = {1, 5, 7, 9}.
F = {1, 7, 13}.
G = {7, 9, 11, 13}.

First tabulate sets C and D. Next tabulate C ∩ D. Then tabulate C ∪ D. U = N.

7 C = {x | $x + 2 < 9$}.
D = {x | $x + 3 > 4$}.

8 C = {x | $x + 14 < 31$}.
D = {x | $x - 2 = 13$}.

9 C = {x | $x > 8$}.
D = {x | $x + 4 < 13$}.

10 C = {x | $x - 3 > 10$}.
D = {x | $x + 6 > 19$}.

11 C = {x | $x + 16 < 30$}.
D = {x | $x - 6 > 6$}.

12 First tabulate sets R, S, and T. Next tabulate R ∩ S ∩ T. Then tabulate R ∪ S ∪ T. U = N.

R = {x | $x + 5 < 6$}.
S = {x | $x - 2 > 2$}.
T = {x | $x + 7 < 13$}.

Sets W, X, Y, and Z are described below. Use the descriptions to make a Venn diagram for each set named in exercises 13 through 18. The universe is the set of all animals.

W is the set of all pets.
X is the set of all dogs.
Y is the set of all rabbits.
Z is the set of all four-legged animals.

13 Z ∪ X
14 W ∪ Z
15 W ∪ Y
16 Z ∪ W
17 Z ∪ Y
18 Y ∪ X

Compound conditions and their solution sets

You have already learned how to find solution sets of conditions. In this lesson you will learn how two conditions can be combined to form a new condition.

A What conditions are expressed in D1? What is the universe for x in each condition? Tabulate the solution set of each condition. Conditions like the ones expressed in D1 are *simple conditions*.

B Study D2. The open sentence expresses a condition formed from two simple conditions and the connective "and."

The connective "and" means that each solution of the condition must satisfy not only $x + 3 > 7$ but also $6 + x < 14$.

Mathematicians use the symbol shown at the right below to express the connective "and." D3 shows how to read and write an open sentence that includes the symbol for "and."

U = {0, 1, 2, . . ., 10}.
A $x + 3 > 7$.
B $6 + x < 14$.

D1

U = {0, 1, 2, . . ., 10}.
$x + 3 > 7$ and $6 + x < 14$.

D2

com pound condition (kom′pound). A condition that is made up of two or more simple conditions and one or more connectives. $y < 6 \wedge 3 < y$ is a compound condition. It contains two simple conditions, $y < 6$ and $3 < y$, and includes the connective "and."

A condition like $x + 3 > 7 \wedge 6 + x < 14$, which is made up of two simple conditions and a connective, is a *compound condition.*

C A standard description of the solution set of $x + 3 > 7 \wedge 6 + x < 14$ is given in D4. Read the description.

Now you will learn how to find the solution set of $x + 3 > 7 \wedge 6 + x < 14$. The solution set of each simple condition is tabulated in D5. The solution set of the compound condition is also tabulated in D5.

D Which members of $\{x | x + 3 > 7\}$ are also members of $\{x | 6 + x < 14\}$?

E Does $\{5, 6, 7\}$ contain all the members of the universe that satisfy both $x + 3 > 7$ and $6 + x < 14$? Is $\{5, 6, 7\}$ the same set as $\{x | x + 3 > 7 \wedge 6 + x < 14\}$?

F Is $\{x | x + 3 > 7 \wedge 6 + x < 14\}$ the same set as $\{x | 6 + x < 14 \wedge x + 3 > 7\}$? Explain your answer.

G Study D6. It shows how to write and read a name of the intersection of $\{x | x + 3 > 7\}$ and $\{x | 6 + x < 14\}$.

REMINDER

The intersection of two sets is the set that contains only those objects that belong to both sets.

See lesson 22, page 83.

H Read the sentence in D7. Does the sentence express a true statement? Explain your answer.

"x plus three is greater than seven and six plus x is less than fourteen."

$x + 3 > 7 \wedge 6 + x < 14.$

D3

$\{x | x + 3 > 7 \wedge 6 + x < 14\}$

D4

$U = \{0, 1, 2, \ldots, 10\}.$

$\{x | x + 3 > 7\} = \{5, 6, 7, 8, 9, 10\}.$

$\{x | 6 + x < 14\} = \{0, 1, 2, 3, 4, 5, 6, 7\}.$

$\{x | x + 3 > 7 \wedge 6 + x < 14\} = \{5, 6, 7\}.$

D5

"The intersection of $\{x | x + 3 > 7\} \cap \{x | 6 + x < 14\}$ the solution set of $x + 3 > 7$ and the solution set of $6 + x < 14$"

D6

$\{x | x + 3 > 7\} \cap \{x | 6 + x < 14\} = \{5, 6, 7\}.$

D7

I Is the intersection of the solution set of $x+3>7$ and the solution set of $6+x<14$ the same set as the solution set of $x+3>7 \wedge 6+x<14$? How do you know?

J Read the open sentence in D8. What two simple conditions must be satisfied by each member of $\{x|x<9 \wedge x+2<8\}$?

K What members of N satisfy $x+2<8$?

L What members of N that satisfy $x+2<8$ also satisfy $x<9$? Are these the only natural numbers that satisfy $x<9 \wedge x+2<8$?

M What members of N are in the intersection of $\{x|x<9\}$ and $\{x|x+2<8\}$? Are these the same natural numbers that satisfy $x<9 \wedge x+2<8$?

N Look at the Venn diagram in D9. How many sets are represented? How is the universe represented? The pictures of the two circles and of their interiors represent sets A and B. Set A is the solution set of a simple condition. Set B is the solution set of another simple condition.

O Which portion of the diagram represents the solution set of a compound condition that includes the connective "and"? Which portion of the diagram represents $A \cap B$?

P The solution set of a compound condition that includes "and" is the intersection of the solution sets of the two simple conditions that form the compound condition. Explain why this statement is true.

Now you will learn to make graphs of the solution sets of conditions like $x+3>7 \wedge 6+x<14$ and $x<9 \wedge x+2<8$.

REMINDER

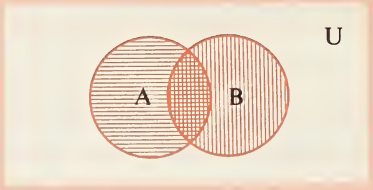
The graph of a set of numbers is a set of dots.

See lesson 13, page 52.

$U = N$.

$x<9 \wedge x+2<8$.

D8

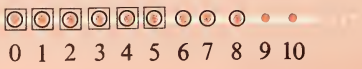


D9

$U = \{0, 1, 2, \ldots, 10\}$.

D10

$U = N$.



D11

A Look at graph A in D10. Do the encircled dots form the graph of $\{x|x+3>7\}$?

B Look at graph B. Do the dots enclosed by squares form the graph of $\{x|6+x<14\}$?

C Graphs A and B are combined in graph C. How can you tell which dots form the graph of $\{x|x+3>7 \wedge 6+x<14\}$?

D Look at D11. Which dots form the graph of $\{x|x<9 \wedge x+2<8\}$?

E $A = \{x|x < 9\}$. $B = \{x|x + 2 < 8\}$. Make a Venn diagram to represent $A \cap B$.

F Which portion of the diagram represents $\{x|x < 9 \wedge x + 2 < 8\}$?

G How do the graphs in D11 and the Venn diagram you made for exercise E show that $\{x|x < 9 \wedge x + 2 < 8\}$ is a subset of $\{x|x < 9\}$? How do the graphs and the diagram show that $\{x|x < 9 \wedge x + 2 < 8\} = \{x|x + 2 < 8\}$?

In earlier lessons you learned that sometimes the solution set of a simple condition can be the empty set or an infinite set. The exercises that follow show that the solution set of a compound condition can also be the empty set or an infinite set.

H Read the open sentence in D12. What members of the universe satisfy $x < 2$?

I What numbers that satisfy $x < 2$ also satisfy $x + 1 > 4$?

J How many members of the universe satisfy $x < 2 \wedge x + 1 > 4$?

K Tabulate $\{x|x < 2 \wedge x + 1 > 4\}$.

L Now look at D13. Tabulate $\{x|x + 1 > x\}$. Is $\{x|x + 1 > x\}$ an infinite set?

M What solutions of $x + 1 > x$ also satisfy $x + 2 > 10$?

N Tabulate $\{x|x + 2 > 10 \wedge x + 1 > x\}$. Is $\{x|x + 2 > 10 \wedge x + 1 > x\}$ an infinite set?

O Suppose you want to find $\{x|x + 4 > 13 \wedge x + 2 = 16\}$. Which is easier to find, the solution set of $x + 4 > 13$, or the solution set of $x + 2 = 16$?

P Tabulate $\{x|x + 4 > 13 \wedge x + 2 = 16\}$.

U = N.
$x < 2 \wedge x + 1 > 4$.

D12

U = N.
$x + 2 > 10 \wedge x + 1 > x$.

D13

Now you know how to find the solution set of a compound condition that includes the connective "and." You also know that this solution set is the intersection of the solution sets of the simple conditions that make up the compound condition.

On your own

For exercises 1 through 6, tabulate the solution set of each compound condition. U = N.

1 $x < 14 \wedge x > 8$.

2 $x < 19 \wedge x + 1 = 11$.

3 $x = 5 \wedge 4 + x = 15$.

4 $x < 13 \wedge 15 - x > 6$.

5 $x + 3 > 12 \wedge 14 < x$.

6 $x > 100 \wedge x + 18 > x$.

For each compound condition expressed in exercises 7, 8, and 9, first make a graph of the set of natural numbers. Extend your picture of the line to contain as many dots as you need for the particular exercise. Encircle the dots in the graph of the solution set of the first simple condition. Draw squares around the dots in the graph of the solution set of the second simple condition. Finally, tabulate the solution set of the compound condition. U = N.

7 $14 = 8 + x \wedge x < 10$.

8 $x + 9 < 17 \wedge x > 2$.

9 $4 + x < 11 \wedge 4 + x > 11$.

For each of exercises 10, 11, and 12, let A be the solution set of one simple condition and let B be the solution set of the other simple condition. For each exercise make a Venn diagram to represent a compound condition formed from the two simple conditions and the connective "and." U = N.

10 The solution set of the compound condition is the same set as B.

11 The solution set of the compound condition is the empty set.

12 The solution set of the compound condition contains at least one member of A. It does not contain all members of either A or B.

For each of exercises 13 through 16, tell which of the Venn diagrams you just made could be used to represent the solution set of the compound condition expressed in the exercise. U = N.

13 $x - 1 > 5 \wedge x + 3 < 13$.

14 $x > 17 \wedge x + 5 < 22$.

15 $x > 20 \wedge x + 15 > 17$.

16 $x + 15 > 15 \wedge x + 2 < 7$.

KEEPING SKILFUL

Multiply

1 372 by 49.
2 609 by 25.
3 132 by 578.
4 903 by 264.
5 352 by 710.
6 405 by 180.

Divide

7 3164 by 24.
8 5000 by 78.
9 6448 by 31.
10 23498 by 62.
11 8742 by 186.
12 4750 by 214.
13 157700 by 456.
14 54032 by 176.

For each of exercises 15 through 20, first tabulate sets F and G. Next tabulate F $\cap$ G. Then tabulate F $\cup$ G. U = N.

15 F = $\{s \mid 3 + s > 4\}$.
G = $\{s \mid 8 - s > 4\}$.

16 F = $\{d \mid d + 24 = 51\}$.
G = $\{d \mid d - 7 > 18\}$.

17 F = $\{c \mid 14 + c > 20\}$.
G = $\{c \mid c < 9\}$.

18 F = $\{n \mid n + 172 < 180\}$.
G = $\{n \mid 294 - n = 276\}$.

19 F = $\{v \mid v - 14 = 25\}$.
G = $\{v \mid 28 + v = 67\}$.

20 F = $\{c \mid c - 42 > 38\}$.
G = $\{c \mid c + 59 > 64\}$.

Problems involving compound conditions

So far, you have solved problems by using only one simple condition. For some problems you used additional information to decide which solutions of the condition could be used to get sensible answers. However, problems like these really involve more than one simple condition. In this lesson you will solve problems by using compound conditions and finding solution sets of these conditions. U = N.

Read the problem in D1. First you will develop a compound condition for this problem.

A How do you know that Mrs. Ames had fewer than 10 hamburger buns to begin with? Use x as a variable for the number of buns she could have had before she bought more.

$x < 10$.

Does the sentence at the left express one simple condition for the problem?

B Mrs. Ames bought a package of 6 buns and then had more than 12. Why do you again use x as a variable for the number of buns she could have had before she bought 6 more?

$x + 6 > 12$.

C Does the sentence at the left express another simple condition for the problem?

$x < 10 \wedge x + 6 > 12$.

D Does the sentence at the left express the compound condition for the problem?

{7, 8, 9, . . .}

E How do you know that the set tabulated at the left is the solution set of $x + 6 > 12$?

Mrs. Ames needed 10 hamburger buns for lunch. She did not have enough buns on hand. So she bought a package of 6. Then she had more than 12 buns. How many buns could she have had before she bought 6 more?

D1

The record for consecutive push-ups in George's gym class is 14. George's own record is fewer than 14. One day George missed his own record by 1, but he still did more than 9 push-ups. What can George's record be?

D2

What numbers greater than 5 can be added to 6 so that each sum is less than 16?

D3

F Can you use each member of $\{x|x+6>12\}$? to get an answer to the problem? Explain your answer.

G How do you know that each number used to get an answer to the problem must be less than 10?

H Tabulate $\{x|x<10 \wedge x+6>12\}$. Remember that each member of $\{x|x<10 \wedge x+6>12\}$ must satisfy each of the simple conditions.

I Is $\{x|x<10 \wedge x+6>12\}$ the intersection ? of $\{x|x<10\}$ and $\{x|x+6>12\}$, or is it the union of the two solution sets?

J Can you use each member of $\{x|x<10 \wedge x+6>12\}$ to get an answer to the problem?

You can use each solution of the compound condition to get an answer to the problem.

Mrs. Ames could have had 7, 8, or 9 buns before she bought 6 more.

K Verify 7 buns as an answer. Are 7 buns fewer than 10 buns? If Mrs. Ames had 7 buns and bought 6 more, would she have had more than 12 buns?

L Verify each of the other answers.

M Read the problem in D2. George's record for consecutive push-ups is fewer than 14. Is $x<14$ a condition for the problem? For what is x a variable?

N The day George missed his record by 1, he still did more than 9 push-ups. Is $x-1>9$ a condition for the problem? For what is x a variable?

O What is the compound condition for the problem in D2?

P Tabulate $\{x|x-1>9\}$ and $\{x|x<14\}$.

Q Which do you find, the intersection or the union of $\{x|x<14\}$ and $\{x|x-1>9\}$?

R Tabulate $\{x|x<14 \wedge x-1>9\}$.

S Give the answers to the problem.

T Verify your answers.

Now you will solve problems that concern numbers only. U = N.

A Read the problem in D3. Is $x>5$ one simple condition for the problem? Is $6+x<16$ the other simple condition? For what is x a variable? What is the compound condition for the problem?

B Tabulate $\{x|6+x<16\}$. Is each member of this solution set an answer to the problem in D3?

C Which members of the set you tabulated satisfy $x>5$?

D Tabulate $\{x|x>5 \wedge 6+x<16\}$.

Now you can use each solution of the compound condition as an answer to the problem.

The numbers are the natural numbers from 6 through 9.

When 3 is added to a number, the sum is 40. When 15 is subtracted from the number, the difference is 25. What is the number?

D4

E Read the problem in D4. Use s as a variable for the number you are to find. What is the compound condition for the problem?

F Tabulate $\{s \mid s + 3 = 40\}$.

G Is the member of $\{s \mid s + 3 = 40\}$ also a member of $\{s \mid s - 15 = 25\}$?

H Tabulate $\{s \mid s + 3 = 40 \wedge s - 15 = 25\}$.

Since there are no solutions of the compound condition, there are no answers to the problem.

Now you know how to solve problems by using compound conditions that include the connective "and."

On your own

For each problem, write a sentence that expresses the compound condition. Use any letter you wish to express the variable. Tabulate the solution set of the compound condition. Then give the answer to the problem. U = N.

1 Bob has 13 model cars, and Jerry has 9 model cars. Pete has fewer model cars than Bob and Jerry have together, but he has more model cars than Bob. How many model cars can Pete have?

2 Three years from now, Bill will still be less than 21 years old. Five years ago, he was more than 9 years old. How old can Bill be now?

3 In April, Henry sold 62 magazine subscriptions. In May, he sold fewer than 25 subscriptions. Altogether he sold more than 81 subscriptions during these two months. How many subscriptions could he have sold in May?

4 Last week, Mr. Ray drove fewer than 1790 miles. If he had driven 175 miles more, he would have driven more than 1900 miles. How many miles could Mr. Ray have driven last week?

5 Mr. Hill bought 144 tomato plants. He set out more than 96 of these plants in one hour, and still had more than 12 plants left to set out. How many tomato plants could he have left to set out?

6 Mrs. Johnson bought 60 postage stamps. She used more than 27 of these stamps, but still had more than 24 stamps left. How many stamps could she have used?

7 Mr. Jones bought 315 pairs of overshoes to sell in his shoe store. He sold more than 269 pairs and then had more than 27 pairs left to sell. How many pairs of overshoes could he have had left?

8 When a number is added to 2950, the sum is 4010. When the number is subtracted from 3000, the difference is 1940. What is the number?

9 What numbers greater than the product of 406 and 850 can be added to 279,379 so that each sum is less than 600,000?

10 What numbers greater than the difference of 5010 and 3465 can be added to 4276 so that each sum is less than 5826?

11 What numbers greater than the quotient of 68,068 and 187 can be added to 576 so that each sum is less than 1000?

12 When 1496 is added to a number, the sum is 2130. When the number is added to 373, the sum is 1000. What is the number?

13 When 9568 is subtracted from a number, the difference is 3627. When the number is added to 1708, the sum is 14,903. What is the number?

APPLYING MATHEMATICS

Some of the problems in this lesson involve simple conditions. Others involve compound conditions. For each problem, write a sentence that expresses the condition for the problem. Tabulate the solution set of the condition and give the answer to the problem. U = N.

1 The area of Lake Huron is 23,010 square miles. The area of Lake Michigan is 22,400 square miles. The area of Lake Michigan is how many square miles less than the area of Lake Huron?

2 Dan scored 5 points during the first quarter of a basketball game. During the whole game, he scored fewer than 24 points. How many points could Dan have scored during the last three quarters of the game?

3 627 votes were cast in a school election. The seventh grade cast 276 of these votes. The eighth and ninth grades cast the rest of the votes. How many votes were cast by the eighth and ninth grades together?

4 David had 98 coins in his collection. After he sold 19 coins, he still had more coins than Arthur. How many coins could Arthur have had?

5 Harold, who is 13 years old, is younger than Jim. The difference in their ages is less than 6 years. How old can Jim be?

6 One year there were 166,629 telephones in use in Nova Scotia and 122,205 telephones in New Brunswick. The total number of telephones in Nova Scotia and New Brunswick was less than the number of telephones in Alberta. What is the least number of telephones there could have been in Alberta that year?

7 In 7 years, Janet will be more than 18 years old. Six years ago, she was less than 9 years old. How old can Janet be now?

8 Dave has 9 Mexican coins. He also has some European coins. He has 17 more European coins than Mexican coins. How many European coins does he have?

9 The weight of the Discoverer V satellite was 1700 pounds. This satellite weighed 1455 pounds more than the Discoverer I satellite. How many pounds did the Discoverer I satellite weigh?

10 Miss Jones had 35 students in her first-period class. She did not have enough history books so that each student could have one. After she obtained 9 more history books, she had more than 37. How many of these books could she have had before she obtained the 9 books?

11 One year in Canada, there were fewer than 8,402,000 persons of voting age. 5,701,000 of these persons voted in the federal election that year. What could be the greatest number of persons of voting age who did not vote in this election?

12 The "Enterprise," an atomic-powered aircraft carrier, is longer than the "Queen Elizabeth," an ocean liner. The difference in the lengths of these ships is less than 71 feet. The length of the "Queen Elizabeth" is 1031 feet. What is the greatest length the aircraft carrier can be?

13 What numbers less than the quotient of 48 and 4 can be subtracted from 13 so that each difference is greater than 3?

14 What numbers less than the difference of 243 and 195 can be added to 64 so that each sum is greater than 112?

15 What numbers less than the sum of 137 and 48 can be subtracted from 211 so that each difference is greater than 19?

16 What numbers greater than the product of 42 and 19 can be added to 124 so that each sum is greater than 912?

CHECKING UP

You should restudy lessons 22 through 25 if you are unable to do the work in these tests.

Test 20

Using the sets tabulated below, tabulate each of the sets named in exercises 1 through 9.

$A = \{1, 2, 3, 4, 5\}$. $D = \{\ \}$.
$B = \{0, 2, 4, 6\}$. $E = \{3\}$.
$C = \{1, 3, 5\}$. $F = \{1, 3, 5\}$.

1 $A \cap B$ **4** $B \cup C$ **7** $F \cap C$
2 $A \cup B$ **5** $E \cap D$ **8** $C \cup F$
3 $C \cap B$ **6** $E \cup D$ **9** $A \cap C$

Test 21

Using the standard descriptions given below, for each of exercises 10 through 15, make a Venn diagram to represent the sets named. $U = N$.

$G = \{x | x + 2 = 14\}$. $J = \{x | x > 7\}$.
$H = \{x | x + 2 < 14\}$. $K = \{x | x < 3\}$.

10 G and H **12** J and H **14** H and K
11 K and G **13** J and K **15** G and J

Test 22

For exercises 16 through 23, $U = N$.

16 Tabulate $\{x | x + 3 < 9\}$.

17 Tabulate $\{x | x - 2 > 1\}$.

18 Using the tabulations that you made for exercises 16 and 17, tabulate
$\{x | x + 3 < 9 \wedge x - 2 > 1\}$.

19 Make a graph of N.

20 On the graph that you just made for exercise 19, encircle the dots that are in the graph of $\{x | x + 2 < 12\}$.

21 On the graphs that you made for exercises 19 and 20, draw squares around the dots that are in the graph of $\{x | 6 + x < 9\}$.

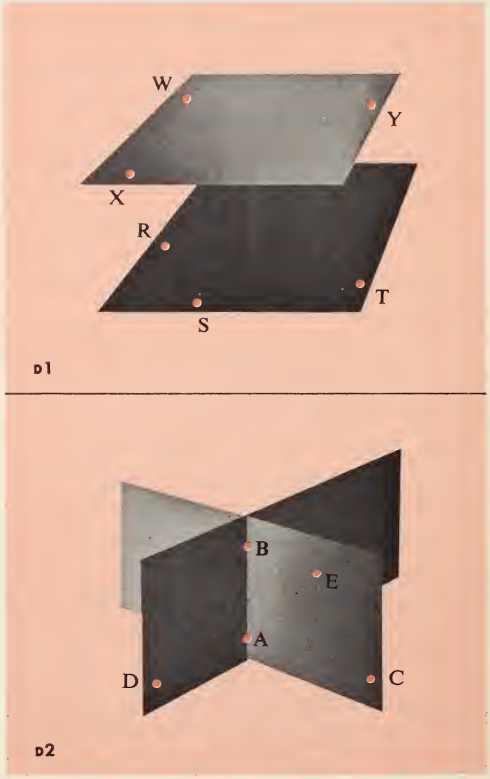
22 Using the graphs that you made for exercises 19, 20, and 21, tabulate $\{x | x + 2 < 12 \wedge 6 + x < 9\}$.

23 How do you know that $\{x | 6 + x < 9\}$ is a subset of $\{x | x + 2 < 12\}$?

Intersection of lines and planes

You know that the intersection of two sets is the set that consists of only those objects that belong to both sets. In this lesson you will learn about the intersection of sets of points.

Each photograph on page 99 represents an idea about sets of points. The objects shown suggest lines and planes. As you study the les-



Intersection of two planes; of a line and a plane; of two lines

son, you can use objects like these to help you understand the ideas presented in this lesson.

A Name the planes represented in D1. Is each plane an infinite set of points?

B Remember that a plane is unlimited in extent. We will assume that the planes represented in D1 have no points in common. In other words, they do not meet. Are planes WXY and RST disjoint sets? Explain your answer.

> REMINDER
> Two sets that have no common members are disjoint sets.
> *See lesson 22, page 84.*

Two planes, like plane WXY and plane RST, that have no points in common, are *parallel planes.*

C Are any two parallel planes disjoint sets? Explain your answer.

D In which photograph on this page do the objects suggest parallel planes?

E You know that the intersection of two sets is also a set. Describe the intersection of the two planes represented in D1.

F If any two planes are parallel, what do you know about the intersection of the two planes?

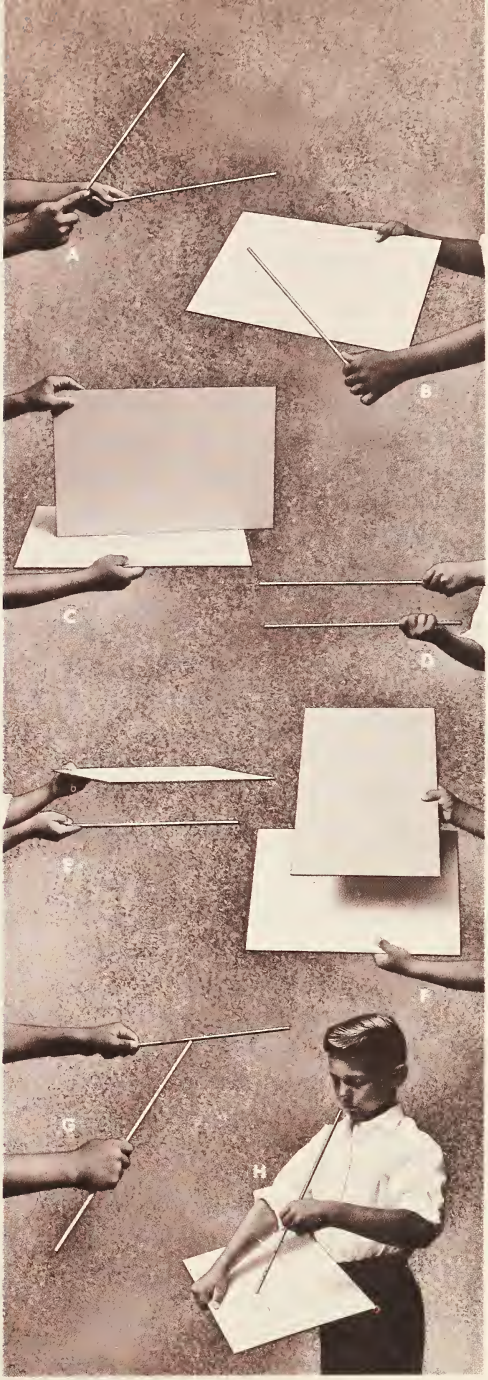
You have been studying an important property of two parallel planes. This property is expressed below.

The intersection of two parallel planes is the empty set.

Next you will learn about planes that are not parallel.

G Name the planes represented in D2. Is A in each of the planes? Is B in each of the planes?

par al lel planes (par′ə lel). Two planes that have no points in common.



H Look again at D2. Explain why $\overleftrightarrow{AB}$ is included in each of the planes represented.

> REMINDER
> If a line contains two points of a plane, the line is included in the plane.
> *See lesson 9, page 34.*

You remember that two sets that have common members are said to meet. Geometric figures that have common members are said to meet, or to *intersect*.

I Look again at D2. Do plane ABD and plane ABC have common members? Do the two planes intersect?

J In which photograph on page 99 do the objects suggest planes that intersect?

K What is the intersection of plane ABD and plane ABC?

L How many planes are determined by points A, B, and E, represented in D2? Explain why E cannot be a common member of plane ABD and plane ABC. Remember that plane ABD and plane ABC are different planes.

Exercises G through K help you to understand an important property of two planes that intersect. This property is expressed below.

The intersection of two planes that meet is a line.

M Three points are in the intersection of two planes. How do you know that the two planes are not parallel? How do you know that the three points are collinear?

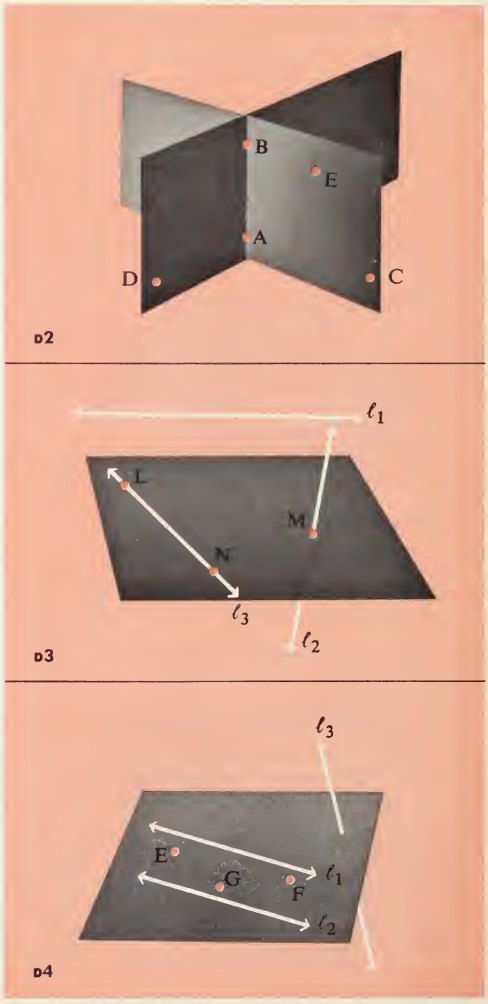
N Describe the intersection of a plane FGH and a plane EGH different from plane FGH.

You have learned that the intersection of two parallel planes is the empty set. You also know that if two planes meet, their intersection is a line. Next you will learn about the intersection of a line and a plane.

D3 represents the three ways in which a line and a plane may be related.

A ℓ_1 and plane LMN do not have any common members. ℓ_1 is parallel to plane LMN. What is the intersection of ℓ_1 and plane LMN?

B ℓ_2 intersects plane LMN. What is the intersection of ℓ_2 and plane LMN?



We say that ℓ_2 intersects plane LMN *in* point M. We mean that point M is the only member common to ℓ_2 and plane LMN.

C Plane LMN contains points L and N. How do you know that ℓ_3 is a subset of plane LMN? What is the intersection of ℓ_3 and plane LMN?

You know that a line is either parallel to or intersects a given plane. The property expressed below concerns a line and a plane that are parallel.

The intersection of a line and a plane that are parallel is the empty set

D In which photograph on page 99 do the objects suggest a line and a plane that are parallel?

The property expressed below concerns a line and a plane that intersect.

If a line and a plane intersect, their intersection either contains exactly one point or is the line.

E In which photograph on page 99 do the objects suggest a line that intersects a plane in one point?

F In which photograph on page 99 do the objects suggest a line and a plane whose intersection is the line?

So far you have learned about the intersection of two planes and the intersection of a line and a plane. Now you will learn about the intersection of two lines.

A Name the lines represented in D4. Which is each line, a finite set of points or an infinite set of points? Does each line extend endlessly in opposite directions?

B Lines ℓ_1 and ℓ_2 represented in D4 are included in the same plane. Name this plane. Is each line a subset of the plane?

C Lines ℓ_1 and ℓ_2 have no points in common. They do not intersect. Are ℓ_1 and ℓ_2 disjoint sets? Describe the intersection of ℓ_1 and ℓ_2.

Two lines like ℓ_1 and ℓ_2, that are subsets of the same plane and that do not intersect, are *parallel lines.*

D In which photograph on page 99 do the objects suggest parallel lines?

E Remember that two parallel lines are included in the same plane. Look again at D4. ℓ_1 and ℓ_3 have no points in common. Are ℓ_1 and ℓ_3 in the same plane? Are ℓ_1 and ℓ_3 parallel? Explain your answers.

Two lines that are not included in the same plane are *skew lines.* ℓ_1 and ℓ_3 are skew lines.

F In which photograph on page 99 do the objects suggest skew lines?

G $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have no points in common. Do you know whether or not $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel? Explain your answer.

H Describe the intersection of two parallel lines. The intersection of two skew lines.

A property of parallel lines and a property of skew lines are expressed below.

The intersection of two parallel lines is the empty set. The intersection of two skew lines is the empty set.

REMINDER

The union of two sets is the set that contains only those objects that belong to either set or to both sets.

See lesson 23, page 87.

par al lel lines (par′ə lel). Two lines, included in the same plane, that do not have any points in common.

skew lines (skū). Two lines not included in the same plane.

I Suppose that $\ell_4 \cup \ell_5$ is not a subset of any one plane. What do you know about ℓ_4 and ℓ_5?

J Suppose that $\ell_6 \cup \ell_7$ is a subset of a plane. Suppose also that $\ell_6 \cap \ell_7 = \{\ \}$. What do you know about ℓ_6 and ℓ_7?

K Now look at D5. $\overleftrightarrow{WS}$ and $\overleftrightarrow{SJ}$ are included in plane WSJ. Are $\overleftrightarrow{WS}$ and $\overleftrightarrow{SJ}$ skew lines? Are $\overleftrightarrow{WS}$ and $\overleftrightarrow{SJ}$ parallel lines?

L What point do $\overleftrightarrow{WS}$ and $\overleftrightarrow{SJ}$ have in common? Are $\overleftrightarrow{WS}$ and $\overleftrightarrow{SJ}$ disjoint sets? Do $\overleftrightarrow{WS}$ and $\overleftrightarrow{SJ}$ intersect?

M Describe the intersection of $\overleftrightarrow{WS}$ and $\overleftrightarrow{SJ}$. Does $\overleftrightarrow{WS} \cap \overleftrightarrow{SJ} = \{S\}$?

$\overleftrightarrow{WS}$ and $\overleftrightarrow{SJ}$ intersect in point S. $\overleftrightarrow{WS}$ and $\overleftrightarrow{SJ}$ are included in the same plane, but they are not parallel. An important property of two lines like $\overleftrightarrow{WS}$ and $\overleftrightarrow{SJ}$ is expressed below.

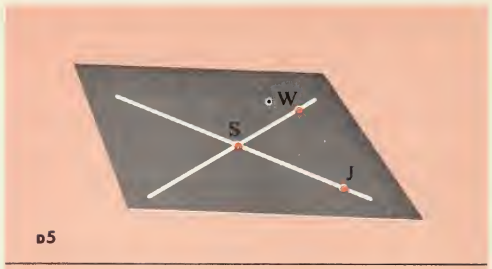
If two lines intersect, their intersection contains exactly one point.

N In which photograph on page 99 do the objects suggest lines that intersect?

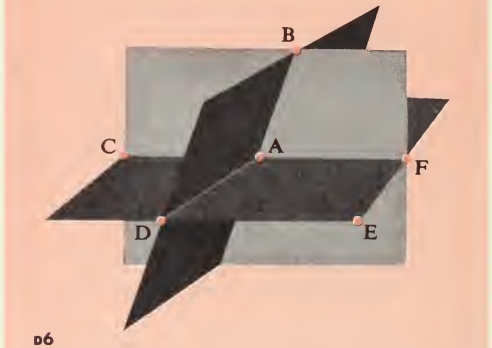
O Suppose that $\ell_8 \cap \ell_9$ is not the empty set. Describe $\ell_8 \cap \ell_9$.

P Two lines, ℓ_{10} and ℓ_{11}, intersect. How many planes are determined by ℓ_{10} and ℓ_{11}?

In this lesson you have learned about the intersection of two sets of points and about geometric figures that are parallel.



D5



D6

D7

On your own

Use D6 as you answer the questions in exercises 1 through 9.

1 What is the intersection of plane ABC and plane ACD?

2 What is the intersection of plane ABD and plane ACD?

3 $\overleftrightarrow{AB}$ is the intersection of what two planes?

4 Tabulate the intersection of $\overleftrightarrow{CA}$ and $\overleftrightarrow{AD}$.

5 How many planes represented in D6 contain A?

6 Which pairs of lines represented in D6 do you think are skew lines?

7 Which pairs of lines represented in D6 do you think are parallel lines?

8 Tabulate the intersection of $\overleftrightarrow{AC}$ and plane ADB.

9 What is the intersection of $\overleftrightarrow{AB}$ and plane ADB?

For exercises 10 through 16, use D7. The geometric figures represented in D7 are all sub-

sets of the same plane. Review lesson 12 if you need to recall certain ideas about a circle.

10 Tabulate the intersection of circle H and ℓ_1.

11 Tabulate the intersection of circle H and ℓ_2.

12 Tabulate the intersection of circle H and ℓ_3.

13 Describe the intersection of the interior of circle H and ℓ_1.

14 Tabulate the intersection of the interior of circle H and ℓ_2.

15 Tabulate the intersection of the interior of circle H and ℓ_3.

16 What is the intersection of ℓ_3 and the exterior of circle H?

17 Make a sketch of circle R. Next make a sketch of a line that intersects circle R in one point. Then make a sketch of a line that intersects circle R in two points.

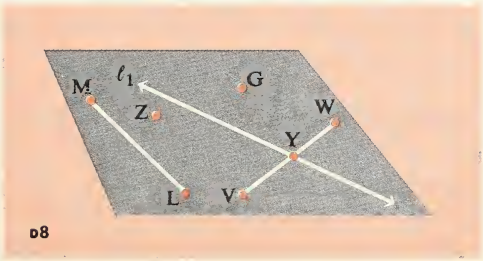
18 Make a sketch of circle S. Then make a sketch of two parallel lines each of which intersects the circle in two points.

Use D8 to help answer the questions in exercises 19 through 21. Review lessons 10 and 11 if you need to recall certain ideas about segments and separation.

19 ℓ_1 separates the plane into two half-planes. Are points L and M in the same half-plane? Tabulate the intersection of $\overline{LM}$ and ℓ_1.

20 Are points V and W in the same half-plane? Tabulate the intersection of $\overline{VW}$ and ℓ_1.

21 How many points are in the intersection of $\overline{ZG}$ and ℓ_1? Are points Z and G in the same half-plane?



D8

22 Make a sketch of circle R. Make a sketch of radius RS. Tabulate the intersection of $\overline{RS}$ and circle R.

23 In the same drawing, make a sketch of chord MN of circle R. Tabulate the intersection of chord MN and circle R.

24 $\overline{LT}$ and $\overline{ST}$ are radii of circle T. Tabulate the intersection of these two radii.

KEEPING SKILFUL

Tabulate the solution set of each condition expressed in exercises 1 through 16.
$U = \{1, 3, 5, 7, 9, 11\}$.

1 $69 + x < 74$.

2 $s + s > 8$.

3 $121 - w < 121$.

4 $2m + 3 = 17$.

5 $108 \div 12 > x$.

6 $2x = 22$.

7 $142 + y = 151$.

8 $v + 4 = 7$.

9 $12 + v > 13$.

10 $4y + 1 = 29$.

11 $27 - p > 17$.

12 $3x < 25$.

13 $216 \div 12 > w$.

14 $w + 159 < 170$.

15 $3y = 2y + y$.

16 $2z + 1 = z$.

Tabulate the solution set of each condition expressed in exercises 17 through 26. U = N.

17 $3806 - n = 1979$.

18 $17 + v < 33$.

19 $36(972) > d$.

20 $18 - b = 18$.

21 $13 - w > 4$.

22 $4497 + 8596 = m$.

23 $143 + c > 200$.

24 $2z + 5 = 23$.

25 $562 + n > 561$.

26 $3f = f$.

Tabulate the solution set of each condition expressed in exercises 27 through 36. U = N.

27 $m > 7 + 4 \wedge m + 9 < 24$.

28 $x > 27 \wedge x + 154 < 192$.

29 $765 \div 45 > s \wedge 80 - s > 59$.

30 $75 + x > 113 \wedge x < 300 - 176$.

31 $w < 15 \wedge w + 435 > 446$.

32 $b > 27(42) \wedge 237 + b < 1380$.

33 $s + 187 = 506 \wedge s - 298 = 21$.

34 $c < 347 + 962 \wedge c + 524 > 1833$.

35 $12m = 228 \wedge 124 + m = 143$.

36 $9068 - 8976 < z \wedge 182 + z > 274$.

Rays and angles

You have studied some geometric figures in previous lessons. In this lesson you will study some other geometric figures called rays and angles.

A Look at D1. Point A separates the line represented in D1 into two half-lines. Are points T and U in the same half-line? Are R and S in the same half-line? Explain your answers. Remember that you are thinking about the two half-lines bounded by A.

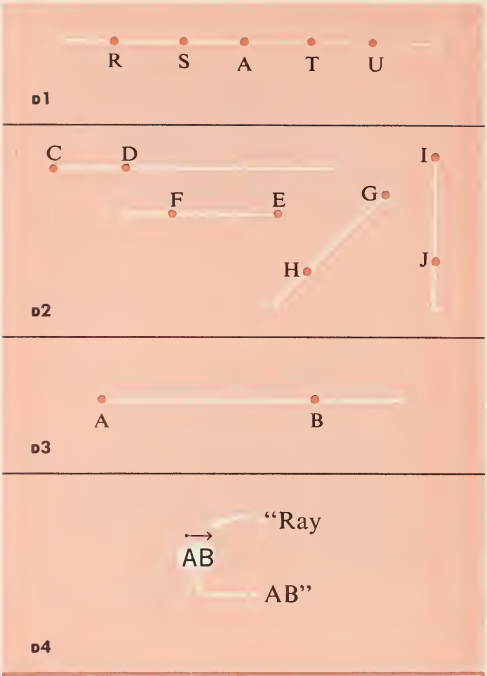
> REMINDER
> A point separates a line into two half-lines. The point is the boundary of each half-line. The boundary is not in either half-line.
> Each half-line is bounded by the point.
> *See lesson 11, page 42.*

B Is each of the half-lines bounded by A an infinite set of points? Does each half-line extend endlessly in one direction?

Look again at D1. Think about the set of points that contains A and all the points in the line to the right of A. This set of points is a *ray*. Point A is the *endpoint* of the ray.

C Describe another ray represented in D1 that has A as its endpoint.

A set of points that contains the boundary of a half-line and all the points in the half-line is a ray. The boundary of the half-line is the endpoint of the ray. A ray is a geometric figure.



ray (rā). A set of points containing the boundary of a half-line and all the points in the half-line. The boundary of the half-line is the **endpoint** of the ray.

D Several rays are represented in D2. Name the endpoint of each ray.

E Name the endpoint of the ray represented in D3. Name another point in the ray.

In naming a ray, we use the name of its endpoint and the name of one other point in the ray. D4 shows how to write and read a name of the ray represented in D3.

F Notice that the name of the endpoint is written first. Notice also the symbol above the letters AB. Why is this symbol a good symbol to use in writing a name of a ray?

G Name the rays represented in D2. How would you write each of these names?

H Is it sensible to ask if $\overrightarrow{CD}$ is longer than $\overrightarrow{EF}$? Explain your answer.

Use D5 to help you answer exercises I through N.

I $\overrightarrow{PQ}$ is the set of points consisting of P and all points in ℓ_1 to the right of P. Describe $\overrightarrow{QP}$.

J What is the endpoint of $\overrightarrow{PQ}$? Of $\overrightarrow{QP}$?

K Is ray PQ the same as ray QP? Explain your answer.

L What is the intersection of $\overrightarrow{PQ}$ and $\overrightarrow{QP}$?

M Is $\overrightarrow{PQ}$ the same as $\overrightarrow{PR}$? Explain your answer.

N What is the union of {P} and the half-line to the left of P?

Next you will use what you know about rays as you study another set of points that is a geometric figure.

A Look at D6. Point B is the common endpoint of $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Point E is the common endpoint of which two rays represented in D6? Point H is the common endpoint of which two rays represented?

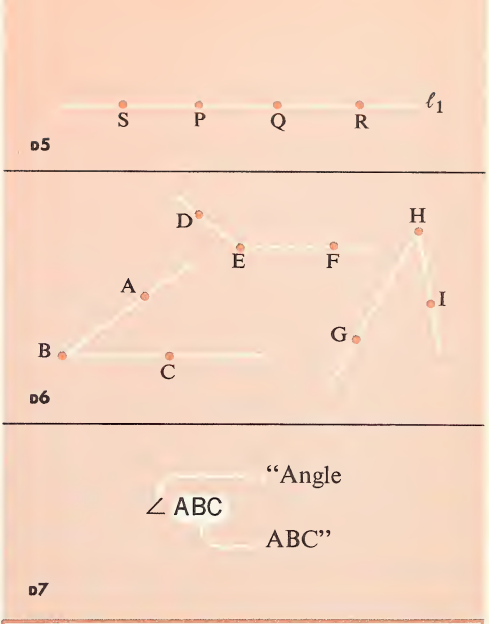
B The set that contains all the points in $\overrightarrow{BA}$ together with all the points in $\overrightarrow{BC}$ is an *angle*. Describe each of the other angles represented in D6.

An angle is the union of two rays that have a common endpoint, but that do not form a line. The common endpoint is the *vertex* of the angle. Each of the rays is a *side* of the angle. Since an angle is a set of points, it is a geometric figure.

C Name the sides and vertex of each angle represented in D6.

D Is each side of an angle a subset of the angle? Explain your answer.

E Are all the points in an angle contained in the same plane? Is an angle a subset of a plane?



an gle (ang′gəl). The union of two rays that have a common endpoint, but that do not form a line. The common endpoint is the **vertex** (vėr′teks) of the angle. Each of the rays is a **side** of the angle.

F Is it sensible to ask which of the two sides of an angle is the longer? Is it sensible to ask which of two angles has longer sides? Explain your answers.

One way of naming an angle is to use the name of its vertex and the names of two other points, one in each side of the angle. D7 shows how to write and read a name of the angle represented at the left in D6.

The vertex of ∠ABC is point B. Notice that the name of the vertex is written between the names of the other two points. Mathematicians always put the name of the vertex in this middle position to indicate clearly which point is the vertex of the angle.

G Notice also the symbol to the left of the letters ABC. Why is this symbol a good symbol to use in writing a name of an angle?

H Look again at D6. Is $\angle ABC$ the same angle as $\angle CBA$?

I Give two names for each of the other angles represented in D6.

J Is $\angle RST$ the same angle as $\angle RTS$? Explain your answer.

Now you will learn about some special pairs of angles.

A Look at D8. Point A is the endpoint of each of the three rays represented in the display. Name these three rays.

Notice that in D8 there are no dots to represent the exact locations of points A, B, C, and D. Point A is the common endpoint of each of the three rays. Point B is some other point in $\overrightarrow{AB}$; point C is some other point in $\overrightarrow{AC}$; and point D is some other point in $\overrightarrow{AD}$. From now on, you will be given the exact locations of points only when you need such information to do your work.

B $\angle DAB$ and $\angle DAC$ are represented in D8. What is their common vertex? What is their common side? Sides AB and AC are not common to $\angle DAB$ and $\angle DAC$.

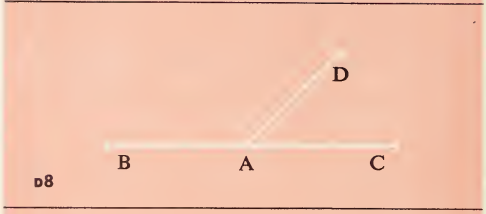
C Points A, B, and C are collinear. Is the union of sides AB and AC a line?

If two angles have a common vertex and a common side, and if the union of the sides that are not common is a line, then the two angles are a *linear pair of angles*. $\angle DAB$ and $\angle DAC$, represented in D8, are a linear pair of angles.

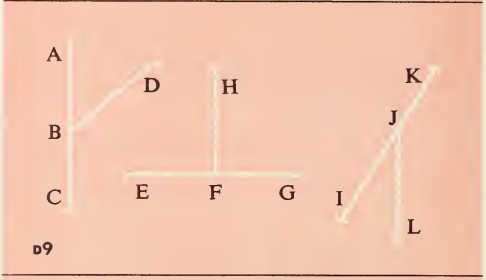
D Look at D9. Points A, B, and C, points E, F, and G, and points I, J, and K are collinear. $\angle ABD$ and $\angle DBC$ are a linear pair of angles.

lin e ar pair of angles (lin′ē ər). Two angles that have a common vertex and a common side, and whose sides that are not common form a line.

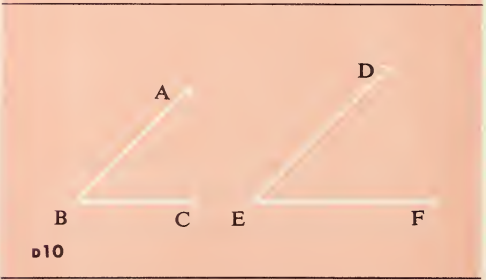
D6



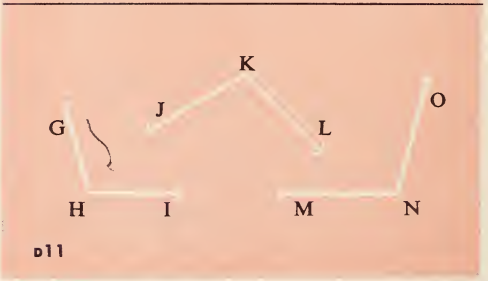
D8



D9



D10



D11

Name two other linear pairs of angles represented in D9.

Each angle of a linear pair of angles is the *supplement* (sup′lə mənt) of the other angle of the pair. Thus, ∠ABD is the supplement of ∠DBC, and ∠DBC is the supplement of ∠ABD.

E Look again at D9. What angle is the supplement of ∠EFH? Of ∠IJL? Of ∠HFG? Of ∠KJL?

Now you will learn that congruence has the same meaning for angles that it has for segments. Congruent angles are "the same size."

F Look at D10. ∠ABC ≅ ∠DEF. Do you think that ∠DEF ≅ ∠ABC?

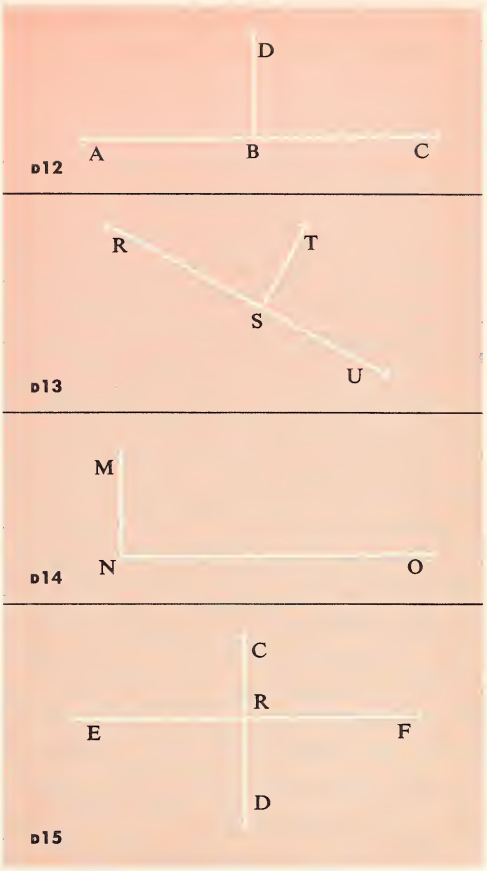
G Look at D11. ∠GHI ≅ ∠JKL. ∠JKL ≅ ? ∠MNO. Do you think that ∠GHI ≅ ∠MNO?

H Points A, B, and C, represented in D12, are collinear. Are ∠ABD and ∠DBC a linear pair of angles? What angle is the supplement of ∠ABD? Of ∠DBC?

Now suppose that ∠ABD ≅ ∠DBC. Then each of these two angles is a *right angle*. An angle that is congruent to its supplement is a right angle.

I Look at D13. Points R, S, and U are collinear. ∠TSU is a right angle. What angle is the supplement of ∠TSU? Is ∠TSU ≅ ∠TSR? Is ∠TSR a right angle? Explain your answers.

J ∠MNO, represented in D14, is a right angle. ? Explain how you would make a sketch of a supplement of ∠MNO. Is ∠MNO congruent to this supplement?



right an gle (rīt ang′gəl). An angle that is congruent to its supplement.

per pen dic u lar lines (pėr′pən dik′ū lər). Two lines that intersect so that right angles are formed.

You have been learning about right angles. You can use what you know about right angles to learn more about lines that intersect.

K Look at D15. Lines CD and EF intersect in point R. Point R is the endpoint of the four rays represented in the display. Name the four rays. Name the four angles that are formed by these rays.

Suppose that the angles formed when $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ intersect are right angles. Then $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ are *perpendicular lines*. Two lines are

perpendicular if they intersect and if the angles formed by these lines are right angles.

L How can you hold two pencils so that they suggest perpendicular lines?

M Name a pair of objects in your classroom that suggest perpendicular lines.

In this lesson you have studied rays and angles. You have learned what right angles are, and you know that perpendicular lines meet to form right angles.

On your own

Study D16. Then, for each of exercises 1 through 7, tell whether the sentence expresses a true statement or a false statement. A, B, C, and D are collinear points.

1 $\overrightarrow{AC} = \overrightarrow{AB}$.

2 $\overrightarrow{BC} = \overrightarrow{CB}$.

3 $\overrightarrow{AD}$ is a subset of $\overleftrightarrow{AD}$.

4 $\overrightarrow{AD}$ is a subset of $\overline{AD}$.

5 B is the endpoint of $\overrightarrow{AB}$.

6 $\overrightarrow{CA} \cup \overrightarrow{CD} = \overleftrightarrow{AD}$.

7 $\overrightarrow{BC} \cap \overrightarrow{CB} = \overline{BC}$.

Use D17 to help you answer exercises 8 through 12.

8 Is $\overrightarrow{OB} \cup \overrightarrow{OC} = \angle COB$? Is $\overrightarrow{OC} \cup \overrightarrow{OA} = \angle AOC$?

9 Name the sides and vertex of $\angle AOC$.

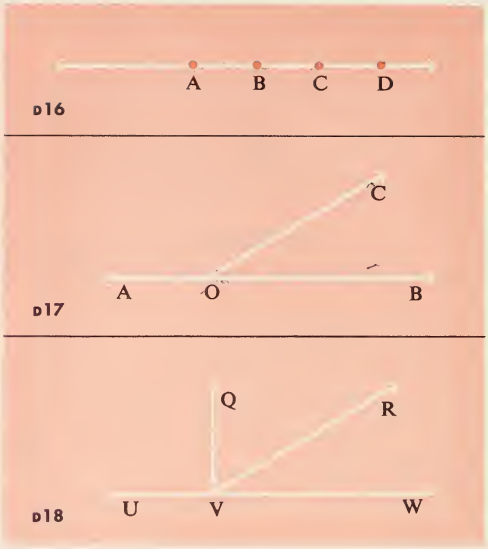
10 Name the sides and vertex of $\angle COB$.

11 Is $\overrightarrow{OC}$ a subset of $\angle COB$? Of $\angle AOC$?

12 $\angle AOC$ and $\angle COB$ have a common side and a common vertex. What else must you know to decide if $\angle AOC$ and $\angle COB$ are a linear pair of angles?

Use the following information for exercises 13, 14, and 15. $\overleftrightarrow{KL}$ and $\overleftrightarrow{MN}$ are any two lines that intersect in point Q. Point Q is between K and L and also between M and N.

13 Name the common vertex of $\angle MQL$ and $\angle NQL$. Name the common side of these two angles.



14 Is $\overrightarrow{QM} \cup \overrightarrow{QN} = \overleftrightarrow{MN}$ a true statement about the sides of $\angle MQL$ and $\angle NQL$ that are not common? Are $\angle MQL$ and $\angle NQL$ a linear pair of angles?

15 Name another linear pair of angles formed by $\overleftrightarrow{KL}$ and $\overleftrightarrow{MN}$. Name the common side of these two angles.

Use D18 to help you with exercises 16 through 19. Points U, V, and W are collinear.

16 Are $\angle RVW$ and $\angle RVU$ a linear pair of angles? Name another linear pair of angles.

17 What is the supplement of $\angle RVW$?

18 What is the supplement of $\angle QVW$?

19 $\angle QVW \cong \angle QVU$. Name two right angles represented in D18.

Use the following information for exercises 20 and 21. $\overleftrightarrow{IJ}$ and $\overleftrightarrow{GH}$ are any two perpendicular lines that intersect in point R. Point R is between I and J and also between G and H.

20 Is $\angle JRG$ a right angle? Is $\angle GRI$ a right angle? Is $\angle JRG \cong \angle GRI$?

21 Name two other right angles formed by $\overleftrightarrow{IJ}$ and $\overleftrightarrow{GH}$.

The triangle

In this lesson you will learn about another plane geometric figure called a triangle.

A Look at D1. A, B, and C are three noncollinear points. How many segments are determined by A, B, and C?

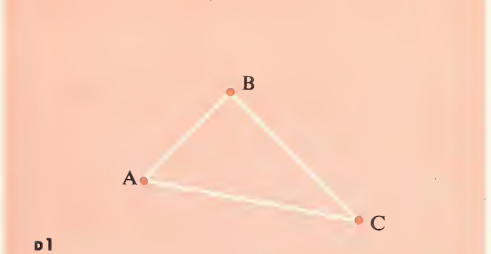
> REMINDER
> Points that are not contained in the same line are noncollinear points.
> *See lesson 9, page 34.*

B Points A, B, and C determine segments AB, BC, and AC. Is the set containing all the points in $\overline{AB}$, all the points in $\overline{BC}$, and all the points in $\overline{AC}$ the union of these three segments? How do you know?

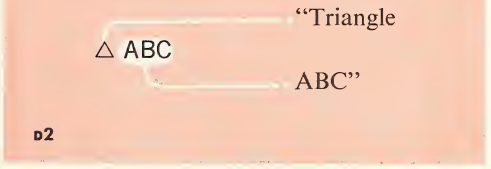
C Look again at D1. Is each of the points A, B, and C a member of the union of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$?

The union of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ is a *triangle*. A triangle is a set of points whose members are all the points contained in the three segments determined by any three noncollinear points. Each of the segments is a *side* of the triangle. Each endpoint of the segments is a *vertex* of the triangle.

> **tri an gle** (trī′ang′gəl). The union of the three segments determined by any three noncollinear points. The three segments are the **sides** of the triangle. The endpoints of the segments are the **vertices** (vėr′tə sēz) of the triangle.



D1



D2

D Name the three sides of the triangle represented in D1. Name the three vertices.

E Is a triangle a subset of more than one plane? Explain your answer.

In naming a triangle, we use the names of its three vertices. Study D2, which shows how to write and read a name of the triangle represented in D1.

F Look again at D1. Is △CBA the same triangle as △ABC? Is △BAC the same triangle as △ABC? Explain your answers.

You know that we sometimes talk about the center of a circle, but you know that the center is not in the circle. In the same way, we talk about the angles of a triangle. We know that a triangle does not include angles. The phrase "angles of a triangle" is only a way of identifying certain angles. In talking about △ABC, for example, we say that ∠BAC is an angle of △ABC. This statement means that each side of ∠BAC includes a side of the triangle. Thus, $\overrightarrow{AB}$ includes $\overline{AB}$, and $\overrightarrow{AC}$ includes $\overline{AC}$.

G Is ∠ABC a subset of △ABC? Explain your answer.

Now you will learn to identify triangles according to the number of congruent sides that each triangle has.

△GHI, △JKL, and △MNO are represented in D3. Notice that there are no dots to represent the vertices of these triangles. Since the endpoints of the sides of a triangle are the vertices of a triangle, it is not necessary to use dots to represent the vertices.

The two sentences below each picture of a triangle in D3 express true statements about the sides of the triangle. Use the pictures and the sentences to help you answer the questions in exercises A through I.

A In △GHI, do you think that $\overline{GH} > \overline{GI}$? How many sides of △GHI are congruent?

B Does △JKL have two congruent sides? What other triangle represented in D3 also has two congruent sides?

C Which triangle has three congruent sides?

A triangle with no two sides congruent is a *scalene triangle.*

D Which triangle represented in D3 is a scalene triangle?

A triangle with two congruent sides is an *isosceles triangle.*

E Which triangles represented in D3 are isosceles triangles?

A triangle with three congruent sides is an *equilateral triangle.*

F Which triangle represented in D3 is an equilateral triangle?

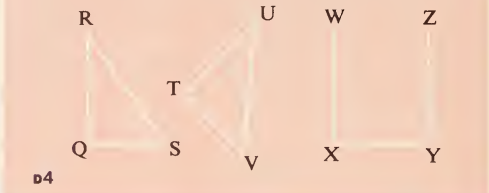
G Is any equilateral triangle also an isosceles triangle? How do you know?

H Is the set of all equilateral triangles a subset of the set of all isosceles triangles? How do you know?

I Is the set of all isosceles triangles a subset of the set of all equilateral triangles? Explain your answer.

J Make a sketch of an equilateral triangle. Make a sketch of an isosceles triangle that is not equilateral. Make a sketch of a scalene triangle.

D3



D4

sca lene triangle (skā lēn′). A triangle with no two of its sides congruent.

i sos ce les triangle (ī sos′ə lēz). A triangle with two of its sides congruent.

e qui lat er al triangle (ē′kwə lat′ər əl). A triangle with its three sides congruent.

right triangle. A triangle with one right angle.

In lesson 27 you learned about right angles. Now you will study right triangles.

K Look at D4. ∠RQS of △QRS is a right angle. Since one angle of △QRS is a right angle, △QRS is a *right triangle.*

L Look again at D4. ∠VTU of △TUV is a right angle. Is △TUV a right triangle?

M Look at the picture of the geometric figure WXYZ in D4. This figure has two right angles, ∠WXY and ∠XYZ. Is the figure a triangle?

N Do you think that a right triangle can have more than one right angle?

O Do you think that a right triangle can also ? be an isosceles triangle? Do you think that a right triangle can also be an equilateral triangle?

P Suppose that U is the set of all triangles. ? Suppose also that A is the set of all isosceles triangles, and that B is the set of all equilateral triangles. Make a Venn diagram to represent these sets. Describe $A \cap B$.

Q Suppose that C is the set of all right trian- ? gles. Then $A \cap C$ is the set of all triangles that are both isosceles and right triangles. On the Venn diagram you made for exercise P, represent set C. Describe $B \cap C$.

R Suppose that D is the set of all scalene tri- ? angles. Describe $C \cap D$.

You have already studied congruent segments and congruent angles. Now you will learn about congruent triangles.

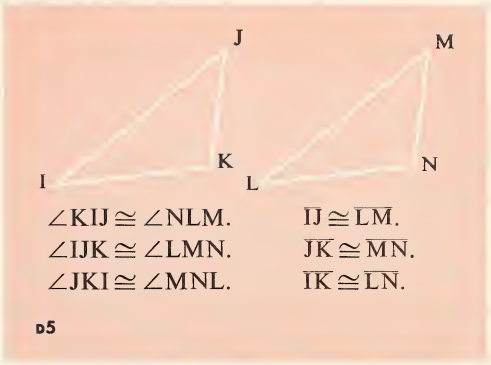
A The sentences in D5 express true statements about the angles and sides of $\triangle IJK$ and $\triangle LMN$ represented in D5. Study D5.

B Is each angle of $\triangle IJK$ congruent to an angle of $\triangle LMN$? Is each angle of $\triangle LMN$ congruent to an angle of $\triangle IJK$?

C Is each side of $\triangle IJK$ congruent to a side of $\triangle LMN$? Is each side of $\triangle LMN$ congruent to a side of $\triangle IJK$?

Each angle of $\triangle IJK$ is congruent to an angle of $\triangle LMN$, and each angle of $\triangle LMN$ is congruent to an angle of $\triangle IJK$. Each side of $\triangle IJK$ is congruent to a side of $\triangle LMN$, and each side of $\triangle LMN$ is congruent to a side of $\triangle IJK$. Therefore, $\triangle IJK$ and $\triangle LMN$ are *congruent triangles*.

con gru ent triangles (kong′grü ənt). Two triangles that have the same size and shape. Suppose that $\triangle ABC$ and $\triangle DEF$ are any two triangles. If $\angle ABC \cong \angle DEF$, $\angle BAC \cong \angle EDF$, $\angle BCA \cong \angle EFD$; and if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{AC} \cong \overline{DF}$, then $\triangle ABC \cong \triangle DEF$.



D5

Suppose that $\triangle ABC$ and $\triangle DEF$ are any two triangles. If $\angle ABC \cong \angle DEF$, $\angle BAC \cong \angle EDF$, $\angle BCA \cong \angle EFD$; and if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{AC} \cong \overline{DF}$, then $\triangle ABC \cong \triangle DEF$.

D Look again at D5. $\triangle IJK \cong \triangle LMN$. Do you think that $\triangle LMN \cong \triangle IJK$?

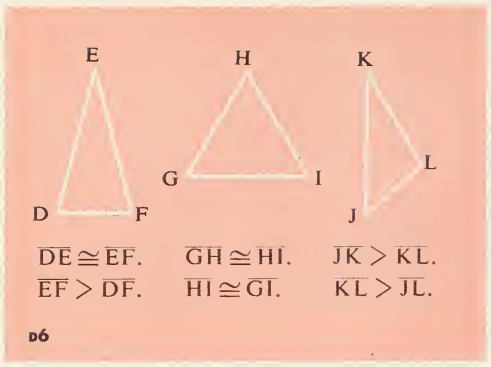
E Suppose that $\triangle LMN$ is congruent to a third triangle, $\triangle OPQ$. Do you think that $\triangle IJK \cong \triangle OPQ$?

Now you know that if two triangles are congruent, they have three pairs of angles that are congruent and three pairs of sides that are congruent.

F Suppose that two triangles have three an- ? gles of one triangle congruent to three angles of the other. Do you think that the three sides of one triangle are congruent to the three sides of the other triangle?

G Suppose that two triangles have three sides ? of one triangle congruent to three sides of the other. Do you think that the three angles of one triangle are congruent to the three angles of the other triangle?

In this lesson you have studied different kinds of triangles. You also know what it means to say that two triangles are congruent.



On your own

1 Three noncollinear points determine exactly how many segments? What geometric figure is the union of these segments?

2 Make a sketch of a triangle determined by noncollinear points R, S, T. Name the sides of this triangle. Name the vertices.

3 $\triangle RST$ is the union of what segments?

4 Is $\overline{RS}$ a subset of $\triangle RST$?

5 Is $\{R\}$ a subset of $\triangle RST$?

6 Is $\angle SRT$ a subset of $\triangle RST$?

For exercises 7 through 12, use the sentences and the pictures in D6 to help you.

7 Is $\triangle DEF$ an isosceles triangle? Is $\triangle DEF$ an equilateral triangle? Explain your answers.

8 In $\triangle DEF$, do you think $\angle EFD \cong \angle FDE$?

9 Is $\triangle GHI$ an isosceles triangle? An equilateral triangle? Explain your answers.

10 In $\triangle GHI$, do you think $\angle HIG \cong \angle IGH$? Do you think $\angle IGH \cong \angle GHI$? Do you think $\angle HIG \cong \angle GHI$?

11 Is $\triangle JKL$ a scalene triangle? How do you know?

12 In $\triangle JKL$, do you think $\angle KLJ \cong \angle LJK$? Do you think $\angle LJK \cong \angle JKL$? Do you think $\angle KLJ \cong \angle JKL$?

13 Suppose that U is the set of all triangles; X is the set of all equilateral triangles; Y is the set of all right triangles. Make a Venn diagram to represent $X \cap Y$. Tabulate $X \cap Y$.

14 Could you make a sketch of a right equilateral triangle? Explain your answer.

15 In $\triangle LMN$, $\angle MLN$ is a right angle. $\overline{LM} \cong \overline{LN}$. Make a sketch of $\triangle LMN$.

16 Is $\triangle LMN$ a right triangle? What other kind of triangle is it?

17 In $\triangle OPQ$, $\angle POQ$ is a right angle, $\overline{QP} > \overline{PO}$, and $\overline{PO} > \overline{OQ}$. Make a sketch of $\triangle OPQ$.

18 Is $\triangle OPQ$ a scalene triangle? What other kind of triangle is it?

19 Make a sketch of a geometric figure with two right angles. Is the figure a triangle?

20 Make a sketch of a geometric figure with three right angles. Is the figure a triangle?

21 Can a triangle have more than one right angle?

22 Suppose that $\triangle RST$ and $\triangle UVW$ are any two triangles such that $\angle RST \cong \angle UVW$, $\angle STR \cong \angle VWU$, $\angle TRS \cong \angle WUV$; and that $\overline{RS} \cong \overline{UV}$, $\overline{ST} \cong \overline{VW}$, $\overline{RT} \cong \overline{UW}$. What do you know about $\triangle RST$ and $\triangle UVW$?

23 Now suppose that $\triangle XYZ$ is any triangle other than $\triangle UVW$ so that $\angle UVW \cong \angle XYZ$, $\angle VWU \cong \angle YZX$, $\angle WUV \cong \angle ZXY$; and that $\overline{UV} \cong \overline{XY}$, $\overline{VW} \cong \overline{YZ}$, $\overline{UW} \cong \overline{XZ}$. Is $\triangle UVW \cong \triangle XYZ$?

24 Use the information given in exercises 22 and 23 to answer the following question. Do you think that $\triangle RST \cong \triangle XYZ$?

25 Suppose that $\triangle BCD$ and $\triangle EFG$ are any two triangles such that $\angle BCD \cong \angle EFG$, $\overline{CB} \cong \overline{FE}$, and $\overline{CD} \cong \overline{FG}$. Do you think that $\overline{BD} \cong \overline{EG}$, that $\angle CDB \cong \angle FGE$, and that $\angle DBC \cong \angle GEF$?

26 Suppose that $\triangle EFG$ and $\triangle HIJ$ are any two triangles such that $\angle EFG \cong \angle HIJ$, $\angle FGE \cong \angle IJH$, and $\overline{FG} \cong \overline{IJ}$. Do you think that $\overline{EG} \cong \overline{HJ}$, that $\overline{FE} \cong \overline{IH}$, and that $\angle GEF \cong \angle JHI$?

Regions of the plane

You already know that a circle separates a plane into two regions. Now you will learn about other geometric figures that separate a plane. The geometric figures that you will study in this lesson are plane figures.

A Look at D1. How many points are contained in the intersection of $\overline{AB}$ and ℓ_1? How do you know that A and B are in the same half-plane?

B Look again at D1. How many points are contained in the intersection of $\overline{BC}$ and ℓ_1? How do you know that B and C are not in the same half-plane?

REMINDER

A line separates a plane into two sets of points. The two sets are called half-planes.

A circle separates a plane into two regions.

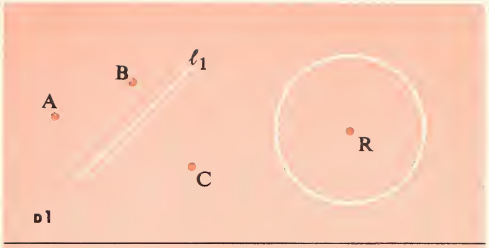
See lesson 11, page 41. See lesson 12, page 49.

C Circle R, represented in D1, separates the plane into two regions. Name these regions.

D Are A, B, and C in the exterior of circle R? What point named in D1 is in the interior of circle R?

E What is the boundary of a half-plane? What is the boundary of the interior of a circle? What is the boundary of the exterior of a circle?

F What is the intersection of a circle and its interior?



D1

D2

G What is the union of a circle, its interior, and its exterior?

H Now look at D2. $\overrightarrow{KJ}$ is one side of $\angle JKL$. What is the other side?

I Is $\overrightarrow{KJ}$ included in $\overleftrightarrow{KJ}$? Is $\overrightarrow{KL}$ included in $\overleftrightarrow{KL}$? Explain your answers.

You know that $\overleftrightarrow{KJ}$ separates plane JKL into two half-planes. One of these half-planes contains point L. This half-plane is represented by the part of D2 that is shaded by pictures of vertical lines.

J Does $\overleftrightarrow{KL}$ also separate plane JKL into two half-planes? One of these half-planes contains point J. How is this half-plane represented in D2?

K Now think of the half-plane represented by ? pictures of vertical lines and the half-plane represented by pictures of horizontal lines. How is the intersection of these two half-planes represented in D2?

The intersection of the two half-planes represented in D2 is the *interior* of ∠JKL. Points in the plane that are not in either ∠JKL or the interior of ∠JKL are in the *exterior* of ∠JKL. The interior and exterior of ∠JKL are regions of the plane. ∠JKL is a boundary of each region.

L Which point named in D2 is in the interior of ∠JKL? Which points named are in the exterior of ∠JKL? Which points named are in ∠JKL?

M Suppose that C is a point in the interior of an angle and that D is a point in the exterior. What do you know about the intersection of $\overline{CD}$ and the angle?

N Name the triangle represented in D3. A triangle has how many sides? A triangle has how many angles?

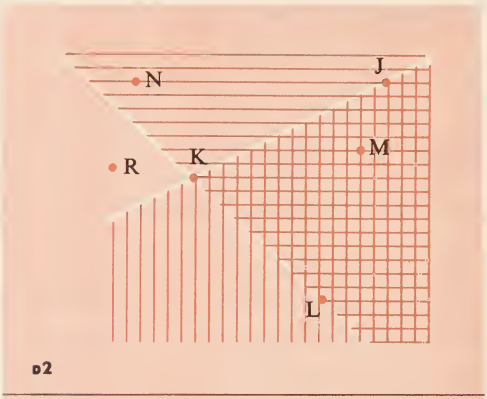
O Think of the angles of the triangle represented in D3. Point B is in the interior of which angle of △ADF? Point H is in the interior of which angle of △ADF? Point R is in the interior of which angle of △ADF?

P Which point named in D3 is in the interiors of all three angles of △ADF? Which point named is in the intersection of the interiors of the three angles of △ADF?

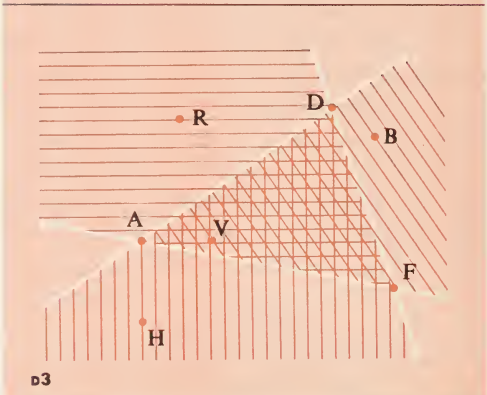
The intersection of the interiors of the angles of a triangle is the *interior of the triangle*. Points that are not in either the triangle or the interior of the triangle are in the *exterior of the triangle*. The interior of a triangle and the exterior of a triangle are regions of the plane. The triangle is a boundary of each of these regions.

Q Look again at D3. Which point named is in the interior of △ADF? Which points named are in △ADF?

R Suppose that M is a point in the interior of a triangle, and that N is a point in the exterior. What do you know about the intersection of $\overline{MN}$ and the triangle?



D2



D3

in te ri or of an angle (in tēr′ē ər). The intersection of two half-planes. If ∠ABC is any angle, then the interior of ∠ABC is the intersection of the following two half-planes: the half-plane that is bounded by line AB and that contains point C and the half-plane that is bounded by line BC and that contains point A.

ex te ri or of an angle (eks tēr′ē ər). The set of points that are not in either the angle or the interior of the angle.

in te ri or of a triangle (in tēr′ē ər). The intersection of the interiors of the angles of the triangle.

ex te ri or of a triangle (eks tēr′ē ər). The set of points that are not in either the triangle or the interior of the triangle.

S What is the intersection of a triangle and its interior?

T What is the union of a triangle, its interior, and its exterior?

You have learned that circles, angles, and triangles separate planes into two regions. Now you will learn about some other geometric figures that separate a plane.

A Each picture in D4 represents a set of coplanar points. Is each set of points a plane figure? Explain your answer.

The plane geometric figure represented by picture A does not intersect itself. This geometric figure is a *simple closed curve*. Each of pictures B, C, D, E, and F also represents a simple closed curve.

Observe that in geometry the word "curve" has a meaning that differs from its meaning in everyday speech. Geometric figures like those represented by pictures A, C, and D are simple closed curves, even though they are the union of segments.

When you make a sketch of a simple closed curve, you will not need to lift your pencil from the paper, and you will not cross a pencil mark that you have already made. You will start and finish at the same place.

B Is a segment a simple closed curve? Is an angle a simple closed curve? Is a triangle a simple closed curve? Is a circle a simple closed curve? Explain your answers.

C Explain why pictures G, H, I, J, K, and L in D4 do not represent simple closed curves.

D Look at pictures A through F in D4. Into ? how many regions does each simple closed curve separate the plane?

Simple closed curves have the property expressed below.

A simple closed curve separates the plane into two regions.

D4

One of the regions is the *interior of the simple closed curve*, and the other region is the *exterior of the simple closed curve*.

E Suppose a plane figure does not separate the ? plane into two regions. What do you know about the figure?

F Suppose one point is in the interior of a ? simple closed curve and one point is in the exterior of the simple closed curve. What do you know about the intersection of the simple closed curve and the segment determined by the two points?

G What is the intersection of the interior of a simple closed curve and the exterior of the simple closed curve? Of a simple closed curve and the interior of the simple closed curve?

H What is the union of a simple closed curve, the interior of the simple closed curve, and the exterior of the simple closed curve?

I Is each of the plane figures represented in D5 a simple closed curve?

J Into how many regions does each geometric figure represented in D5 separate the plane?

In each simple closed curve represented in D5, each point of the simple closed curve is also a member of a segment included in the curve. A simple closed curve that is the union of segments is a *simple polygon*. The segments are sides of the polygon. The endpoints of the segments are vertices of the polygon.

In this book we will use the word "polygon" to mean "simple polygon." As you continue with your work in mathematics, you will learn about polygons that are not simple polygons. You will then learn how to describe a simple polygon without using the words "simple closed curve."

K Is a triangle a polygon? How do you know?

L Three polygons are represented in D6. What are the vertices and sides of polygon ABCDE? What are the vertices and sides of polygon FGHIJK? Notice that the names of the vertices are used to name a polygon.

M Look again at D6. Point S is in the interior of polygon LMNR. Which other point named is in the interior of polygon LMNR? What points named in D6 are in the exterior of polygon LMNR?

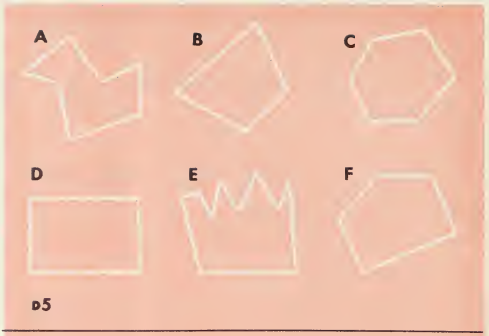
N Polygon LMNR is the union of what segments?

Now you know that a simple closed curve separates a plane into two regions. You also know that a polygon is a special kind of simple closed curve.

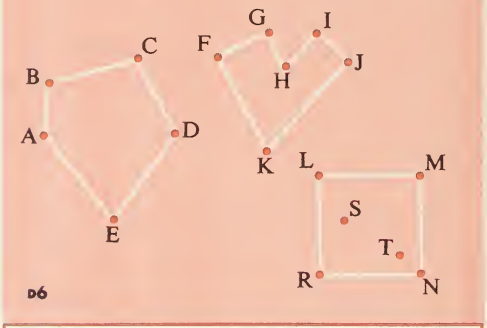
On your own

Use D7 as you answer the questions in exercises 1 through 10.

1 What points named in D7 are in the interior of $\angle CED$?



D5



D6

simple polygon (sim′pəl pol′i gon). A simple closed curve that is the union of segments. The segments are the sides of the polygon. The endpoints of the segments are the vertices of the polygon.

2 What points named are in $\angle DAB$?

3 What point named is in the intersection of the interior of $\angle DAB$ and the interior of $\angle CED$?

4 What point named is in the intersection of the exterior of $\angle DAB$ and the exterior of $\angle CED$?

5 Tabulate $\angle DAB \cap \angle CED$.

6 What points named are in the union of $\angle DAB$ and $\angle CED$?

7 What point named is in the interior of $\triangle CDE$?

8 What points named are in the exterior of $\triangle CDE$?

9 What point named is in the intersection of the interior of $\angle CED$ and the exterior of $\angle DAB$?

10 What points named are in the union of $\triangle CDE$ and the interior of $\triangle CDE$?

Use D8 as you answer the questions in exercises 11 through 20.

11 Polygon ABCDEF has how many sides? Name the sides of this polygon.

12 Name the angles of polygon ABCDEF.

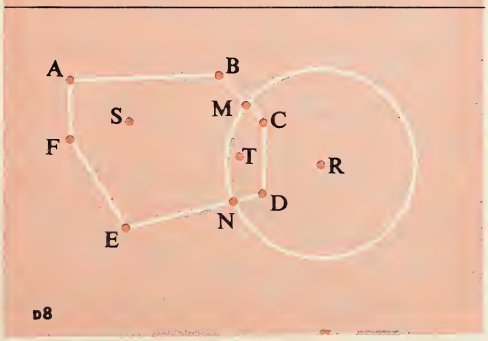
13 Tabulate circle R $\cap$ polygon ABCDEF.

14 What point named in D8 is in the intersection of the interior of circle R and the interior of polygon ABCDEF?

15 How many points named are in the intersection of the exterior of circle R and the exterior of polygon ABCDEF?

16 What points named are in the union of circle R and polygon ABCDEF?

D7



D8

17 What points named are in the union of the interior of polygon ABCDEF and the interior of circle R?

18 What points named are in the union of the exterior of circle R and the exterior of polygon ABCDEF?

19 How many points are in the intersection of $\overline{ST}$ and polygon ABCDEF? Are both S and T in the interior of polygon ABCDEF?

20 How many points are in the intersection of $\overline{SR}$ and polygon ABCDEF?

KEEPING SKILFUL

Make one sketch for exercises 1 through 7. The universe is the plane that includes circle W.

1 Make a sketch of circle W.

2 Make a sketch of radius WS.

3 Locate G in the interior of circle W and F in the exterior of circle W.

4 Write a standard description of circle W.

5 Write a standard description of the interior of circle W.

6 Write a standard description of the exterior of circle W.

7 Describe the intersection of circle W and $\overline{GF}$.

8 A, B, and C are collinear points. Suppose that B is between A and C. Write a sentence that expresses a true statement comparing $\overline{AB}$ and $\overline{AC}$.

9 Suppose that C is between A and B. Write a sentence that expresses a true statement comparing $\overline{AB}$ and $\overline{AC}$.

10 Suppose that C and B are the same point. Write a sentence that expresses a true statement comparing $\overline{AB}$ and $\overline{AC}$.

11 No three of the points A, B, C, D, E, F, and G are collinear. How many of the segments determined by these points contain point D?

SPECIAL CHALLENGE

Read the problem in D1. Then study D2 and D3. They show possible locations of farms A, B, and C and of wells J, K, and L.

A Make sketches similar to those in D2 and D3.

B Try to show how each farmer could arrange the pipes from his farm to each of the wells. Remember that no two pipes can cross.

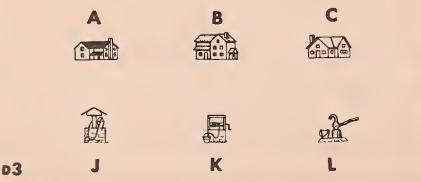
C Make other sketches to show different locations of the farms and wells. In each sketch, try to show how the pipes can be arranged.

D Is it possible for the farmers to arrange the pipes under the conditions given in the problem? Use what you know about simple closed curves to explain your answer.

Each of three farmers wants to run a separate pipe from his farm to each of three wells. However, all the pipes must be at the same level, and no two of the pipes can cross. Is it possible for the farmers to arrange the pipes under these conditions?

D1

D2



D3

CHECKING UP

The numeral within parentheses tells you what page to turn to if you need help with an exercise.

Test 23

For each of exercises 1 through 4, choose the word from the following list that most clearly expresses the idea described by the exercise.

equilateral	scalene
isosceles	skew
line	vertex

1 Common endpoint of two sides of a polygon (116)

2 Triangle with exactly two congruent sides (110)

3 Triangle with no two congruent sides (110)

4 Intersection of two planes that are not parallel (100)

For each of exercises 5, 6, and 7, choose the word from the following list that most clearly expresses the idea described by the exercise.

angle	perpendicular
endpoint	ray
parallel	triangle

5 Simple polygon (116)

6 Disjoint planes (98)

7 Half-line and its boundary (104)

Test 24

What words or symbols best complete exercises 8 through 16?

8 Sets that have common members are sets that ——. (85)

9 —— is made up of two or more simple conditions and one or more connectives. (91)

10 The intersection of sets may be represented by —— diagrams. (84)

11 The union of set A and the empty set is ——. (89)

12 Two lines that are not included in the same plane are ~~~ lines. (101)

13 Segments AB, BC, CD, DE, and EA are the sides of polygon ~~~. (116)

14 The intersection of the empty set and a set that has 5 members is a set that has ~~~ members. (86)

15 The intersection of two sets is a ~~~ of each of the sets. (86)

16 A simple closed curve separates the plane into ~~~ regions. (115)

Test 25

A list of symbols or expressions is given for each exercise. From each list choose the expression or symbol that correctly completes the exercise.

17 The symbol ~~~ expresses the idea of intersection. (83)

a $\cup$ b $\cong$ c $\angle$ d $\cap$

18 The symbol ~~~ expresses the connective "and." (90)

a $\triangle$ b $\wedge$ c $\rightarrow$ d $<$

19 If the intersection of two lines that are included in the same plane is the empty set, then the two lines ~~~. (101)

a meet
b are skew lines
c are parallel
d intersect

20 ~~~ is the same as $\angle JKL$. (105)

a $\overline{KJ} \cup \overline{KL}$
b $\angle JLK$
c $\overrightarrow{KJ} \cup \overrightarrow{KL}$
d $\overline{KJ} \cap \overline{LK}$

21 Each angle of a linear pair of angles is ~~~ the other angle of the pair. (107)

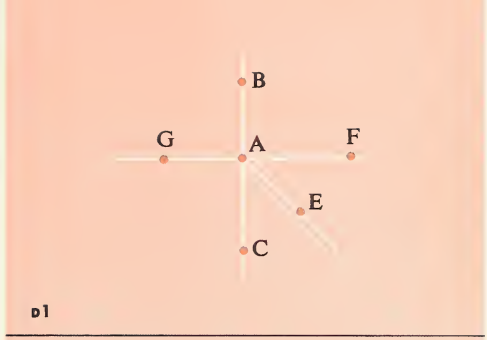
a the supplement of
b congruent to
c the right angle of
d perpendicular to

22 A triangle is the union of three ~~~ determined by three noncollinear points. (109)

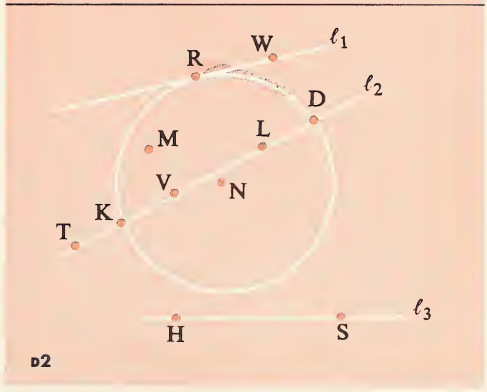
a rays
b segments
c lines
d half-lines

23 The greatest number of right angles a triangle may have is ~~~. (110)

a 3 b 2 c 1 d 0



D1



D2

Test 26

Use D1 to help you answer exercises 24 through 28. Lines BC and GF are perpendicular.

24 Name the sides of $\angle EAC$. (105)

25 Name the common side of $\angle CAF$ and $\angle GAC$. (106)

26 What angle and $\angle FAE$ are a linear pair of angles? (106)

27 Name a supplement of $\angle EAC$. (107)

28 Four right angles are represented in D1. Name these angles. (107)

Use D2 to help you with exercises 29 through 32. N is the center of the circle represented in the display.

29 Tabulate $\ell_1 \cap$ circle N. (103)

30 Tabulate $\ell_2 \cap$ circle N. (103)

31 Tabulate $\ell_3 \cap$ circle N. (103)

32 What points named are in the intersection of ℓ_2 and the interior of circle N? (103)

Test 27

Tabulate the solution set of each condition expressed in exercises 33 through 40. U = N.

33 $x > 7 \wedge x < 15$. (90)

34 $x > 4(3) \wedge x - 12 < 5$. (90)

35 $x + 3 < 14 \wedge 27 - x > 12$. (90)

36 $m > 6 \wedge m - 4 < 21$. (90)

37 $x + 6 > 13 \wedge x - 2 > 5$. (90)

38 $10 + x > 14 \wedge x + 3 < 5$. (90)

39 $9 - x > 8 \wedge x + 13 < 21$. (90)

40 $14 + x = 27 \wedge 27 - x = 13$. (90)

Test 28

For each problem below, first write a sentence that expresses the condition for the problem. You may use any letter as a name for the variable. Next tabulate the solution set of the condition. Then give the answer to the problem. U = N.

41 During a basketball game, Tom scored 18 points, and Bob scored 9 points. Jerry scored more points than Tom, but fewer than the total number of points scored by Tom and Bob. How many points could Jerry have scored during the game? (94)

42 Pat had fewer than 75 old pennies. He sold 15 of them to Harold, but still had more than 47 left. How many pennies could Pat have had before he sold the 15 to Harold? (94)

43 During the summer, Ray read fewer than 19 books. If he had read 9 more books, he would have read more than 26 books. How many books did Ray read during the summer? (94)

44 What numbers are less than 78 and greater than the sum of 19 and 31? (95)

45 What numbers less than 9 can be added to 15 so that each sum is greater than 20? (95)

46 What numbers less than 26 can be subtracted from 347 so that each difference is greater than 315? (95)

47 What numbers greater than 13 can be added to 98 so that each sum is less than 111? (95)

CHECKING UP

The numeral within parentheses tells you what page to turn to if you need help with an exercise.

Test 29

What words or symbols best complete exercises 1 through 9?

1 The sentence $95 - 47 > 45$ expresses ~~~ statement. (13)

2 The set of natural numbers that are greater than 155,000,000 is ~~~ set. (23)

3 ~~~ has infinitely many radii. (46)

4 The boundary of a half-plane is ~~~. (41)

5 {A, D} is ~~~ of {A, B, C, D, E}. (11)

6 ~~~ separates space into half-spaces. (41)

7 If three different points are ~~~, then one of the points is between the other two. (37)

8 $\overline{FG}$ is a segment whose ~~~ are F and G. (38)

9 A variable in a condition may be replaced only by a member of the ~~~. (16)

Test 30

Tabulate the solution set of each condition expressed in exercises 10 through 16.

U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

10 $3x < 9$. (18)

11 $5 + c \neq 7$. (18)

12 $12 - y = 2$. (18)

13 $x - 1 > 5$. (18)

14 $k < 4 \wedge k > 1$. (90)

15 $x + 2 > 5 \wedge 2x > 8$. (90)

16 $x < 10 \wedge x + 2 = 2 + x$. (90)

Test 31

From the list that follows each exercise, choose the expression that correctly completes the exercise.

17 $17 \times 5 = 85$ is ~~~. (17)

a a condition for equality
b a condition for inequality
c an equation
d an inequality

18 $13 - 5 \neq x$ is ~~~. (21)

a a condition for equality
b a condition for inequality
c an equation
d an inequality

19 In the condition $m - 7 < 32$, the m is a ~~~. (15)

a placeholder **c** numeral
b variable **d** number

20 {1, 3, 5, . . ., 1,000,001} is ~~~. (23)

a a finite set **c** the set of all odd numbers
b a condition **d** an infinite set

21 A ~~~ is the boundary of two half-lines. (42)

a point **b** plane **c** finite set **d** line

Test 32

For each problem, first write a sentence that expresses the condition for the problem. You may use any letter as a name for the variable. Next tabulate the solution set of the condition. Then give the answer to the problem. U = N.

22 After Phil found 7 golf balls, he had 23 golf balls. How many golf balls did he have before he found the 7? (68)

23 Kay has 57 autographs of famous persons. Betty has 24 more autographs than Kay. How many autographs does Betty have? (68)

24 Joyce had more than 21 picture postcards. After she mailed 17 of these cards, she had fewer than 9 left. How many postcards could she have had before she mailed the 17? (94)

25 What numbers can be added to 43 so that each sum is greater than 51? (78)

26 What numbers greater than 5 can be subtracted from 24 so that each difference is greater than 13? (94)

27 What numbers less than the difference of 300 and 172 can be added to 280 so that each sum is greater than 355? U = N.

Test 33

Multiply	Divide
28 38 by 5. (55)	**33** 423 by 9. (58)
29 76 by 18. (55)	**34** 480 by 32. (58)
30 124 by 86. (55)	**35** 6322 by 58. (58)
31 699 by 413. (55)	**36** 40120 by 472. (59)
32 207 by 390. (55)	**37** 251740 by 307. (59)

Unit 3

Conditions in two variables

lesson page

30 Ordered pairs 122
31 Sets of ordered pairs 126
32 Graphs of sets of ordered pairs 130
33 Conditions and graphs 134
34 The Cartesian set N × N 138
35 Charts and solution sets 143
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37 Simple conditions for problems 151
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39 Compound conditions for more complex problems 156
40 Conditions for abstract problems 160

3 30 Exploring ideas

Ordered pairs

For some time now you have been studying sets and conditions. In this lesson and in several of the lessons that follow, you will apply what you have learned to sets and conditions that differ in some ways from those you have already studied.

A Look at D1. Think of the capitals whose names are listed in D1 as a set of cities. Name this set of cities set A. Tabulate A.

B Think of the countries whose names are listed as a set of countries. Name this set of countries set B. Tabulate B.

C ? Is it possible to decide whether the idea expressed by the sentence in D2 is true or false? Explain your answer.

D How many different variables are there in the condition expressed in D2? How many different variables were there in the conditions that you studied in previous lessons?

E You will use the members of set A as replacements for m in the condition expressed in D2. Why can you use Paris as a replacement for m? Why should you not use Spain as a replacement for m?

F When you replace m by Paris, you obtain the condition expressed in D3. Why is it not possible to decide whether the idea expressed by the sentence in D3 is true or false?

G You will use the members of set B as replacements for n in the condition expressed in D2. Why can you use Spain as a replacement for n? Why should you not use Paris as a replacement for n?

CAPITAL CITIES	COUNTRIES
Paris	Norway
Ottawa	France
Madrid	Spain
	Canada

D1

---

m is the capital of n.

D2

Paris is the capital of n.

D3

---

m is the capital of Spain.

D4

Paris is the capital of Spain.

D5

H When you replace n by Spain, you obtain the condition expressed in D4. Is it possible to decide whether the idea expressed by the sentence in D4 is true or false? Explain your answer.

I When a condition contains two variables, must you make a replacement for each variable in order to obtain a statement?

J Now replace m by Paris and n by Spain. Do you obtain the statement expressed in D5? Is the statement true?

You used Paris as a replacement for m and Spain as a replacement for n in the condition "m is a capital of n." Notice that you used a pair of objects to obtain the statement expressed in D5.

K The first object of the pair, Paris, is the *first component* of the pair. The first component is a member of what set?

L The second object of the pair, Spain, is the *second component* of the pair. The second component is a member of what set?

In the exercises above, you used a city for the first component and a country for the second component of a pair of objects. A pair of objects whose components occur in a special order is an *ordered pair*.

M Look at D6. Notice that the names of the components of the ordered pair Paris, Spain are written within parentheses. D6 also shows how to read and write the name of this ordered pair.

N Think about the condition "*m* is the capital of *n*." When you replace *m* by a member of {Paris, Ottawa, Madrid} and *n* by a member of {Norway, France, Spain, Canada}, do you obtain a statement? Do the sentences in D7 express all the statements you can obtain?

O For each statement expressed in D7, write the name of the ordered pair that was used to make replacements. Which do you write first, the name of the replacement for *m* or the name of the replacement for *n*?

P Which of the statements expressed in D7 are true?

Q What ordered pairs were used to obtain true statements?

R Read the symbols (Paris, Norway) and ? {Paris, Norway}. How do the ideas expressed by these symbols differ?

Now you will study some conditions that concern numbers. You will also learn about ordered pairs whose components are numbers.

A Read the open sentence in D8. What are the variables in the condition expressed by the sentence?

"The ordered pair

(Paris, Spain)

whose first component
is Paris and

whose second component
is Spain"

D6

A Paris is the capital of Norway.
B Paris is the capital of France.
C Paris is the capital of Spain.
D Paris is the capital of Canada.
E Ottawa is the capital of Norway.
F Ottawa is the capital of France.
G Ottawa is the capital of Spain.
H Ottawa is the capital of Canada.
I Madrid is the capital of Norway.
J Madrid is the capital of France.
K Madrid is the capital of Spain.
L Madrid is the capital of Canada.

D7

$x < y$.
$A = \{1, 2, 3, 4\}$.
$B = \{2, 4, 6\}$.

D8

or dered pair (ôr′dərd pãr). A pair of objects in which the objects occur in a special order. The object that occurs first is the **first component** (kəm pō′nənt) of the ordered pair. The object that occurs second is the **second component** of the ordered pair.

B Also read the tabulations of sets A and B in D8. You will use the members of set A as replacements for x in $x < y$. You will use the members of set B as replacements for y in $x < y$.

C To obtain a statement from a condition in two variables, you must make replacements for how many variables?

D Do the sentences in D9 express all the statements that you can obtain from $x < y$ when you use the members of set A and set B, tabulated in D8, as replacements for x and y?

E Think of the statement expressed by the first sentence in D9. (1, 2) was used in making replacements to obtain this statement. D10 shows a short and convenient way to read and write the name of this ordered pair.

F Select four of the statements expressed in D9. For each statement, name the ordered pair that was used in making replacements for x and y.

G Which statements expressed in D9 are true? What ordered pairs were used to obtain true statements?

H The sentences $x + x = 4$ and $x + y = 4$ express two conditions. How many different variables are there in each condition? How do the two conditions differ?

$x < y$.
$A = \{1, 2, 3, 4\}$.
$B = \{2, 4, 6\}$.

D8

$1 < 2$.	$2 < 2$.	$3 < 2$.	$4 < 2$.
$1 < 4$.	$2 < 4$.	$3 < 4$.	$4 < 4$.
$1 < 6$.	$2 < 6$.	$3 < 6$.	$4 < 6$.

D9

(1, 2) "The ordered pair one, two"

D10

In this lesson you have learned that ordered pairs are used to make replacements for two variables.

On your own

Write a name for each of the ordered pairs described in exercises 1 through 5.

1 The ordered pair whose first component is Robert Louis Stevenson and whose second component is *Treasure Island*

2 The ordered pair whose first component is Mary and whose second component is guitar

3 The ordered pair whose first component is 15° and whose second component is 8 P.M.

4 The ordered pair whose first component is Larry and whose second component is 141

5 The ordered pair whose first component is 1 and whose second component is 13

Four conditions are expressed in exercises 6 through 9. Use the members of set C, tabulated below, as replacements for x. Use the members of set D, tabulated below, as replacements for y. For each exercise write sentences to express all the statements that you can obtain from the condition. Then write "T" after each sentence that expresses a true statement.

$C = \{2, 4, 6\}$. $D = \{5, 10, 15\}$.

6 $6 + x = y$.

7 $x + 5 < y$.

8 $x + 4 = y$.

9 $y > x + 1$.

10 In each ordered pair expressed below, the second component is 3 times the first component. One of the components is missing. Copy each symbol and supply the name of the missing component.

(1,) (4,) (, 9) (2,) (, 24)

11 In each ordered pair expressed below, the first component is 3 more than the second component. One of the components is missing. Copy each symbol and supply the name of the missing component.

(8,) (, 18) (12,) (, 29) (, 1)

APPLYING MATHEMATICS

The problems in this lesson are similar to the problems that you have studied so far in this book. For each problem, first write a sentence that expresses the condition for the problem. You may use any letter to name the variable. Next tabulate the solution set of the condition. Then give the answer to the problem. The universe is given for each problem.

1 Miss Larson had $1405 in her savings account. She withdrew some of this money and then had $890 left in her account. How much money did she withdraw from this account? U = N.

2 Martin scored 17 points during a basketball game. Together, Martin and Leo scored fewer than 32 points during the game. How many points could Leo have scored? U = C.

3 A restaurant owner had 315 dinner plates. He bought fewer than 150 new dinner plates and then had more than 434. How many new dinner plates could he have bought? U = N.

4 After Charles delivered 117 handbills, he had fewer than 43 handbills left to deliver. What is the greatest number of handbills that Charles could have had to deliver? U = N.

5 Susan collects unusual buttons. After her aunt had given her 18 buttons, Susan had 205 buttons in all. How many buttons had she collected before her aunt gave her the 18? U = N.

6 During a two-week business trip, Mr. Foster drove more than 913 miles. He drove 547 miles the first week. What is the least number of miles he could have driven the second week? U = N.

7 A survey stated that there were 6495 more private pilots than commercial pilots in Canada. There were 2548 commercial pilots at that time. How many private pilots were there in Canada at the time of the survey? U = N.

8 A merchant had more than 210 suits to sell. After he sold 89 of these suits, he had fewer than 131 suits left to sell. How many suits could he have had before he sold the 89? U = N.

9 Walter had 92¢ when he went to the school store. He spent some of this money and still had more than 34¢ left. How much money could he have spent? U = C.

10 Nine years ago, Mr. Roberts was less than 39 years old. Fifteen years from now, he will be more than 56 years old. How old can he be now? U = C.

11 In 1923, there were 680,772 fewer automobiles registered in Canada than in 1943. The number of automobiles registered in 1943 was 1,193,847. How many automobiles were registered in 1923? U = N.

12 The 2 P.M. temperature at an airport was 71 degrees above zero. The temperature had risen less than 16 degrees since 10 A.M., when it was less than 60 degrees. What could the temperature have been at 10 A.M.? U = N.

13 What numbers can be subtracted from 62 so that each difference is greater than 53? U = N.

14 When 39 is subtracted from a number, the difference is 27. When the number is added to 39, the sum is 105. What is the number? U = N.

15 What number is the product of 148 and 203? U = N.

16 What numbers are less than the sum of 49 and 86? U = N.

17 What numbers can be added to 7224 so that each sum is greater than 7401? U = N.

18 What numbers less than the product of 13 and 18 can be added to 76 so that each sum is greater than 295? U = N.

19 What numbers greater than 75 can be subtracted from 3212 so that each difference is greater than 3137? U = N.

Sets of ordered pairs

You have already learned what ordered pairs are and how to use them to make replacements in conditions in two variables. In this lesson you will learn about universes and solution sets whose members are ordered pairs.

A How many variables are there in the condition expressed in D1?

You know that you use ordered pairs to make replacements for two different variables in the same condition. Each component of an ordered pair is a member of a set of objects. Let us agree to use the members of set A, tabulated in D2, as replacements for a. Let us agree to use the members of set B as replacements for b.

B Think about the condition that is expressed in D1. Is (Labor Day, March) one ordered pair that you can use to make replacements for a and b?

C What are two other ordered pairs that you can use to make replacements for a and b?

To find all the ordered pairs that can be used to make replacements for a and b, match a member of A with each member of B in turn. Then match another member of A with each member of B, and so on. D3 shows one way to do this.

When each member of A is matched with each member of B, the result is a set of ordered pairs. This set is tabulated in D4.

Since the set tabulated in D4 was formed by matching each member of A with each member of B, you may use the symbol below at the right as a name of this set. D5 shows how to write and read this name of this set of ordered pairs.

$A \times B$

D What does the name $A \times B$ tell you about the first component of each ordered pair in the set? What does it tell you about the second component of each ordered pair in the set?

$A \times B$ is the *Cartesian set* produced from sets A and B.

E Look again at D2. How many members are in A? How many members are in B? How many members are in the Cartesian set $A \times B$?

a is a holiday in b.

D1

A = {Thanksgiving, Labor Day, Christmas}.
B = {March, June, September, December}.

D2

D3

{(Thanksgiving, March), (Thanksgiving, June), (Thanksgiving, September), (Thanksgiving, December), (Labor Day, March), (Labor Day, June), (Labor Day, September), (Labor Day, December), (Christmas, March), (Christmas, June), (Christmas, September), (Christmas, December)}

D4

F If you know the number of members in a set M and the number of members in a set N, how can you find the number of members in the Cartesian set $M \times N$?

G Is (June, Thanksgiving) a member of $A \times B$? Is (June, Thanksgiving) a member of $B \times A$? Is $B \times A$ the same set as $A \times B$? Explain your answers.

H Now you will use the members of $A \times B$ to make replacements for *a* and *b* in "*a* is a holiday in *b*." When you replace *a* by the first component of each ordered pair, and *b* by the second component of the same ordered pair, do you obtain the statements that are expressed in D6?

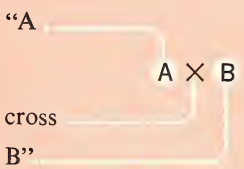
I Which statements expressed in D6 are true?

J Which members of the Cartesian set $A \times B$ are used to obtain true statements from "*a* is a holiday in *b*"?

You may tabulate the solution set of "*a* is a holiday in *b*" as shown in D7.

K Is each member of the solution set an ordered pair? Is the solution set a set of ordered pairs?

Car te sian set (kär tē′zhən). The set of all ordered pairs that can be formed by matching each member of one set in turn with each member of a second set. The Cartesian set $S \times R$ is the set of all ordered pairs that can be formed by matching each member of S in turn with each member of R. The members of S are the first components of the ordered pairs, and the members of R are the second components. If $S = \{1, 2\}$ and $R = \{4, 5\}$, then $S \times R = \{(1, 4), (1, 5), (2, 4), (2, 5)\}$.



D5

A Thanksgiving is a holiday in March.
B Thanksgiving is a holiday in June.
C Thanksgiving is a holiday in September.
D Thanksgiving is a holiday in December.
E Labor Day is a holiday in March.
F Labor Day is a holiday in June.
G Labor Day is a holiday in September.
H Labor Day is a holiday in December.
I Christmas is a holiday in March.
J Christmas is a holiday in June.
K Christmas is a holiday in September.
L Christmas is a holiday in December.

D6

{(Labor Day, September), (Christmas, December)}

D7

In exercises H through K, you used the members of A × B to make replacements for a and b in "a is a holiday in b." A × B is the universe for the pair of variables a and b. You may use the symbol below at the right to show that you are working with an ordered pair of variables. D8 shows a short way of reading this symbol.

(a, b)

L Is the solution set of "a is a holiday in b" a ? subset of the universe for (a, b)? Explain your answer.

Now you will study Cartesian sets in which each member is an ordered pair of numbers.

A How many variables are there in the condition expressed in D9?

We will use the symbol (x, y) to express the ordered pair of variables for $x > y$.

B What do you need before you can make replacements for x and y?

We will use the members of set X, tabulated in D10, as replacements for the first component of (x, y). We will use the members of set Y as replacements for the second component.

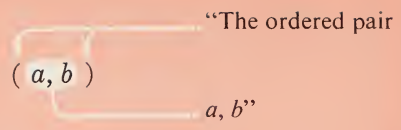
C D11 shows how each member of X may be matched with each member of Y. How many ordered pairs are formed?

D The ordered pairs that are formed when each member of X is matched with each member of Y are the members of the Cartesian set X × Y. The set X × Y is the universe for (x, y). Tabulate X × Y.

E When you use the first components of the members of the universe to make replacements for x in $x > y$ and the second components to make replacements for y, do you obtain the statements expressed in D12?

F Which statements expressed in D12 are true?

G Which ordered pairs in X × Y make true statements from $x > y$?



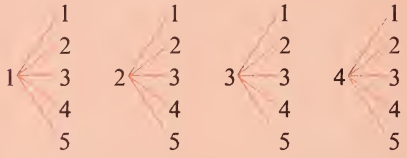
D8

$x > y$.

D9

X = {1, 2, 3, 4}.
Y = {1, 2, 3, 4, 5}.

D10



D11

Universe for (x, y) = X × Y.
$x > y$.

A $1 > 1$.	K $3 > 1$.
B $1 > 2$.	L $3 > 2$.
C $1 > 3$.	M $3 > 3$.
D $1 > 4$.	N $3 > 4$.
E $1 > 5$.	O $3 > 5$.
F $2 > 1$.	P $4 > 1$.
G $2 > 2$.	Q $4 > 2$.
H $2 > 3$.	R $4 > 3$.
I $2 > 4$.	S $4 > 4$.
J $2 > 5$.	T $4 > 5$.

D12

Universe for $(x, y) = S \times T$.
$S = \{1, 3, 5\}$.
$T = \{2, 4, 6\}$.
$y = x + 1$.

D13

$A = \{1, 2, 3\}$.

D14

$A \times A = \{(1, 1), (1, 2), (1, 3),$
$(2, 1), (2, 2), (2, 3),$
$(3, 1), (3, 2), (3, 3)\}$.

D15

Universe for $(m, n) = A \times A$.
$m < 2 + n$.

D16

H Tabulate the solution set of $x > y$.

I Is the solution set of $x > y$ a subset of the universe for (x, y)? Explain your answer.

J Now look at D13. How is the Cartesian set $S \times T$ obtained? Tabulate $S \times T$.

K What condition is expressed in D13? What is the universe for the variables?

L The members of what set will be used as replacements for x? The members of what set will be used as replacements for y?

M Write the sentences that express all the statements obtained when you make replacements for x and y in $y = x + 1$.

N Which of the sentences that you wrote express true statements?

O Tabulate the solution set of $y = x + 1$.

P Is the solution set a subset of $S \times T$?

Q Look again at D13. Suppose that $S \times T$ is the universe for (y, x) rather than for (x, y). The members of what set would be used as replacements for y? The members of what set would be used as replacements for x?

It is also possible to produce a Cartesian set from only one set of objects.

A Set A is tabulated in D14. Make a diagram to show how each member of A may be matched with itself and with each of the other members of A.

When each member of A is matched with itself and with each of the other members of A, ordered pairs are formed. These ordered pairs are the members of the Cartesian set $A \times A$. This set is tabulated in D15.

B $A \times A$ is the *Cartesian set of A*. For each ordered pair that is a member of $A \times A$, both components are members of the same set. What is this set?

C What condition is expressed in D16? What is the universe for (m, n)?

D Look again at D15. Is $(1, 3)$ a member of $A \times A$? Can you use $(1, 3)$ to make replacements for (m, n) for $m < 2 + n$?

E If you use $(1, 3)$, how do you know that you should replace m by 1 and n by 3?

F Think about the statement you obtain when you use $(1, 3)$ to make replacements in the condition. Is $1 < 2 + 3$ a true statement? Is $(1, 3)$ a solution of $m < 2 + n$?

G Use each member of $A \times A$ to make replacements for m and n. For each statement obtained, decide whether or not the statement is true. Then tabulate the solution set of $m < 2 + n$.

H Is the solution set of $m < 2 + n$ a subset of $A \times A$?

In this lesson you have formed sets of ordered pairs. You have used members of sets of ordered pairs to make replacements in conditions

in two variables. You have also found the solution sets of the conditions.

On your own

1 Set A, tabulated below, is a set of rivers. Set B is a set of continents. Tabulate A × B.

A = {Amazon, Danube, Nile}.

B = {Africa, Asia, Europe, Australia}.

2 Sets M, P, and R are tabulated below. Tabulate M × P, P × R, and M × R.

M = {7, 11, 6, 4}.

P = {1, 100, 15, 10, 5}.

R = {1, 2}.

3 Is (100, 11) a member of M × P?

4 Tabulate P × M. Is (100, 11) a member of P × M?

5 Tabulate M × M. How many members are there in M × M?

6 G × H is tabulated below. Tabulate G. Tabulate H.

G × H = {(1, 2), (1, 4), (1, 6), (1, 8), (1, 10), (3, 2), (3, 4), (3, 6), (3, 8), (3, 10)}.

Which of the sets tabulated in exercises 7, 8, and 9 are subsets of G × H?

7 {(1, 2), (1, 8), (3, 2), (3, 4)}

8 {1, 2, 3, 4}

9 {(3, 8), (1, 8), (4, 3)}

G × H, tabulated in exercise 6, is the universe for (x, y) for the conditions expressed in exercises 10 through 17. Tabulate the solution set of each condition.

10 $x + 3 = y$.

11 $x < y$.

12 $x > y$.

13 $y > x + 2$.

14 $y < x + 1$.

15 $2x > y$.

16 $x + y = x + 4$.

17 $2x < 2y$.

18 Explain how {10, 1}, (10, 1), and {(10, 1)} differ.

19 Suppose that {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)} is the universe for the variables in $s > r$. What else must you know before you can find the solution set of $s > r$?

3 32 Exploring ideas

Graphs of sets of ordered pairs

You have studied sets of ordered pairs and found solution sets of conditions in two variables. Now you will learn to make graphs of universes and solution sets whose members are ordered pairs.

A Look at D1. To obtain statements from $x + 2 < y$, you must make replacements for two variables. What are the variables?

B Tabulate A × B. Then use each member of A × B to make replacements for x and y in $x + 2 < y$. Write the sentences that express the statements you obtain.

C Which of the sentences that you wrote express true statements?

D Tabulate the solution set of $x + 2 < y$. Is each solution a member of A × B? Is the solution set a subset of A × B?

Another way to represent the universe for (x, y) and the solution set of $x + 2 < y$ is to make a graph. First, pictures of two number lines are drawn, as shown in D2. Each of these number lines is an *axis* (ak′sis). Notice that the two axes (ak′ sēz) are perpendicular. The horizontal number line is the *first axis*. The vertical number line is the *second axis*.

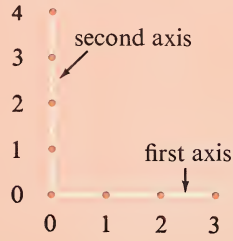
The first component of each ordered pair in A × B is associated with a point in the first axis. The second component of each ordered pair in A × B is associated with a point in the second axis.

E Look at D3. Each dot represents a point associated with a member of A × B. Is point D

Universe for $(x, y) = A \times B$.
$A = \{0, 1, 2, 3\}$.
$B = \{0, 1, 2, 3, 4\}$.
$x + 2 < y$.

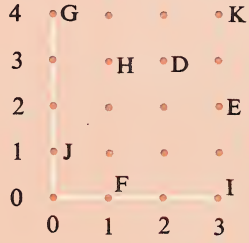
D1

Universe for $(x, y) = A \times B$.



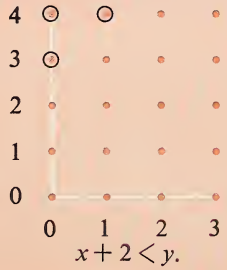
D2

Universe for $(x, y) = A \times B$.



D3

Universe for $(x, y) = A \times B$.



$x + 2 < y$.

D4

located directly above point 2 in the first axis? Is it located directly to the right of point 3 in the second axis? Point D is associated with (2, 3). Is (2, 3) a member of $A \times B$?

F Is point E located directly above point 3 in the first axis? Is it located directly to the right of point 2 in the second axis? With what ordered pair is point E associated? Is this ordered pair also a member of $A \times B$?

G Now look at the dot representing point F. Is the location of point F the same as the location of point 1 in the first axis? Is point F located directly to the right of point 0 in the second axis? What ordered pair is associated with point F?

H Look at the dot representing point G. Is point G located directly above point 0 in the first axis? Is the location of point G the same as the location of point 4 in the second axis? With what ordered pair is point G associated? Is this ordered pair a member of $A \times B$?

I For points H, I, J, and K, name the ordered pair associated with each point.

J Is each member of $A \times B$ associated with a point represented by a dot? Does each dot represent a point associated with a member of $A \times B$?

The set of dots that represent the points associated with the members of $A \times B$ is a graph of $A \times B$.

K How many members are there in $A \times B$? How many dots are in the graph of $A \times B$?

L Which dots in D4 form a graph of $A \times B$?

Now you are prepared to decide whether or not the encircled dots in D4 form a graph of the solution set of $x + 2 < y$.

M The solution set of $x + 2 < y$ is $\{(0, 3), (0, 4), (1, 4)\}$. Is each solution associated with a point whose dot is encircled? Does each encircled dot represent a point associated with a solution of $x + 2 < y$?

The encircled dots represent the points associated with the solutions of $x + 2 < y$. The set of encircled dots is a graph of the solution set of $x + 2 < y$.

N Is the graph of the solution set of $x + 2 < y$? a subset of the graph of $A \times B$? Explain your answer.

In the following exercises you will work with other conditions in two variables to give you more experience in using sets of ordered pairs.

A Look at D5. Tabulate $A \times A$. Then use each member of $A \times A$ to make replacements for x and y in $2x > y$. For each statement obtained, decide whether or not the statement is true. Then tabulate the solution set of $2x > y$.

B Is the solution set a subset of the universe?

A graph can be used to represent the universe for (x, y) and the solution set of $2x > y$.

C Look at D6. The numbers that are replacements for x are associated with points in which axis? The numbers that are replacements for y are associated with points in which axis?

D Is each member of $A \times A$ associated with a point represented by a dot? Does each dot represent a point associated with a member of $A \times A$?

E Does D6 show a graph of the universe?

Now you can decide whether or not the encircled dots in D6 form a graph of the solution set of $2x > y$. Use the solution set you tabulated for exercise A to help you decide.

F Is each solution of $2x > y$ associated with a point whose dot is encircled? Are there any encircled dots that do not represent points associated with solutions of $2x > y$?

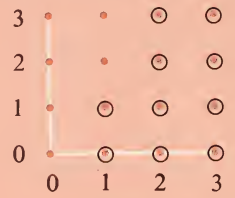
G Does the set of encircled dots form a graph of the solution set of $2x > y$?

H D6 shows two graphs. The set of all dots is a graph of the universe. The set of encircled dots is a graph of the solution set of $2x > y$.

Universe for $(x, y) = A \times A$.
$A = \{0, 1, 2, 3\}$.
$2x > y$.

D5

Universe for $(x, y) = A \times A$.



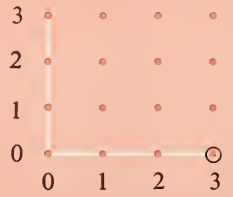
$2x > y$.

D6

Universe for $(x, y) = A \times A$.
$A = \{0, 1, 2, 3\}$.
$y + 3 = x$.

D7

Universe for $(x, y) = A \times A$.



$y + 3 = x$.

D8

Is the graph of the solution set a subset of the graph of the universe?

I Look at D7. Use each member of $A \times A$ to make replacements for x and y in $y + 3 = x$. Decide whether or not each statement obtained is true.

J Tabulate the solution set of $y + 3 = x$. Is the solution set a subset of the universe?

K Is a graph of the universe for (x, y) shown in D8? Explain your answer.

L Does the set of encircled dots form a graph of the solution set of $y + 3 = x$? Explain your answer.

M Is the graph of the solution set of $y + 3 = x$ a subset of the graph of the universe?

You have learned to make a graph of a universe when the universe is a set of ordered pairs. You have also learned to make graphs of solution sets of conditions in two variables.

On your own

1 D9 shows a graph of a set of ordered pairs. First tabulate the set of ordered pairs that are associated with all the points represented by dots in the graph. Then tabulate the set of ordered pairs that are associated with the points represented by encircled dots.

2 For D10, use the instructions given for D9.

3 For D11, tabulate the set of ordered pairs associated with the points represented by dots.

Each set tabulated in exercises 4, 5, and 6 is a subset of the set you tabulated for exercise 3. For each exercise make a copy of the graph that is shown in D11. Then make a graph of the subset.

4 $\{(1, 3)\}$

5 $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\}$

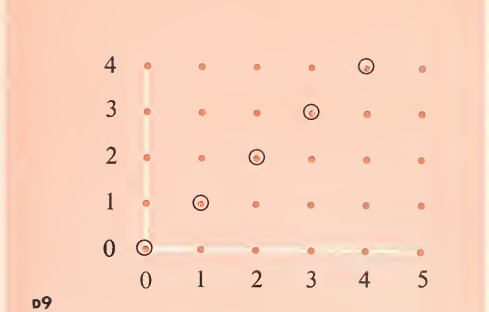
6 $\{(0, 4), (1, 4), (2, 4), (3, 4), (4, 4)\}$

7 Set A and set B are tabulated below. Tabulate $A \times B$.

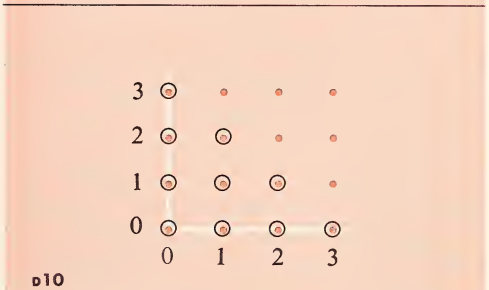
$A = \{0, 1, 2, 3, 4\}$. $B = \{0, 1, 2, 3\}$.

8 Now use $A \times B$ as the universe for (x, y) for $x = y + 1$. Tabulate the solution set of $x = y + 1$.

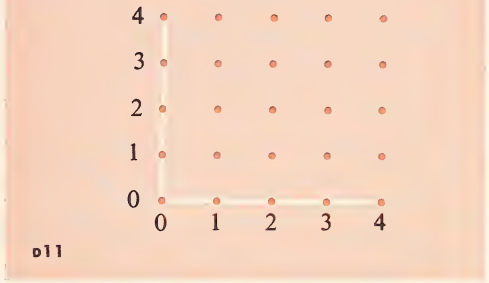
9 Make a graph of the solution set of $x = y + 1$. Remember that the graph of the solution set of $x = y + 1$ is shown as a subset of the graph of the universe.



D9



D10



D11

For the conditions expressed in exercises 10 through 13, use the universe for (x, y) that is tabulated below. First tabulate the solution set of each condition. Then, for each condition, make graphs of the universe and the solution set.

Universe for $(x, y) = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 1), (2, 2), (2, 3), (3, 0), (3, 1), (3, 2), (3, 3), (4, 0), (4, 1), (4, 2), (4, 3)\}$.

10 $x = y$.

11 $x > y$.

12 $2y = x$.

13 $y + x < 5$.

3 33 Exploring ideas

Conditions and graphs

You have learned to use a set of ordered pairs as a universe. In this lesson you will learn how to find solution sets of conditions in one variable when the universe is a set of ordered pairs.

A What condition is expressed in D1? What is the universe for the variable? Is the universe a finite set?

B Tabulate the solution set of the condition expressed in D1. The solution set has how many members? Is the solution set a finite set?

C Why is the solution of $x = 3$ a number and ? not an ordered pair of numbers?

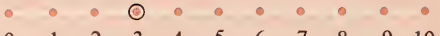
D Does the graph of the universe in D1 include a graph of the solution set of $x = 3$? Explain your answer.

E $A \times A$, tabulated in D2, is the universe you will use in the following exercises. How do the members of $A \times A$ differ from the members of the universe tabulated in D1?

When the universe is a set of ordered pairs, you can use any two different variables to refer to the members of the universe. In this lesson we will use (x, y) to refer to the members of the universe. This means that a Cartesian set is the universe for a pair of variables, x and y, in a condition, and that the first components are the replacements for x and the second components are the replacements for y.

F Even though the universe is a set of ordered pairs, you can find the solution set of a condition in only one variable. How many variables are there in $x = 3$?

$U = \{0, 1, 2, \ldots, 10\}$.



$x = 3$.

D1

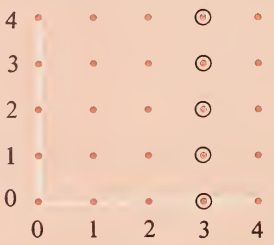
$A = \{0, 1, 2, 3, 4\}$.

$A \times A = \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (1, 0), (1, 1), (1, 2), (1, 3), (1, 4), (2, 0), (2, 1), (2, 2), (2, 3), (2, 4), (3, 0), (3, 1), (3, 2), (3, 3), (3, 4), (4, 0), (4, 1), (4, 2), (4, 3), (4, 4)\}$.

$x = 3$.

D2

Universe for $(x, y) = A \times A$.



$x = 3$.

D3

G How do you know that x in $x = 3$ may be re- ? placed by the first component of each ordered pair in $A \times A$?

H Does $x = 3$ contain a variable that may be replaced by the second component of each member of $A \times A$?

I Look at D3. Which dots form a graph of $A \times A$?

J Do the encircled dots in D3 represent points associated with ordered pairs? What is the first component of each of these ordered pairs? Tabulate this set of ordered pairs.

K Is the condition $x = 3$ satisfied by each of the ordered pairs contained in the set you tabulated for exercise J?

L Do any other ordered pairs in the universe satisfy the condition?

The set of ordered pairs that you tabulated for exercise J is the solution set of $x = 3$. The first component of each of these ordered pairs is 3. The condition does not make a requirement about the second component of each solution. Therefore, each ordered pair in the universe whose first component is 3 is a solution of $x = 3$.

M Which dots in D3 form a graph of the solution set of $x = 3$?

N Is the solution set of $x = 3$ a finite set? Is the solution set a subset of the universe? Explain your answers.

Next you will find the solution set of a condition for inequality in one variable. The universe is a set of ordered pairs.

A Look at D4. What is the universe for (x, y)? Tabulate the universe.

B What requirement does $y < 4$ make about the first component of each member of the solution set?

C What requirement does $y < 4$ make about the second component of each member of the solution set?

D Look at D5. Which dots form a graph of $A \times B$?

E Tabulate the set of ordered pairs associated with the points represented by encircled dots.

F Is $y < 4$ satisfied by each ordered pair in this set?

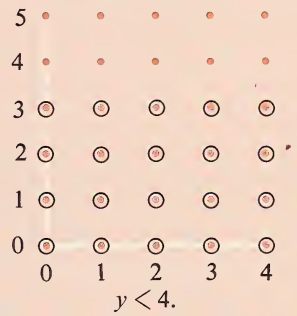
G Do any other ordered pairs in $A \times B$ satisfy $y < 4$?

H Is the set of ordered pairs you tabulated for exercise E the solution set of $y < 4$? Explain your answer.

Universe for $(x, y) = A \times B$.
$A = \{0, 1, 2, 3, 4\}$.
$B = \{0, 1, 2, 3, 4, 5\}$.
$y < 4$.

D4

Universe for $(x, y) = A \times B$.



$y < 4$.

D5

Universe for $(x, y) = R \times R$.
$R = \{0, 1, 2, 3\}$.
$y + 3 > 4$.

D6

I Which dots in D5 form a graph of the solution set of $y < 4$? Is the solution set of $y < 4$ a subset of the universe?

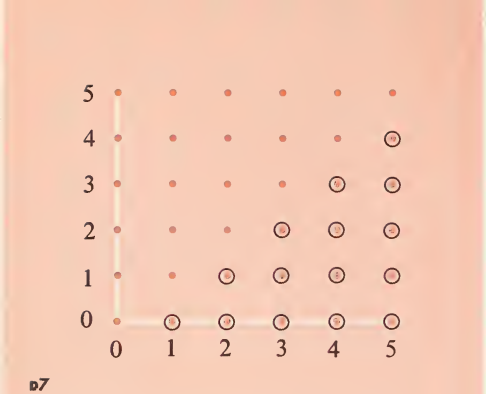
J Look at D6. What condition is expressed? What is the universe for (x, y)?

K Make a graph of $R \times R$.

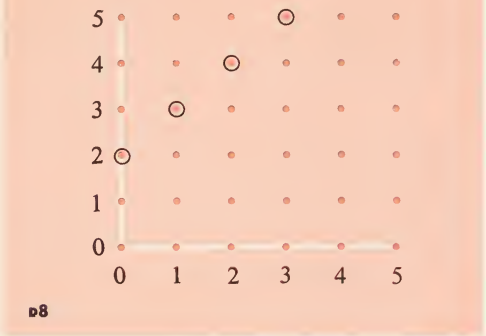
L Tabulate the solution set of $y + 3 > 4$. Remember that an ordered pair whose second component satisfies $y + 3 > 4$ is a solution of $y + 3 > 4$.

M Why is the solution set of $y + 3 > 4$ a set of ordered pairs?

N In the graph of $R \times R$ that you made for exercise K, make a graph of the solution set of $y + 3 > 4$.



D7



D8

Next you will see how you can obtain a condition when you are given a graph of its solution set.

A A graph of a universe for (x, y) consists of all the dots in D7. A graph of the solution set of a condition consists of the encircled dots. Tabulate this solution set.

B Think about the members of the solution set that you have tabulated. Is the first component of each member less than the second component? Is the first component of each member greater than the second component?

C Can the set you tabulated in exercise A be the solution set of $x < y$? Can it be the solution set of $x > y$? Explain your answers.

D A graph of a universe for (x, y) consists of all the dots in D8. A graph of the solution set of a condition consists of the encircled dots. Tabulate this solution set.

E What number can be added to the first component of each member so that the sum equals the second component?

F The set you tabulated in exercise D is the solution set of a condition. What is a condition that has this solution set?

In this lesson you have learned that when the universe is a set of ordered pairs, the solution set of a condition in one variable is also a set of ordered pairs. You have also learned how to obtain a condition by studying a graph of its solution set.

On your own

The universe for (x, y) for the conditions expressed in exercises 1 through 6 is tabulated below. Tabulate the solution set of each condition. For each condition, make a graph of the universe and a graph of the solution set.

Universe for $(x, y) = \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (1, 0), (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 0), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (3, 0), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (4, 0), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5)\}$.

1 $x = 2$.
2 $x < 2$.
3 $x > 2$.
4 $x = y$.
5 $y + 3 < 6$.
6 $y + 2 = x$.

The universe for (x, y) for the conditions expressed in exercises 7 through 12 is the same as the universe you used for the exercises above. Tabulate the solution set of each condition.

7 $x + 3 = 4$.
8 $y + 1 > 3$.
9 $x + 2 < 5$.
10 $y + 1 = 3$.
11 $x + 1 = 4$.
12 $x < y$.

13 D9 shows a graph of a set of ordered pairs. The dots form a graph of the universe for (x, y). A graph of the solution set of a condition consists of the encircled dots. Tabulate the

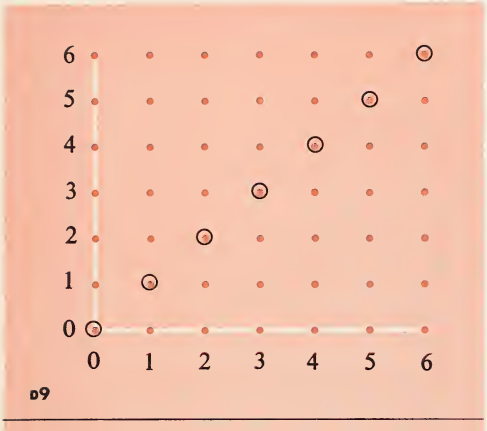
solution set. Then write a sentence expressing a condition for which this set of ordered pairs is the solution set.

14 For D10, use the instructions given for D9.

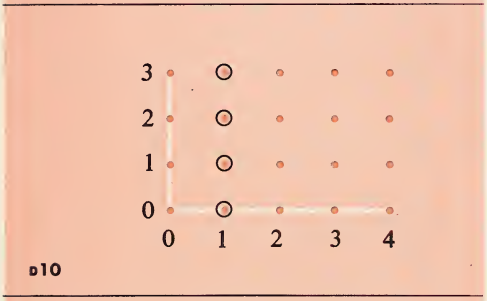
15 For D11, use the instructions given for D9.

16 For D12, use the instructions given for D9.

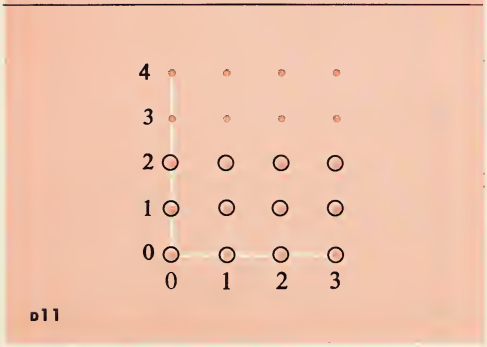
17 For D13, use the instructions given for D9.



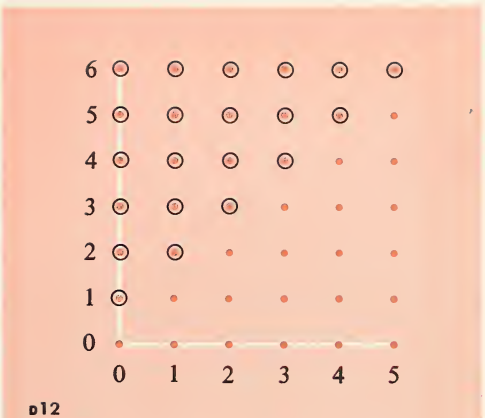
D9



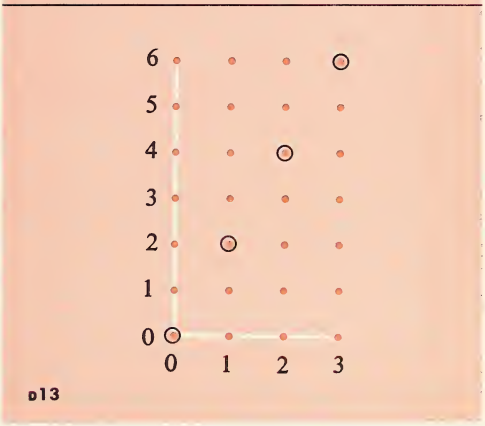
D10



D11



D12



D13

KEEPING SKILFUL

Find the difference.

1 7136, 5929
2 6851, 5983
3 31408, 8267
4 12040, 92
5 50000, 36105
6 42367, 38298

Multiply

7 726 by 38.
8 905 by 41.
9 826 by 600.
10 315 by 287.
11 604 by 190.
12 271 by 305.
13 7872 by 96.

Divide

14 6490 by 27.
15 14212 by 34.
16 12816 by 178.
17 21720 by 235.
18 35534 by 109.
19 108373 by 519.
20 62400 by 400.

The Cartesian set N × N

You have already used standard descriptions to express solution sets of conditions in one variable. In this lesson, you will use standard descriptions to express solution sets of conditions in two variables. You will also learn about the Cartesian set of the set of natural numbers, and you will use this set as a universe in making graphs of solution sets.

A Assume that the universe you are using is a set whose members are ordered pairs of natural numbers. Read the description of a solution set in D1. The braces are used instead of what words? The picture of a vertical line is used instead of what words?

B What part of the description expresses the condition that must be satisfied by each member of the solution set? Suppose the universe is for (x, y). Does the sentence $x > y$ also express this condition?

D2 shows how to read a standard description of the solution set of $x > y$.

C What condition is satisfied by each member of $\{(x, y) | x > y\}$? How do you know that $\{(x, y) | x > y\}$ is the solution set of $x > y$?

D Read each standard description in D3. What is the universe for the variables? What condition is satisfied by each member of A? Of B? Of C? Of D?

E Tabulate each of the sets described in D3.

F Read sentences A and B in D4. Does sentence A express a true statement? Does sentence B express a true statement? Explain your answers.

{all ordered pairs | the first component of each ordered pair is greater than the second component}

D1

"The set whose members are $\{(x, y) \mid x > y\}$ all ordered pairs x, y that satisfy the condition that x is greater than y"

D2

Universe for $(x, y) = \{(7, 7), (7, 2), (2, 7), (2, 2), (6, 7), (6, 2)\}$.

$A = \{(x, y) | x = y\}$.
$B = \{(x, y) | x = 2\}$.
$C = \{(x, y) | x > 8 - y\}$.
$D = \{(x, y) | y > 3\}$.

D3

Universe for $(x, y) = \{(7, 7), (7, 2), (2, 7), (2, 2), (6, 7), (6, 2)\}$.

A $\{(x, y) | x < y\} = \{(2, 7), (6, 7)\}$.
B $\{(x, y) | x = 5\} = \{\ \}$.

D4

Now you will learn more about making graphs when the universe is a set of ordered pairs.

A Suppose that the universe for (x, y) is the set of all ordered pairs such that both components of each ordered pair are natural numbers. Which of the following are members of the universe?

$(2, 3)$ $(\frac{1}{2}, 6)$ $(500, 3)$ $(7, 7)$

B Name the set that contains all the first components of the ordered pairs in the universe described in exercise A. Name the set that contains all the second components of the ordered pairs.

C Is $N \times N$ a name for the universe described in exercise A? Explain your answer.

D Is the Cartesian set $N \times N$ an infinite set? Explain your answer.

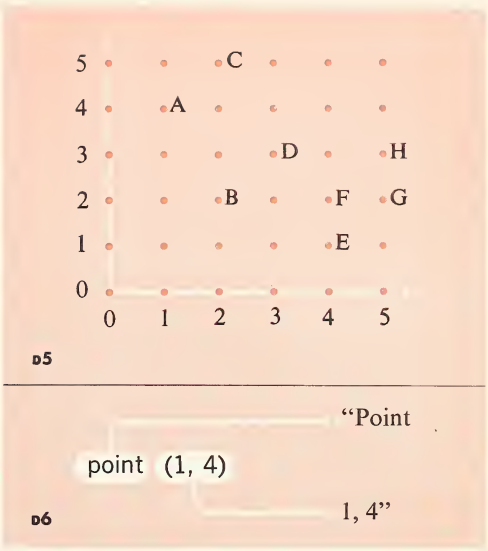
E In lesson 13 you learned how to make a graph of N. Now you will learn how to make a graph of $N \times N$. You will use natural-number lines as the first and second axes. In D5, what do the three small dots to the right of the picture of the first axis tell you? What do the three small dots above the picture of the second axis tell you?

F Point A, represented in D5, is associated with $(1, 4)$, which is a member of $N \times N$. D6 shows how to write and read another name for point A. What is another name for each of points B through H, represented in D5?

G Now think about these eight members of $N \times N$. Is each component of each member less than 6?

H The graph in D5 does not contain the dots that represent points $(6, 0)$, $(7, 0)$, and $(8, 0)$. Does the graph contain the dots that represent points $(0, 6)$, $(0, 7)$, and $(0, 8)$?

I Suppose the picture of the first axis were extended to contain the dot representing point 8.



Could you then use dots to locate points $(6, 0)$, $(7, 0)$, and $(8, 0)$?

J Also suppose the picture of the second axis were extended to contain the dot representing point 8. Could you then use dots to locate points $(0, 6)$, $(0, 7)$, and $(0, 8)$?

K How many members of $N \times N$ have 0 as a second component? Imagine that you could locate all the points that are associated with the members of $N \times N$ that have 0 as a second component. In which axis would these points be located?

L How many members of $N \times N$ have 0 as a first component? Imagine that you could locate all the points that are associated with the members of $N \times N$ that have 0 as a first component. Where would these points be located?

M How many members of $N \times N$ have 1 as a first component? Describe the location of all the points associated with these members.

N How many members of $N \times N$ have 9 as a second component? Describe the location of all the points associated with these members.

O Look at D7. Imagine that the pictures of both the first axis and the second axis have been extended to contain dots representing all points through 100. For each point named, tell where you would place its dot in the graph: (0, 10), (1, 17), (9, 3), (20, 4), (5, 93), (6, 75), (100, 5), (99, 8).

Mathematicians call a graph like the one in D7 a graph of the Cartesian set of the set of natural numbers. It is a graph of $N \times N$.

D7 shows just a few of the dots in the graph of $N \times N$. Since $N \times N$ is an infinite set, the graph of $N \times N$ is an infinite set of dots. To obtain more dots in the graph, more dots could be put in the pictures of the axes.

When we make a graph of $N \times N$, we will place three small dots at the end of the picture of each axis. These dots indicate that the graph of $N \times N$ is an infinite set.

P Does D8 also show a graph of $N \times N$? How do you know?

Now you will see how you can use a graph of $N \times N$ in making a graph of the solution set of a condition.

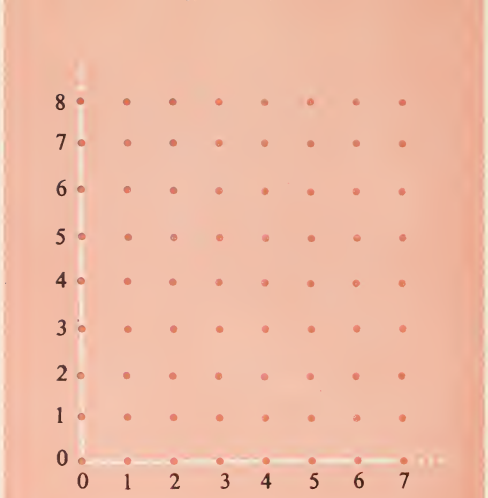
A Read the standard description in D8. Is the set described the solution set of $y = x$?

B Can you make a graph of $\{(x, y) | y = x\}$ by using a graph of the universe? How do you know?

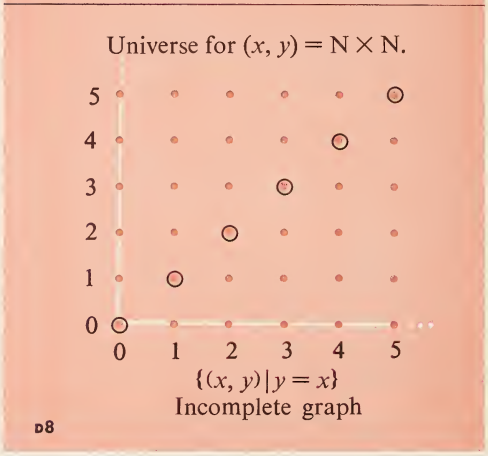
C Look at the encircled dots in the graph in D8. Name the ordered pairs associated with the points represented by these dots. Does each of these ordered pairs satisfy $y = x$?

D Is (98, 98) a solution of $y = x$? In the graph in D8, is there a dot that locates point (98, 98)? What would you have to do before you could place a dot to locate this point?

E Name five other members of $\{(x, y) | y = x\}$ whose points are not represented by dots in the graph in D8.



D7



D8

F The words "incomplete graph" tell you that the entire graph of the solution set of a condition is not shown. Why are the words "incomplete graph" used in D8? How do you know that the solution set of $y = x$ is an infinite set?

Now you will learn how to make graphs of the solution sets of $y > x$ and $y < x$. The universe for (x, y) is $N \times N$.

G Look at D9. The encircled dots are in the graph of $\{(x, y) | y = x\}$. Are any of these dots in the graph of $\{(x, y) | y > x\}$?

H Look at the dots that are below the encircled dots. Select any two of these dots and name the points they represent. Are the points you named associated with ordered pairs? Are these ordered pairs members of the solution set of $y > x$? Explain your answer.

I Should the graph of the solution set of $y > x$ contain any dots below the encircled dots?

J Look at the dots enclosed by squares. Are these dots in the graph of $\{(x, y) | y > x\}$? Explain your answer.

K Does (3, 100) satisfy $y > x$? Does (19, 47) satisfy $y > x$? How do you know that the solution set of $y > x$ is an infinite set?

L Does D9 include all possible dots in the graph of $\{(x, y) | y > x\}$? Why are the words "incomplete graph" used in D9?

M Study D10. How are the dots in the graph of the solution set of $y = x$ indicated? How are the dots in the graph of the solution set of $y < x$ indicated? How is the position of the dots in the graph of $\{(x, y) | y < x\}$ related to the position of the dots in the graph of $\{(x, y) | y = x\}$?

N Is the solution set of $y < x$ an infinite set? Why are the words "incomplete graph" used?

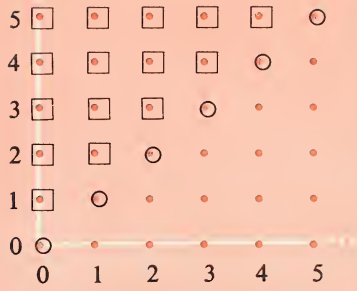
O Look at D11. Does the display show all the dots in the graph of $\{(x, y) | x + y = 4\}$? Is $\{(x, y) | x + y = 4\}$ an infinite set? A finite set?

P How could you use the graph of the solution set of $x + y = 4$ to make a graph of the solution set of $x + y < 4$? To make a graph of the solution set of $x + y > 4$?

Q Is the solution set of $x + y < 4$ a finite set? Is the solution set of $x + y > 4$ a finite set? Explain your answers.

Now you can use a standard description to express the solution set of a condition in two variables. You also can use a graph of N × N in making graphs of solution sets.

Universe for $(x, y) = N \times N$.

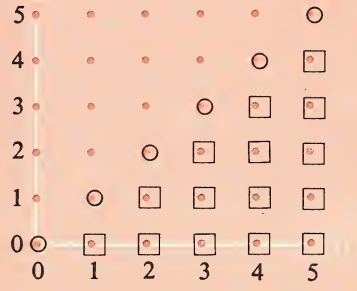


$\{(x, y) | y > x\}$

Incomplete graph

D9

Universe for $(x, y) = N \times N$.

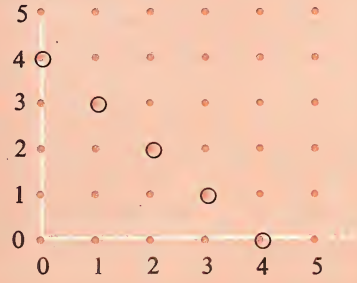


$\{(x, y) | y < x\}$

Incomplete graph

D10

Universe for $(x, y) = N \times N$.



$\{(x, y) | x + y = 4\}$

D11

On your own

Write a standard description of each set described in exercises 1 and 2.

1 $\{(x, y) |$ the first component of each ordered pair equals the result of subtracting 14 from the second component of that ordered pair$\}$

2 $\{(x, y) |$ the second component of each ordered pair equals the result of subtracting 14 from the first component of that ordered pair$\}$

3 Sets S and T are tabulated below. Tabulate $S \times T$.

$S = \{5, 6, 7, 8, 9, 10\}$.

$T = \{20, 21\}$.

The universe for (x, y) in exercises 4 through 9 is $S \times T$, which you tabulated for exercise 3. For each condition expressed, write a sentence that gives two names of the solution set. One name should be a standard description. The other name should be a tabulation of the solution set. For example, for exercise 4 you would write the following sentence:

$\{(x, y) | x = 27 - y\} = \{(7, 20), (6, 21)\}$.

4 $x = 27 - y$.

5 $x > 27 - y$.

6 $x < 27 - y$.

7 $y = 20$.

8 $y - 12 = x$.

9 $3x < y + 9$.

For each of exercises 10 through 17, follow these instructions: First make a graph of $N \times N$. Then make a graph of the solution set described in the exercise. Tell whether this set is finite or infinite. If the entire graph of the solution set is not shown, write the words "incomplete graph." The universe for (x, y) is $N \times N$.

10 $\{(x, y) | x = 4\}$

11 $\{(x, y) | x < 4\}$

12 $\{(x, y) | y = 4\}$

13 $\{(x, y) | y > 4\}$

14 $\{(x, y) | x + y = 6\}$

15 $\{(x, y) | x + y > 6\}$

16 $\{(x, y) | y = 2 + x\}$

17 $\{(x, y) | y > 2 + x\}$

18 How is the graph of $\{(x, y) | x < 4\}$ related to the graph of $\{(x, y) | x = 4\}$?

19 How is the graph of $\{(x, y) | x + y > 6\}$ related to the graph of $\{(x, y) | x + y = 6\}$?

KEEPING SKILFUL

For exercises 1 through 10, tell whether each sentence expresses a condition or a statement.

1 $29 < 600 \div 24$.

2 $n = 861 + 79$.

3 $34 - s > 18$.

4 $6(15) > 6(14)$.

5 $3 + 1 = 7 - 3$.

6 $d = 19 + 32$.

7 $8 + 6 = 2(7)$.

8 $w > 24 + 9$.

9 $15 - 6 < 3 + 6$.

10 $m > 3m$.

11 Which sentences in exercises 1 through 10 express true statements? Which express false statements?

12 Tabulate the solution set of each condition expressed in exercises 1 through 10. $U = N$.

13 Set R and set S are tabulated below. Tabulate $R \times S$.

$R = \{0, 1, 2, 3\}$.

$S = \{0, 1, 2\}$.

For each condition expressed in exercises 14 through 21, the universe for (f, g) is set $R \times S$, which you tabulated for exercise 13. First tabulate the solution set of each condition. Then, for each condition, make a graph of the universe and of the solution set.

14 $f + g = 4$.

15 $f + g < 3$.

16 $f + 2 < 4$.

17 $f + g > 2$.

18 $7 - f > 5$.

19 $f + 1 = g$.

20 $f = g$.

21 $3 - g = f$.

For each of exercises 22 through 26, first tabulate sets K and L. Next tabulate $K \cap L$. Then tabulate $K \cup L$. $U = N$.

22 $K = \{c | 39 - c > 27\}$.
$L = \{c | 42 + c < 51\}$.

23 $K = \{v | 2103 - v = 1926\}$.
$L = \{v | v + 624 < 631\}$.

24 $K = \{d | d + 76 > 92\}$.
$L = \{d | 121 + d < 140\}$.

25 $K = \{b | b - 95 > 17\}$.
$L = \{b | 739 + b > 766\}$.

26 $K = \{m | 304 - m = 268\}$.
$L = \{m | m + 435 = 471\}$.

Charts and solution sets

You have learned to find solution sets of conditions in two variables by making replacements for each of the variables, and deciding whether or not each statement that you obtained was true. You have also learned how to make graphs of universes and of solution sets of conditions. Now you will find solution sets by first replacing one of the variables in the condition and then determining the replacement for the other variable.

A Look at D1. What condition is expressed? What is the universe for (x, y)? Is $A \times A$ a set of ordered pairs? Tabulate $A \times A$.

B Is $A \times A$ an infinite set? Is it a finite set?

C By completing the chart in D2, you will learn a convenient way to obtain the solution set of $y = 5 - x$. The ordered pairs named in the completed chart will be solutions of the condition. Is each number named in the row opposite the letter x the first component of at least one member of $A \times A$? Are these numbers the only numbers that can be used as replacements for x in $y = 5 - x$? How do you know?

D Replace x in $y = 5 - x$ by 0. What replacement for y satisfies $y = 5 - 0$?

E Look again at the chart in D2. Why is the numeral 5 placed in the row opposite the letter y and directly below the numeral 0?

F Is (0, 5) a member of the solution set of $y = 5 - x$? How do you know?

G Copy the chart in D2. Use each of the other numbers named in the row opposite the letter x as a replacement for x in $y = 5 - x$. For each condition you obtain, find the replacement for y that satisfies the condition. Complete the chart by writing the name of each replacement for y.

H What are the six ordered pairs that you can obtain from your completed chart? Is each of these ordered pairs a member of $A \times A$? Are there any other members of $A \times A$ that satisfy $y = 5 - x$?

I Tabulate $\{(x, y) \mid y = 5 - x\}$.

Notice that each first component of the members of $A \times A$ is a first component of a solution of $y = 5 - x$. Now you will see that each first component of the members of the universe is not always a first component of a solution.

J Look at D3. What is the universe for (x, y)? Is $B \times B$ an infinite set? Is it a finite set?

D1

Universe for $(x, y) = A \times A$.
$A = \{0, 1, 2, 3, 4, 5\}$.
$y = 5 - x$.

D2

Universe for $(x, y) = A \times A$.
$y = 5 - x$.

x	0	1	2	3	4	5
y	5					

D3

Universe for $(x, y) = B \times B$.
$B = \{0, 1, 2, 3, 4, 5, 6\}$.
$x + y = 4$.

x	0	1	2	3	4
y					

K Replace x in $x + y = 4$ by 5. Do you obtain $5 + y = 4$? Is there a member of B that satisfies $5 + y = 4$?

L Why do you not use either 5 or 6 as a replacement for x in $x + y = 4$?

M Look at the chart in D3. Why are the numbers named in the row opposite the letter x the only numbers that you use as replacements for x in $x + y = 4$?

N Copy the chart in D3. Use each number named in the top row as a replacement for x in $x + y = 4$. For each condition you obtain, find the replacement for y that satisfies the condition. Then complete the chart.

O How many ordered pairs can you obtain from the chart? Is each of these ordered pairs a member of $B \times B$? Are there any other members of $B \times B$ that satisfy $x + y = 4$?

P Tabulate $\{(x, y) | x + y = 4\}$.

Q Make a graph of $\{(x, y) | x + y = 4\}$.

You have used charts to help you determine the solutions of conditions for equality. Now you will use a chart to help you determine the solutions of a condition for inequality.

A What condition for inequality is expressed in D4? What is the universe for (x, y)?

B Use 4 as a replacement for x in $x + y < 4$. What condition do you obtain? Are there any members of B that satisfy $4 + y < 4$?

C Why do you not use a number greater than 3 as a replacement for x in $x + y < 4$?

D Replace x in $x + y < 4$ by 0. What condition do you obtain? Is 0 the first component of more than one ordered pair that satisfies $x + y < 4$? Explain your answer.

E What members of B satisfy $0 + y < 4$? Where are these members named in the chart in D4?

F Now replace x by 1. Does any member of B that is greater than 2 satisfy $1 + y < 4$?

Universe for $(x, y) = B \times B$.
$B = \{0, 1, 2, 3, 4, 5, 6\}$.
$x + y = 4$.

x	0	1	2	3	4
y					

D3

Universe for $(x, y) = B \times B$.
$B = \{0, 1, 2, 3, 4, 5, 6\}$.
$x + y < 4$.

x	0	1	2	3
y	0			
	1			
	2			
	3			

D4

G What members of B satisfy $1 + y < 4$? Where will you write the names of these members in your chart?

H Copy and complete the chart in D4.

I From the chart, you obtain ten ordered pairs that satisfy $x + y < 4$. What universe does each of these ordered pairs belong to?

J Tabulate $\{(x, y) | x + y < 4\}$.

K Make a graph of $\{(x, y) | x + y < 4\}$.

So far in this lesson, each universe you have used has been a finite Cartesian set. In the following exercises, you will use the Cartesian set $N \times N$ as the universe.

A What condition is expressed in D5? What numbers can be used as replacements for r in $s = 2r + 1$?

B Is the set of numbers that can be used as replacements for r an infinite set?

C When you replace r by 50, you obtain $s = 2(50) + 1$. What member of N satisfies this condition?

You know that each member of N can be used as a replacement for r. Since it is impossible to put the name of each number that can be used as a replacement for r in a chart, use only the names of the numbers from 0 through 4 in your chart.

D Make and complete a chart for $s = 2r + 1$.

E Tabulate the set of ordered pairs that you obtain from your chart. Is each member of this set a solution of $s = 2r + 1$?

F Are the members of the set you tabulated for exercise E the only members of $\{(r, s) | s = 2r + 1\}$? Explain your answer. Remember that the universe for (r, s) is $N \times N$.

G What condition is expressed in D6?

H Make a chart that names some of the solutions of $s < 2r + 1$. Use only the numbers from 0 through 4 as replacements for r.

I Tabulate the set of ordered pairs that you obtain from your chart. Is this set the solution set of $s < 2r + 1$? Explain your answer.

J Is $\{(r, s) | s < 2r + 1\}$ a finite set? Is it an infinite set?

K Make a graph of $N \times N$ that contains all the dots that represent points associated with the ordered pairs you tabulated for both exercises E and I. Encircle the dots in the graph of $\{(r, s) | s = 2r + 1\}$. Use squares to enclose the dots in the graph of $\{(r, s) | s < 2r + 1\}$.

L What condition is expressed in D7? Make a chart to help you determine some of the members of $\{(r, s) | s > 2r + 1\}$. Use the numbers from 0 through 4 as replacements for r. Do not find any members of $\{(r, s) | s > 2r + 1\}$ in which the second components are greater than 10.

M Are the ordered pairs that you obtain from your chart the only ordered pairs that satisfy $s > 2r + 1$?

Now you know how to make and use charts to help you determine the solutions of conditions in two variables.

On your own

For each condition expressed in exercises 1 through 6, the universe for (x, y) is $D \times D$. $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. For each condition, make a chart to help you determine the members of the solution set. Then tabulate the solution set. As replacements for x, use only those members of D that are first components of members of the solution set. For exercise 2, for example, you should not use 5 as a replacement for x because $3(5) = 15$, and 15 is not in the universe for y. There is no member of D that satisfies $3(5) = y$.

1 $y = 8 - x$.
2 $3x = y$.
3 $y = x + 1$.
4 $y + x < 3$.
5 $x + 4 = y$.
6 $y = x + 5$.

For each condition expressed in exercises 7 through 11, the universe for (r, s) is $N \times N$. Each chart names a few of the possible replace-

Universe for $(r, s) = N \times N$.
$s = 2r + 1$.

D5

Universe for $(r, s) = N \times N$.
$s < 2r + 1$.

D6

Universe for $(r, s) = N \times N$.
$s > 2r + 1$.

D7

ments for one of the variables. Copy each chart and write in the missing numerals.

7

$s + 17 = r.$						
r						
s	0	1	2	25	93	104

8

$s = 4r + 2.$									
r	0	3	4	6	12	17	95	118	213
s									

9

$4r - 2 > s.$				
r	1	2	3	4
s				

10

$3s + r = 35.$								
r								
s	1	2	4	6	8	9	10	11

11

$r + s < 8.$				
r	0	2	4	6
s				

For each condition expressed in exercises 12 through 17, the universe for (x, y) is $N \times N$. For each exercise, make a chart to help you determine some of the members of the solution set. Use the numbers from 0 through 6 as replacements for x. Tabulate the set of ordered pairs that you obtain from your chart.

12 $x = y - 7$.
13 $y = x + 2$.
14 $2x + 3 = y$.
15 $7 - x > y$.
16 $21 - 3x = y$.
17 $y - 2 = x$.

For each of exercises 12, 15, and 17, make a graph of the solution set of the condition expressed in the exercise. Encircle only the dots that represent points associated with the ordered pairs you can obtain from your charts.

KEEPING SKILFUL

For exercises 1 through 6, round each of the numbers named to the next greater thousand, to the next lesser hundred, and to the next greater ten.

1 3276
2 4721
3 6892
4 19,929
5 36,993
6 5550

Find the sum.

7 368, 705, 296
8 4823, 1960, 382
9 6510, 7396, 982, 4968
10 7727, 88, 9276, 338
11 36805, 72412, 65120
12 210, 87654, 2341, 72
13 16315, 7024, 62657, 9824

Multiply

14 324 by 65.
15 805 by 94.
16 214 by 465.
17 617 by 840.
18 809 by 321.
19 354 by 605.

Divide

20 3668 by 57.
21 9360 by 36.
22 18098 by 94.
23 11890 by 245.
24 21384 by 132.
25 26546 by 241.

For each of exercises 26 through 33, tabulate the solution set of the condition. U = N.

26 $87 + c < 95 \wedge 103 - c > 98$.
27 $w > 288 \div 24 \wedge w + 34 < 53$.
28 $2463 - f = 1987$.
29 $32 - d > 23 \wedge d + 24 > 30$.
30 $s < 34(6) \wedge 27 - s > 9$.
31 $m + m > 8$.
32 $n - 14 > 16 \wedge 47 - n > 14$.
33 $b + 324 > 419 \wedge b > 59 + 67$.

For each of exercises 34 through 39, tabulate the set named. Sets J, K, and L are tabulated below.

$J = \{(0, 1)\ (1, 2),\ (2, 3),\ (3, 4)\}$.
$K = \{(3, 4)\}$.
$L = \{(1, 0),\ (1, 2),\ (3, 2),\ (2, 3)\}$.

34 $L \cup J$
35 $K \cap L$
36 $K \cap J$
37 $J \cup K$
38 $J \cap L$
39 $K \cap K$

Compound conditions in two variables

You have studied compound conditions in one variable. In this lesson you will study compound conditions in two variables and find the solution sets of these conditions.

A Look at D1. What is the universe for (x, y)?

B Does the open sentence in D1 express a compound condition? Explain your answer. How many variables are there in this condition?

C How do you know that the solution set of $x + y = 6 \wedge y = 2x$ is a set of ordered pairs?

D How do you know that each solution of the compound condition must satisfy both $x + y = 6$ and $y = 2x$?

> REMINDER
> Each solution of a compound condition that has two simple conditions and the connective "and" must satisfy both simple conditions.
> *See lesson 24, page 90.*

First you will use a graph of the solution set of $x + y = 6$ and a graph of the solution set of $y = 2x$ to find a graph of $\{(x, y) | x + y = 6 \wedge y = 2x\}$.

E Look at D2. How are the dots in the graph of $\{(x, y) | x + y = 6\}$ indicated? How are the dots in the graph of $\{(x, y) | y = 2x\}$ indicated?

F Which dot is in the graph of both the solution set of $x + y = 6$ and the solution set of $y = 2x$? What ordered pair is associated with the point represented by this dot?

G Does $(2, 4)$ satisfy both $x + y = 6$ and $y = 2x$? Is $(2, 4)$ the only ordered pair that satisfies both $x + y = 6$ and $y = 2x$?

H Tabulate $\{(x, y) | x + y = 6 \wedge y = 2x\}$.

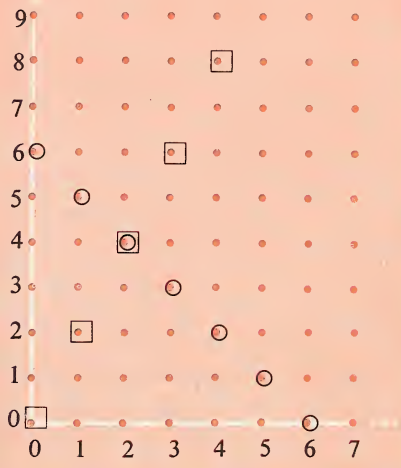
Now you will study the intersection of the solution sets of two simple conditions in two variables.

> REMINDER
> The intersection of two sets is the set that contains only those objects that belong to both sets.
> *See lesson 22, page 83.*

Universe for $(x, y) = \mathrm{N} \times \mathrm{N}$.
$x + y = 6 \wedge y = 2x$.

D1

Universe for $(x, y) = \mathrm{N} \times \mathrm{N}$.



$\{(x, y) | x + y = 6 \wedge y = 2x\}$

D2

I Tabulate the intersection of the solution sets of $x + y = 6$ and $y = 2x$. Is this the same set as the set you tabulated for exercise H?

J Does the following sentence express a true statement? $\{(x, y) | x + y = 6 \wedge y = 2x\} = \{(x, y) | x + y = 6\} \cap \{(x, y) | y = 2x\}$.

Next you will find the solution set of another compound condition in two variables. This time you will use a chart to help you.

K For the condition expressed in D3, what is the universe for (x, y)?

L Why do you first find the solution set of $x + y < 7$ instead of the solution set of $y > x$? Why do you not use a number greater than 6 as a replacement for x in $x + y < 7$?

M A chart like the one begun in D4 will help you determine the solution set of $x + y < 7$. Copy and complete this chart.

N Do the ordered pairs that you obtain from the chart form the solution set of $x + y < 7$?

You can use your chart to select the members of $\{(x, y) | x + y < 7\}$ that satisfy $y > x$.

O What does the condition $y > x$ require? Which members of $\{(x, y) | x + y < 7\}$ satisfy $y > x$? Does each of these ordered pairs satisfy both $x + y < 7$ and $y > x$?

P Tabulate $\{(x, y) | x + y < 7 \wedge y > x\}$. Is this set a finite set?

Q Why is $\{(x, y) | x + y < 7\} \cap \{(x, y) | y > x\}$ the same set as the set you tabulated for exercise P?

Sometimes one of the simple conditions in a compound condition has only one variable. Exercises A through M will help you find the solution set of such a compound condition.

A Look at D5. What is the universe for (x, y)?

B Which simple condition in $x + y < 8 \wedge x + 1 = 4$ contains only one variable? Why do you first find the solution set of $x + 1 = 4$ instead of the solution set of $x + y < 8$?

Universe for $(x, y) = \mathrm{N} \times \mathrm{N}$.
$x + y < 7 \wedge y > x$.

D3

Universe for $(x, y) = \mathrm{N} \times \mathrm{N}$.
$x + y < 7$.

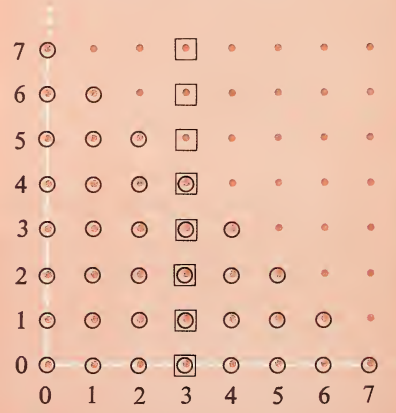
x	0	1	2	3	4	5	6
y	0						
	1						
	2						
	3						
	4						
	5						
	6						

D4

Universe for $(x, y) = \mathrm{N} \times \mathrm{N}$.
$x + y < 8 \wedge x + 1 = 4$.

D5

Universe for $(x, y) = \mathrm{N} \times \mathrm{N}$.



$\{(x, y) | x + y < 8 \wedge x + 1 = 4\}$

D6

C Is $x + 1 = 4$ satisfied by each ordered pair in $N \times N$ whose first component is 3? Is the solution set of $x + 1 = 4$ an infinite set?

Next you must decide which ordered pairs that satisfy $x + 1 = 4$ also satisfy $x + y < 8$.

D Replace x in $x + y < 8$ by 3. Which members of N satisfy $3 + y < 8$?

E Which ordered pairs in $N \times N$ have a first component of 3 and a second component that satisfies $3 + y < 8$? Are these the only ordered pairs in $N \times N$ that satisfy the compound condition?

F Tabulate $\{(x, y) | x + y < 8 \wedge x + 1 = 4\}$. Is this the same set as $\{(x, y) | x + y < 8\} \cap \{(x, y) | x + 1 = 4\}$?

G The graphs of $\{(x, y) | x + y < 8\}$ and $\{(x, y) | x + 1 = 4\}$ are shown in D6. In which graph are the dots encircled? Is this graph incomplete? In which graph are the dots enclosed by squares? Is this graph incomplete?

H How is the graph of $\{(x, y) | x + y < 8 \wedge x + 1 = 4\}$ indicated? Is this graph incomplete?

I Look at D7. Which simple condition in $y + 2 > x \wedge y + 1 < 7$ has only one variable?

J Is $y + 1 < 7$ satisfied by each ordered pair in $N \times N$ whose second component is a number from 0 through 5?

K You can use a chart like the one begun in D8 to help you decide which ordered pairs that satisfy $y + 1 < 7$ also satisfy $y + 2 > x$. Why does the chart first show the names of the replacements for y, instead of for x? Why do you use only the numbers from 0 through 5 as replacements for y? Copy and complete the chart.

L Does each ordered pair that you obtain from your chart satisfy $y + 2 > x \wedge y + 1 < 7$?

M Tabulate $\{(x, y) | y + 2 > x \wedge y + 1 < 7\}$.

So far in this lesson the solution sets of the compound conditions have had one or more members, but they have not been infinite sets. The solution set of a compound condition may also be an infinite set or the empty set.

N Look at D9. What does the first simple condition, $x + y < 6$, require? What does the second simple condition, $x + y = 7$, require?

O Is it possible for the sum of two numbers to be both less than 6 and equal to 7?

P Tabulate $\{(x, y) | x + y < 6 \wedge x + y = 7\}$.

Q Look at D10. Does each ordered pair whose first component is the same as its second component satisfy $x = y$? Is $\{(x, y) | x = y\}$ an infinite set?

Universe for $(x, y) = N \times N$.
$y + 2 > x \wedge y + 1 < 7$.

D7

Universe for $(x, y) = N \times N$.
$y + 2 > x$.

y	0	1	2	3	4	5
x						

D8

Universe for $(x, y) = N \times N$.
$x + y < 6 \wedge x + y = 7$.

D9

Universe for $(x, y) = N \times N$.
$x = y \wedge x + y > 1$.

D10

R Does (0, 0) satisfy $x + y > 1$? Does (1, 1) satisfy $x + y > 1$?

S Is $x = y \wedge x + y > 1$ satisfied by each ordered pair whose components are the same and also greater than 0?

T Is $\{(x, y) | x = y \wedge x + y > 1\}$ an infinite set?

Now you know how to use charts and graphs to help determine solution sets of compound conditions in two variables.

On your own

For each of exercises 1 through 4, make a graph of N × N and graphs of the solution sets of both simple conditions. Include only as many dots as you need to determine whether the solution set of the compound condition is a finite or an infinite set. If the solution set is infinite, label your graph "incomplete graph." If the solution set is a finite set, tabulate it. The universe for $(x, y) = N \times N$.

1 $x + y = 5 \wedge y = x + 1$.

2 $x > 4 \wedge y > 3$.

3 $x + y > 3 \wedge x + 3 = y$.

4 $x + y < 4 \wedge x + y > 1$.

For each of the compound conditions expressed in exercises 5 through 17, tabulate the solution set. The universe for $(x, y) = N \times N$.

5 $x + y < 5 \wedge 2x + y = 7$.

6 $x + y = 325 \wedge x + 42 = 205$.

7 $125 - x = y \wedge y - 25 = 52$.

8 $2x - 16 = y \wedge x - 1 = 7$.

9 $x - y = 17 \wedge x - 12 = 9$.

10 $2x + y < 5 \wedge x + 1 < 8$.

11 $y - x = 751 \wedge y + 15 = 801$.

12 $x = 13 + y \wedge y < 7$.

13 $x - y < 6 \wedge x + 3 = 39$.

14 $x + y < 8 \wedge y > 2x$.

15 $x + y = 16 \wedge 2x = y$.

16 $140 - x = y \wedge x + 3 = 21$.

17 $x - y = 32 \wedge y + 4 < 7$.

CHECKING UP

Use lessons 30 through 36 if you need to restudy the ideas expressed in these tests.

Test 34

Using sets A and B, tabulated below, tabulate each set named in exercises 1 through 6.
$A = \{2, 4, 7\}$. $B = \{3, 4\}$.

1 $A \times B$ 2 $B \times A$ 3 $A \times A$ 4 $B \times B$

5 $\{(x, y) | 2y > x + 3\}$. The universe for (x, y) is $A \times B$.

6 $\{(x, y) | x < y\}$. The universe for (x, y) is $B \times A$.

Test 35

7 The encircled dots in the display below form a graph of a set of ordered pairs. Tabulate this set.

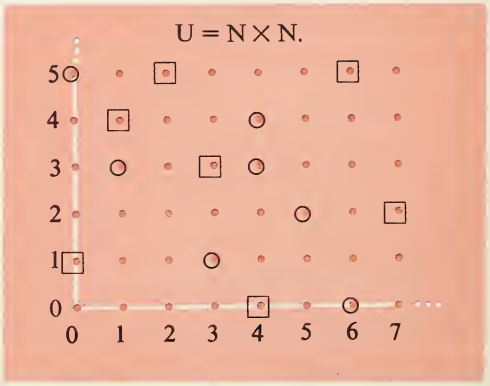
8 The dots enclosed by squares also form a graph of a set of ordered pairs. Tabulate this set.

For exercises 9, 10, and 11, the universe for $(x, y) = N \times N$.

9 Make a chart to help you determine the members of the solution set of $7 - x = y \wedge x < 8$.

10 Tabulate $\{(x, y) | 7 - x = y \wedge x < 8\}$.

11 Make a graph of N × N. Encircle the dots in the graph of $\{(x, y) | 7 - x = y \wedge x < 8\}$.



Simple conditions for problems

You have learned how to find solution sets of conditions in two variables. In this lesson you will learn how to solve problems by using simple conditions in two variables.

Read the problem in D1. First you will develop the condition for the problem. $U = N \times N$.

A Do you know how many fish George caught? Use x as a variable for the number of fish George could have caught.

x

B Do you know how many fish Steve caught? Use y as a variable for the number of fish Steve could have caught.

y

C Why is $x + y$, and not $x - y$, used in the condition? What does $x + y$ refer to?

$x + y$

D Why is the idea of "less than" used in the condition? How many variables are there in the condition?

$x + y < 5$.

E Now you can find the solution set of the condition. What is the universe for (x, y)? Will the solution set be a set of ordered pairs?

F How do you know that George caught fewer than 5 fish? What numbers do you use as replacements for x in $x + y < 5$?

G Look at the chart in D2. The numbers named opposite the letter x are the numbers you use as replacements for x in $x + y < 5$. Why do you not use 5 as a replacement for x?

H Copy and complete the chart in D2.

D1

George and Steve went fishing. Together, they caught fewer than 5 fish. How many fish could each of the boys have caught?

D2

Universe for $(x, y) = N \times N$.
$x + y < 5$.

x	0	1	2	3	4
y					

I Tabulate $\{(x, y) | x + y < 5\}$.

J What does the first component of each member of the solution set refer to? What does the second component of each member refer to?

Since you know the ordered pairs that satisfy the condition for the problem in D1, you can now answer the question asked in the problem. You use the solutions of $x + y < 5$ to get the answers to the problem.

If George caught 0 fish, Steve could have caught from 0 through 4 fish.

If George caught 1 fish, Steve could have caught from 0 through 3 fish.

K What are the other answers to the problem in D1?

L You can verify the answer you obtain from (0, 0) by answering the following question: If George caught 0 fish and Steve caught 0 fish, could George and Steve together have caught fewer than 5 fish?

M Verify the answers you obtain from (1, 3), (2, 2), (3, 1), and (4, 0).

Read the problem in D3. You will first develop the condition for the problem. $U = C \times C$. Remember that $C = \{1, 2, 3, \ldots\}$. The ordered pairs that are members of $C \times C$ do not have 0 as a first or a second component.

A Is 8 the number of stamps that Judy bought? 8

B Do you know how many stamps Judy used? Use m as a variable for the number of stamps she could have used. m

C Why is $8 - m$, and not $8 + m$, used in the condition? $8 - m$

D Do you know how many stamps Judy had left? Use n as a variable for the number of stamps that she could have had left. n

E Why is the idea of equality used in the condition? $8 - m = n$.

F Now you can find the solution set of $8 - m = n$. What is the universe for (m, n)?

G What is the greatest number of postage stamps Judy could have used? What is the least number of postage stamps Judy could have used?

H Why do you not use 0 as a replacement for m in the condition? Why do you not use 8 as a replacement for m?

I What numbers do you use as replacements for m in $8 - m = n$?

J Copy and complete the chart in D4.

K Tabulate $\{(m, n) \mid 8 - m = n\}$.

L What does the first component of each member of the solution set refer to? What does the second component of each member refer to?

M What are the answers to the problem in D3? Remember to use each solution of $8 - m = n$ to get an answer.

Judy bought 8 postage stamps. She used some, but not all, of these stamps. How many stamps could she have used, and how many stamps could she have had left?

D3

Universe for $(m, n) = C \times C$.
$8 - m = n$.

m	1	2	3	4	5	6	7
n							

D4

Now you know how to solve problems by using simple conditions in two variables.

On your own

For each problem, first write a sentence that expresses the condition for the problem. Use whatever letters you wish as names for the variables. Tell what each variable refers to in the problem. Then make a chart to help you determine the ordered pairs that satisfy the condition. Finally, tabulate the solution set and answer the problem. The universe is given for each problem.

1 Dave and Ted built 5 model boats. How many boats could each of the boys have built? $U = N \times N$.

2 Ernie had 9 fishhooks. He gave some, but not all, of them to Tom. He kept the others for himself. How many fishhooks could he have given to Tom, and how many could he have kept for himself? $U = C \times C$.

3 Martha bought fewer than 6 lemons and limes. How many lemons and how many limes could she have bought? $U = C \times C$.

4 Mr. Phillips picked 4 bushels of apples. He sold some, but not all, of the 4 bushels. How

many bushels of apples could he have sold, and how many could he have had left? $U = C \times C$.

5 Together, Louise and Connie made fewer than 7 posters. How many posters could each of the girls have made? $U = C \times C$.

6 Jane had 6 tickets to sell. She sold as many of them as she could and returned the others. How many tickets could she have sold, and how many tickets could she have returned? $U = N \times N$.

7 During the summer, Larry works at a grocery store on Fridays and Saturdays. He works a total of 8 hours during these two days. How many hours can he work on each of these days? $U = C \times C$.

8 Mrs. Ray took 7 pies to a bake sale. Some, but not all, of the pies were cherry, and the others were lemon. How many pies of each kind could she have taken to the bake sale? $U = C \times C$.

9 Harold had 10¢. He spent some, but not all, of this money. How much money could he have spent, and how much could he have had left? $U = C \times C$.

Compound conditions for problems

You have solved problems by using simple conditions in two variables. In this lesson you will learn to solve problems by using compound conditions in two variables.

Read the problem in D1. The exercises that follow will help you develop the compound condition for the problem. $U = N \times N$.

Finding answers to problems involving compound conditions in which one simple condition has two variables and the other condition has one variable

Together, Helen and Sue have 22 phonograph records. If Helen had 3 more, she would have 12 records. How many records does each of the girls have?

D1

A Do you know how many records Helen has? Use r as a variable for the number of records Helen has.

r

B Do you know how many records Sue has? Use s as a variable for the number of records Sue has.

s

C Why is $r + s$, and not $r - s$, used in the condition? What does $r + s$ refer to?

$r + s$

D Why is the idea of equality used in the condition? Does the sentence at the right above express one of the simple conditions for the problem?

$r + s = 22.$

E Why is $r + 3$, and not $s + 3$, used in the other simple condition?

$r + 3$

F Why is the idea of equality used in the condition? Does the sentence at the right above express the other simple condition for the problem?

$r + 3 = 12.$

G Does the sentence below express the compound condition for the problem?

$$r + s = 22 \wedge r + 3 = 12.$$

Now you can find the solution set of the compound condition. Remember that each member of this set must satisfy each of the simple conditions.

H What is the universe for (r, s)?

I What replacement for r satisfies $r + 3 = 12$?

J Is 9 the first component of each solution of $r + s = 22 \wedge r + 3 = 12$? How do you know?

K Use 9 as a replacement for r in $r + s = 22$. What number satisfies $9 + s = 22$?

L Tabulate $\{(r, s) | r + s = 22 \wedge r + 3 = 12\}$.

M To what does the first component of the solution of the compound condition refer? To what does the second component refer?

Since you know the solution of the compound condition for the problem, you can use this solution to get the answer to the problem.

Helen has 9 records, and Sue has 13 records.

N You can verify this answer by answering the following questions: If Helen has 9 records and Sue has 13 records, do Helen and Sue together have 22 records? If Helen had 3 records more than the 9 she has, would she have 12 records?

Read the problem in D2. First you will develop the compound condition for the problem. $U = C \times C$.

A Do you know Karen's age now? Use x as a variable for the number of years Karen's age is now. x

B Do you know Jean's age now? Use y as a variable for the number of years Jean's age is now. y

C Why is the idea of "greater than" used in the condition? Does the sentence at the right express one of the simple conditions for the problem? $x > y$.

D Why is $x - 5$, and not $x + 5$, used in the other simple condition? What does $x - 5$ refer to? $x - 5$

E Why is the idea of equality used in the condition? Does the sentence at the right below express the other simple condition for the problem? $x - 5 = 9$.

F Does the sentence at the right express the compound condition for the problem? $x > y \wedge x - 5 = 9$.

Karen is older than Jean. Five years ago Karen was 9 years old. How old is Karen now? How old can Jean be now?

D2

On Saturday, Tom and Ed painted fewer than 24 screens. Tom painted 7 screens in the morning and 6 screens in the afternoon. How many screens could Ed have painted on Saturday?

D3

G Now you can find the solution set of the compound condition. What is the universe for (x, y)?

H What replacement for x satisfies $x - 5 = 9$?

I How do you know that 14 is the first component of each solution of $x > y \wedge x - 5 = 9$?

J Use 14 as a replacement for x in $x > y$. The second components of the ordered pairs you are to find must satisfy $14 > y$. What members of C satisfy $14 > y$?

K Tabulate $\{(x, y) | x > y \wedge x - 5 = 9\}$.

L To what does the first component of each solution of the compound condition refer? To what does the second component of each solution refer?

You can use the solutions of the compound condition to get the answers to the problem.

Karen is 14 years old, and Jean can be from 1 through 13 years old.

Read the problem in D3. You will develop the condition for this problem. $U = C \times C$.

A Use w as a variable for the number of screens that Tom painted. Use z as a variable for the number of screens that Ed could have painted. What is the universe for (w, z)?

B You know that Tom and Ed painted fewer than 24 screens on Saturday. Is $w + z < 24$ one of the simple conditions for the problem?

C You also know that Tom painted 7 screens in the morning and 6 screens in the afternoon. Is $w = 7 + 6$ also a simple condition for the problem?

D What is the compound condition for the problem in D3?

E What replacement for w satisfies $w = 7 + 6$?

F What is the first component of each member of $\{(w, z) | w + z < 24 \wedge w = 7 + 6\}$?

G Use 13 as a replacement for w in $w + z < 24$. The second components of the ordered pairs you are to find must satisfy $13 + z < 24$. What members of C satisfy $13 + z < 24$?

H Tabulate $\{(w, z) | w + z < 24 \wedge w = 7 + 6\}$.

I To what does the first component of each solution of the compound condition refer? To what does the second component of each solution refer?

J Read the problem in D3 again. Notice that you are asked to find only the number of screens that Ed could have painted. Which components of the solutions of the compound condition do you use to get the answers to the problem?

Now you can answer the question asked in the problem.

Ed could have painted from 1 through 10 screens on Saturday.

Now you know how to solve problems by using compound conditions in two variables.

On your own

For each problem, first write a sentence that expresses the compound condition for the problem. Use whatever letters you wish as names for the variables. Tell what each variable refers to. Then tabulate the solution set of the compound condition and use the solutions to get the answers to the problem. The universe is given for each problem.

1 Mrs. King paid \$455 for a new stove and freezer. If she had paid \$49 less for the stove, she would have paid \$139 for it. How many dollars did she pay for each of the appliances? $U = N \times N$.

2 Together, Larry and Tom have fewer than 15 fossils. Tom has more than 10. How many fossils can each of the boys have? $U = C \times C$.

3 Jane is older than Helen. In 9 years, Jane will be 21 years old. How old is Jane now? How old can Helen be now? $U = C \times C$.

4 During a football game, Jay scored more points than Don. Jay scored 6 points during the first half of the game and 7 points during the second half. How many points could Don have scored during the game? $U = N \times N$.

5 Betty spent 11 days in Alberta and Saskatchewan. She spent more than 6 days in Alberta. How many days could she have spent in each of these provinces? $U = C \times C$

6 Mrs. Ross made 6 more aprons than potholders. After she gave 8 of the aprons away, she had 5 aprons left. How many aprons and how many potholders did she make? $U = N \times N$.

7 Fewer than 8 students received a grade of A on an English examination. More than 3 boys received a grade of A. How many boys and how many girls could have received a grade of A on this examination? $U = N \times N$.

8 Together, Sandy and Ruth made 48 sandwiches. Ruth made 9 peanut butter sandwiches and 13 cheese sandwiches. Sandy made the other sandwiches. How many did Sandy make? $U = N \times N$.

9 In two days Beth read a 150-page book. The second day she read fewer than 7 pages. How many pages could she have read on each of the two days? $U = C \times C$.

10 Lake Michigan is 307 miles long. Lake Michigan is 101 miles longer than Lake Huron. Lake Huron is 144 miles shorter than Lake Superior. What is the length of Lake Huron? Of Lake Superior? $U = N \times N$.

11 One year American tourists spent a total of $2,640,000,000 in the foreign countries that they visited. They spent $380,000,000 in Canada and $365,000,000 in Mexico. How many dollars were spent by American tourists in countries other than Canada and Mexico? $U = N \times N$.

KEEPING SKILFUL

Make one sketch for exercises 1 through 8.

1 Make a sketch of $\overleftrightarrow{RS}$.

2 Make a sketch of $\overleftrightarrow{MN}$ that intersects $\overleftrightarrow{RS}$ in point Q.

3 Name the sides and vertex of each angle formed by $\overleftrightarrow{MN}$ and $\overleftrightarrow{RS}$.

4 Name the common vertex of $\angle RQN$ and $\angle RQM$. Name the common side of these two angles.

5 Describe the union of $\overrightarrow{QM}$ and $\overrightarrow{QN}$.

6 Name four linear pairs of angles.

7 What is a supplement of $\angle RQN$? What is a supplement of $\angle RQM$?

8 Describe the union of $\overrightarrow{QR}$ and $\overrightarrow{QS}$.

Make one sketch for exercises 9 through 14.

9 Make a sketch of a triangle determined by noncollinear points B, C, and D.

10 $\triangle CBD$ is the union of what segments?

11 Locate point K in the interior of $\angle BCD$ and in the exterior of $\triangle CBD$.

12 Locate point F in the interior of $\angle CDB$ and in the interior of $\angle CBD$.

13 Locate point G in the interior of $\angle CDB$ and between C and B.

14 Locate point W in the interior of $\angle CBD$ and the exterior of $\angle BCD$.

Compound conditions for more complex problems

In this lesson you will learn to solve problems that are different from those you have had so far. You will use compound conditions in which each simple condition has two variables.

Read the problem in D1. The exercises that follow will help you develop the compound condition for the problem. $U = N \times N$.

A Do you know how many trees Earl planted? Use x as a variable for the number of trees Earl planted. x

B Do you know how many trees Earl's father planted? Use y as a variable for the number of trees Earl's father planted. y

C Why is $x + y$, and not $x - y$, used in the condition? What does $x + y$ refer to? $x + y$

D Why is the idea of equality used in the condition? Does the sentence at the right above express one of the simple conditions for the problem? $x + y = 7$.

E Why is $x + 3$, and not $y + 3$, used in the other simple condition for the problem? $x + 3$

F Does the sentence at the right above express the other simple condition for the problem? What does y refer to? $x + 3 = y$.

G Does the sentence at the right express the compound condition for the problem? $x + y = 7 \wedge x + 3 = y$.

Finding answers to problems involving compound conditions in which each simple condition has two variables

Earl and his father planted 7 trees in all. If Earl had planted 3 more trees, he would have planted the same number as his father. How many trees did each of them plant?

D1

Universe for $(x, y) = \mathrm{N} \times \mathrm{N}$.
$x + y = 7$.

x	0	1	2	3	4	5	6	7
y								

D2

There are fewer than 9 students in the cast of the spring play. The number of boys in the cast is greater than the number of girls. How many boys and how many girls can be in the cast?

D3

Now you can find the solution set of the compound condition.

H What is the universe for (x, y)?

I Look at D2. You can use a chart like this to help determine the ordered pairs that are solutions of $x + y = 7$. Why are the numbers from 0 through 7 the only numbers that are used as replacements for x in $x + y = 7$?

J Copy and complete the chart in D2.

K Tabulate $\{(x, y) | x + y = 7\}$.

Now you can find the solutions of $x + y = 7$ that are also solutions of $x + 3 = y$.

L Which member of the set you tabulated in exercise K has a second component that is 3 more than the first component?

M Tabulate $\{(x, y) | x + y = 7 \wedge x + 3 = y\}$.

N To what does the first component of the solution of the compound condition refer? To what does the second component refer?

Since you know the solution of the compound condition for the problem, you can use this solution to answer the question asked in the problem.

Earl planted 2 trees, and his father planted 5 trees.

O You can verify the answer you get from (2, 5) by answering the following questions: If Earl planted 2 trees and his father planted 5 trees, did they plant 7 trees in all? If Earl had planted 3 more trees, would he have planted the same number as his father?

Read the problem in D3. First you will develop the compound condition for the problem. $\mathrm{U} = \mathrm{N} \times \mathrm{N}$.

A Do you know how many boys are members of the cast? Use m as a variable for the number of boys who are in the cast. m

B Do you know how many girls are members of the cast? Use n as a variable for the number of girls who are in the cast. n

C Why is $m + n$, and not $m - n$, used in the condition? $m + n$

D Why is the idea of "less than" used in the condition? Does the sentence at the right express one of the simple conditions for the problem? $m + n < 9.$

E Why is $m > n$, and not $n > m$, the other simple condition for the problem? $m > n.$

F Does the sentence at the right express the compound condition for the problem? $m + n < 9 \wedge m > n.$

Now you can find the solution set of the compound condition.

G What is the universe for (m, n)?

H Look at D4. You can use a chart like this to help determine the ordered pairs that are solutions of $m + n < 9$. Why are the numbers from 0 through 8 the only numbers that are used as replacements for m in $m + n < 9$?

I Use the number 0 as a replacement for m in $m + n < 9$. What replacements for n satisfy $0 + n < 9$?

J Copy and complete the chart in D4.

With the help of your chart, you can find the solutions of $m + n < 9$ that are also solutions of $m > n$.

K Which of the ordered pairs expressed in your chart have first components that are greater than their second components?

L Tabulate $\{(m, n) | m + n < 9 \wedge m > n\}$.

M To what does the first component of each solution of the compound condition refer? To what does the second component refer?

You can use the solutions of the compound condition to get the answers to the problem.

If the number of boys in the cast is 1, the number of girls in the cast is 0.

If there are 2 boys in the cast, there is either 0 or 1 girl in the cast.

If there are 3 boys in the cast, there are 0, 1, or 2 girls in the cast.

N What are the other answers to the problem?

Universe for $(m, n) = \mathrm{N} \times \mathrm{N}$.
$m + n < 9$.

m	0	1	2	3	4	5	6	7	8
n									

D4

During the school baseball season, Bob and Jack together hit fewer than 8 home runs. If Bob had hit 5 fewer home runs, he would have hit the same number of home runs as Jack. How many home runs could each of the boys have hit?

D5

Read the problem in D5. You will develop the compound condition for this problem. $\mathrm{U} = \mathrm{N} \times \mathrm{N}$.

A Do you know how many home runs Bob hit? Use a as a variable for the number of home runs that Bob hit. a

B Do you know how many home runs Jack hit? Use b as a variable for the number of home runs that Jack hit. b

C Does the sentence below at the right express one of the simple conditions for the problem? Why is $a + b$, and not $a - b$, used in the condition? Why is the idea of "less than" used in the condition? $a + b < 8.$

D Why is $a - 5$, and not $b - 5$, used in the other simple condition for the problem? $a - 5$

E Does the sentence at the right below express the other simple condition for the problem? What does b refer to? $a - 5 = b.$

F Does the sentence at the right express the compound condition for the problem? $a + b < 8 \wedge a - 5 = b.$

Now you can find the solution set of the compound condition.

G What is the universe for (a, b)?

Universe for $(a, b) = \mathrm{N} \times \mathrm{N}$.
$a + b < 8$.

a	0	1	2	3	4	5	6	7
b								

D6

H You can use a chart like the one in D6 to help determine the ordered pairs that are members of the solution set of $a + b < 8$. Why are the numbers from 0 through 7 the only numbers that are used as replacements for a in $a + b < 8$?

I Use the number 0 as a replacement for a in $a + b < 8$. What replacements for b satisfy $0 + b < 8$?

J Copy and complete the chart in D6.

K How do you know that each ordered pair that satisfies the compound condition must have a first component that is equal to, or greater than, 5?

L Tabulate $\{(a, b) | a + b < 8 \wedge a - 5 = b\}$.

M To what does the first component of each solution of the compound condition refer? To what does the second component refer?

N Give the answers to the problem in D5.

Now you know how to solve problems that involve compound conditions in two variables.

On your own

For each problem, write a sentence that expresses the compound condition for the problem. To express the variables, use any letters you wish. Tell what each variable refers to in the problem. Then tabulate the solution set of the compound condition and use the solutions to get the answers to the problem. The universe is given for each problem.

1 Lucy answered all 8 questions on a test. She had more correct than incorrect answers. How many correct and how many incorrect answers could she have had? $\mathrm{U} = \mathrm{N} \times \mathrm{N}$.

2 Sarah spent less than 5¢ for an eraser and a pencil. She spent less for the pencil than for the eraser. How many cents could she have spent for each item? $\mathrm{U} = \mathrm{C} \times \mathrm{C}$.

3 Nova Scotia has 4 more senators in the Canadian Senate than Newfoundland has. Together, these two provinces have 16 senators. How many senators does each of these provinces have? $\mathrm{U} = \mathrm{N} \times \mathrm{N}$.

4 Mr. Hayes bought fewer than 15 postage stamps. He bought the same number of 4-cent stamps as 5-cent stamps. How many stamps of each kind could he have bought? $\mathrm{U} = \mathrm{C} \times \mathrm{C}$.

5 In an election for class treasurer, Dale received 11 more votes than Harvey. Together, the boys received 21 votes. How many votes did each of the boys receive? $\mathrm{U} = \mathrm{N} \times \mathrm{N}$.

6 Jane and Alice sold fewer than 9 boxes of candy in all. If Alice had sold 4 fewer boxes, she would have sold the same number as Jane. How many boxes of candy could each of the girls have sold? $\mathrm{U} = \mathrm{N} \times \mathrm{N}$.

7 Babe Ruth hit 9 more home runs during the 1918 baseball season than during the 1917 season. He hit only 13 home runs during these two seasons. How many home runs did he hit during each of the two seasons? $\mathrm{U} = \mathrm{N} \times \mathrm{N}$.

8 Henry and Jim each sold the same number of newspapers. Together, they sold fewer than 21 papers. How many papers could each of the boys have sold? $\mathrm{U} = \mathrm{C} \times \mathrm{C}$.

9 During a professional football game, the Ottawa Rough Riders and the Edmonton Eskimos scored a total of 22 points. The Rough Riders scored 10 more points than the Eskimos. How many points did each of the teams score? $\mathrm{U} = \mathrm{N} \times \mathrm{N}$.

SPECIAL CHALLENGE

Think about {a, b, c}. The subsets of this set are { }, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, and {a, b, c}. Notice that the empty set and the set itself are both subsets of {a, b, c}. In later lessons you will learn why the empty set is a subset of every set.

A How many subsets does a set with three members have?

B Tabulate all the subsets of {a, b}. Remember to include the empty set and the set itself. How many subsets does a set with two members have?

C Tabulate all the subsets of {a, b, c, d}. How many subsets does a set with four members have?

D Tabulate all the subsets of {a, b, c, d, e}. How many subsets does a set with five members have?

E How many subsets do you think each of the following sets will have? A set with six members. A set with seven members. A set with ten members.

For exercises F and G, use what you know about the intersection of sets to help you.

F Nine Canadian football teams compete for the Grey Cup. Three sports writers picked the following teams to win pre-season games on July 28.

{Edmonton, Regina, Ottawa, Hamilton}
{Winnipeg, Ottawa, Montreal, Edmonton}
{Hamilton,Edmonton,Vancouver,Montreal}

No writer picked Toronto to win. Calgary did not play on July 28. Determine the league teams that played each other on that day.

G There are six points and four lines. Each point is contained in two and only two lines. Each pair of lines has one and only one member in common. Make a sketch to represent these six points and four lines.

3 40 Exploring problems

Conditions for abstract problems

In lessons 37, 38, and 39, you used ordered pairs to obtain answers to problems about persons and things. In this lesson, you will learn that when problems concern numbers only, you can use ordered pairs as answers to the problems. $U = N \times N$.

A Read the problem in D1. Use x as a variable for the first number. Use y as a variable for the second number. Is $x + y = 7$ the condition for the problem?

B Tabulate $\{(x, y) | x + y = 7\}$. Each solution of the condition is an answer to the problem.

C Is the sum of 0 and 7 the number 7? Is (0, 7) an answer to the problem? What are the other answers?

D Read the problem in D2. Use x as a variable for the first number. Use y as a variable for the second number. Is $x + y = 8$ one of the simple conditions for the problem? Is $y < 2x$ the other simple condition?

E Does the sentence in D3 express the compound condition for the problem in D2?

F Tabulate $\{(x, y) | x + y = 8\}$.

G What does the condition $y < 2x$ require? Which members of the set that you tabulated for exercise F satisfy $y < 2x$?

H Tabulate $\{(x, y) | x + y = 8 \wedge y < 2x\}$.

I Is each solution of $x + y = 8 \wedge y < 2x$ an answer to the problem in D2?

J Read the problem in D4. Use x as a variable for the first number. Use y as a variable for the second number.

The sum of two numbers is 7. What can the numbers be?

D1

The sum of two numbers is 8. The second number is less than the product of 2 and the first number. What can the numbers be?

D2

$$x + y = 8 \wedge y < 2x.$$

D3

A first number is less than a second number. When 3 is subtracted from the first number, the difference is 5. What can the numbers be?

D4

$$x < y \wedge x - 3 = 5.$$

D5

The sum of two numbers is 10. The first number is equal to the product of 3 and the second number. What are the numbers?

D6

$$m + n = 10 \wedge m = 3n.$$

D7

K Does the sentence in D5 express the compound condition for the problem?

L What replacement for x satisfies the simple condition $x - 3 = 5$?

M How do you know that 8 is the first component of each solution of $x < y \wedge x - 3 = 5$?

N What does the condition $x < y$ require?

O Are (8, 9) and (8, 123) solutions of $x < y \wedge x - 3 = 5$? How do you know that the solution set of $x < y \wedge x - 3 = 5$ is an infinite set?

P Is each ordered pair named in exercise O an answer to the problem in D4? Give three other answers to the problem.

Q Give the standard description of the solution set of the condition for the problem.

R Read the problem in D6. Use m as a variable for the first number. Use n as a variable for the second number.

S Does the sentence in D7 express the compound condition for the problem?

T Tabulate $\{(m, n) | m + n = 10\}$.

U Do any of the ordered pairs that you named in exercise T have a first component that is equal to the product of 3 and the second component? Are there any ordered pairs in $N \times N$ that satisfy $m + n = 10 \wedge m = 3n$?

V Tabulate $\{(m, n) | m + n = 10 \wedge m = 3n\}$.

W Are there any members of the universe that are answers to the problem in D6?

Now you know how to solve problems that are about numbers only.

On your own

For each problem, first write a sentence that expresses the condition for the problem. If the solution set of the condition is finite, tabulate the solution set. If the solution set is infinite, write a standard description of the solution set. Use a as a variable for the first number. Use b as a variable for the second number. The universe for $(a, b) = N \times N$.

1 The sum of two numbers is 15. What can the numbers be?

2 The sum of two numbers is less than 20. The first number is equal to the second number. What can the numbers be?

3 The sum of two numbers is 78. Fifteen less than the first number is 26. What are the numbers?

4 The sum of two numbers is 10. The first number is greater than the second number. What can the numbers be?

5 When a first number is subtracted from a second number, the difference is 43. Fifty-eight more than the first number is 321. What are the numbers?

6 A first number is 5 more than a second number. The first number is also 2 less than 8. What are the numbers?

7 When 7 is subtracted from a first number, the difference is 4. The first number is equal to the product of 3 and the second number. What are the numbers?

8 When a number less than 4 is added to a second number, each sum is less than 5. What can the numbers be?

9 When a number greater than the product of 9 and 4 is added to a second number, each sum is greater than 40. What can the numbers be?

10 When a second number is subtracted from a first number, the difference is 9. The sum of the two numbers is 21. What are the numbers?

11 A first number is less than a second number. The second number is 8 less than 13. What can the numbers be?

12 A first number is equal to the quotient of 645 and 129. When a second number is subtracted from the first number, each difference is greater than 6. What can the numbers be?

13 The sum of two numbers is 21. The second number is equal to the product of 2 and the first number. What are the numbers?

APPLYING MATHEMATICS

In this lesson you will solve problems similar to the ones you have studied so far in this book. For each of problems 1 through 20, first write a sentence that expresses the condition for the problem. Use any letters you wish as names for the variables. Tell what each variable refers to in the problem. Then tabulate the solution set of the condition and give the answer to the problem. The universe is given for each problem.

1 A car dealer had more than 17 new cars to sell. He sold 8 of these cars and then had fewer than 12 new cars left to sell. How many new cars could he have had to begin with? $U = N$.

2 Nine students belong to the science club at Woodworth School. How many boys and how many girls can be members of this club? $U = C \times C$.

3 After Mr. Foster wrote a check for $186, he had $418 left in his checking account. How many dollars did Mr. Foster have in his checking account before he wrote the check for $186? $U = N$.

4 Carol is 5 years younger than Ann. In 14 years, Carol will be 21 years old. How old is each of the girls now? $U = N \times N$.

5 Altogether, Mary bought fewer than 5 yards of silk cloth and cotton cloth. How many yards of each kind of cloth could she have bought? $U = C \times C$.

6 One day, Bill and his father together plowed fewer than 28 acres of land. Bill plowed 6 acres in the morning and 5 acres in the afternoon. How many acres could Bill's father have plowed that day? $U = C \times C$.

7 Rachel collects foreign dolls. She has more than 12 of them now. If she had 5 fewer dolls, she would have fewer than 11 in all. How many dolls can Rachel have now? $U = N$.

8 There are as many boys as girls on the staff of the school paper. There are fewer than 23 students on the staff. How many girls and how many boys can be on the staff? $U = C \times C$.

9 One Saturday, Mr. Baker sold 7 men's suits and overcoats. He sold fewer overcoats than suits. How many of each could he have sold? $U = C \times C$.

10 Together, Andy and Peter sold 22 magazines. Andy sold 8 more magazines than Peter. How many magazines did each of the boys sell? $U = N \times N$.

11 Jim had 427 stamps in his collection. After he sold some of them, he still had more than 369 stamps. How many stamps could he have sold? $U = C$.

12 Nine years from now, Beverly will be more than 22 years old. Six years ago, she was less than 10 years old. How old can Beverly be now? $U = N$.

13 Sound travels at different speeds in different materials. In air, the speed of sound is about 1088 feet per second; in hard wood, it is about 12,620 feet per second. Sound travels about how many feet per second faster in hard wood than in air? $U = N$.

14 Susan had 91¢. She spent 12¢ for a comb and 25¢ for some ribbon. How many cents did she have left after she bought the comb and ribbon? $U = N \times N$.

15 If Judy had 15 more phonograph records, she would have more than 37 records. If she had 12 fewer records, she would have fewer than 15 records. How many phonograph records can Judy have now? $U = N$.

16 According to a recent survey, two of the largest cities in the world were Tokyo and London. The population of Tokyo was 9,311,774, and the population of London was 8,222,340. What was the total population of these two cities at the time of the survey? $U = N$.

17 Elizabeth has finished more experiments for science class than Nancy has. When Elizabeth finishes 8 more, she will have finished 21 experiments. How many experiments has Elizabeth finished? How many can Nancy have finished? $U = N \times N$.

18 There were 71,719 persons who entered Canada as immigrants in 1946. In 1956, there were 93,138 more people who entered Canada as immigrants than in 1946. How many people entered Canada as immigrants in 1956? $U = N$.

19 After John bought 13 new stamps, he still had fewer than 145 stamps. If he had bought 35 new stamps, he would have had more than 161 stamps. How many stamps could John have had to begin with? $U = N$.

20 The area of Waterton Lakes National Park is 203 sq. mi. Kootenay National Park is 340 sq. mi. larger than Waterton Lakes park and 468 sq. mi. larger than Elk Island National Park. What are the areas of Kootenay and Elk Island parks? $U = N \times N$.

For each of problems 21 through 25, first write a sentence that expresses the condition for the problem. Then tabulate the solution set of the condition.

21 The sum of two numbers is 9. What can the numbers be? $U = N \times N$.

22 What numbers are greater than the product of 466 and 238? $U = N$.

23 What numbers greater than 78 can be added to 75 so that each sum is less than 160? $U = N$.

24 The sum of two numbers is less than 17. The first number is equal to the product of 2 and the second number. What can the numbers be? $U = N \times N$.

25 When the first of two numbers is subtracted from the second, the difference is 5. The first number is less than 16. What can the numbers be? $U = N \times N$.

CHECKING UP

The small numeral within parentheses tells what page to turn to for help if you need it.

Test 36

For exercises 1 through 6, write "T" for each sentence that expresses a true statement. Write "F" for each sentence that expresses a false statement.

1 (2, 5) and (5, 2) are the same ordered pair. (123)

2 If the universe for (x, y) is $N \times N$, then the solution set of $x + 2 = 7$ is a set of ordered pairs. (134)

3 If $A = \{1, 3, 9\}$, and $B = \{9, 1, 3\}$, then $A \times B = B \times A$. (129)

4 If C has 3 members, and D has 4 members, then $C \times D$ has 7 members. (127)

5 Each member of $\{(x, y) | x = 2y \wedge x + y = 42\}$ satisfies $x = 2y$. (147)

6 If the universe for (x, y) is $N \times N$, then the solution set of $x < 5$ is a finite set. (138)

7 $A = \{0, 1, 2\}$. Tabulate $A \times A$. (129)

8 $B = \{2, 4\}$, and $C = \{1, 3, 5, 7\}$. Tabulate $C \times B$. (126)

The graph in D1 is a complete graph of $X \times Y$. Use the graph for exercises 9 through 14.

9 Tabulate set X. (130)

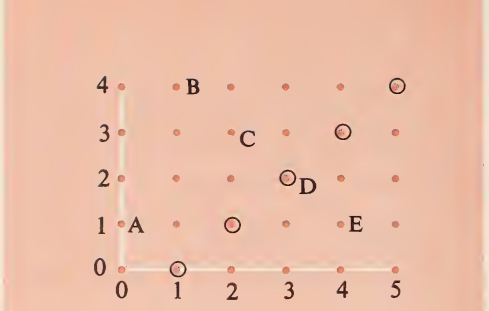
10 Tabulate set Y. (130)

11 Point A is associated with what ordered pair? (131)

12 Which letter names the point associated with (4, 1)? (131)

13 Which letter names the point associated with (2, 3)? (131)

14 The encircled dots form the graph of the solution set of a condition that has x and y as variables. The universe for (x, y) is $X \times Y$. Tabulate the solution set of this condition. (132)



D1

Universe for $(x, y) = N \times N$.

$x + y < 6$.

x	0	1	2	3	4	5
y						

D2

Test 37

For each of exercises 15 through 18, the universe for (a, b) is $R \times S$. $R = \{0, 1, 2, 3\}$. $S = \{0, 1, 2\}$.

15 Write a standard description of the solution set of $b = a + 1$. (138)

16 Tabulate the solution set of $b = a + 1$. (132)

17 Make a graph of $R \times S$. (130)

18 In your graph, encircle each dot that represents a point associated with a solution of $b = a + 1$. (132)

19 Copy and complete the chart in D2. Then tabulate the solution set of the condition expressed in D2. The universe for (x, y) is $N \times N$. (148)

For each of the compound conditions expressed in exercises 20 through 24, tabulate the solution set. The universe for (x, y) is $N \times N$.

20 $x < 5 \wedge y = x + 5$. (147)

21 $x + y = 9 \wedge y = 2x$. (147)

22 $2y + 3 > 3x \wedge y + 5 < 9$. (147)

23 $y + x = 5 \wedge x + y > 8$. (147)

24 $4x + 2 = y \wedge 8 + x < 15$. (147)

Test 38

For each problem, write a sentence that expresses the condition for the problem. Use any letters you wish as names for the variables. Tabulate the solution set of the condition and then give the answer to the problem. The universe is given for each problem.

25 Mary and Donna together ate 4 sandwiches for lunch. How many sandwiches could each of the girls have eaten? $U = C \times C$. (151)

26 Together, Barbara and Kathy took 18 cupcakes to a bake sale. Barbara took 4 more cupcakes than Kathy. How many cupcakes did each of the girls take to the bake sale? $U = N \times N$. (156)

27 Eight inches of snow fell during January and February. Less than 5 inches of snow fell during February. How many inches of snow could have fallen during each of these months? $U = C \times C$. (153)

28 During a softball game, Henry had more hits than Steve. Together, the two boys had 6 hits. How many hits could each of the boys have had? $U = N \times N$. (156)

29 A farmer has 32 more steers than hogs. If he had 47 more steers, he would have 172. How many steers and how many hogs does the farmer have? $U = N \times N$. (153)

30 The sum of two numbers is less than 12. The second number is twice the first number. What can the numbers be? $U = N \times N$. (160)

31 When the first of two numbers is subtracted from the second number, the difference is 7. Two more than the first number is 11. What are the numbers? $U = N \times N$. (160)

32 The sum of two numbers is less than 5. What can the numbers be? $U = N \times N$. (160)

33 When the first of two numbers is subtracted from the second number, the difference is 9. The first number is less than 7. What can the numbers be? $U = N \times N$. (160)

CHECKING UP

The small numeral within parentheses tells what page to turn to for help if you have trouble with an exercise.

Test 39

For exercises 1 through 7, write "T" for each exercise in which a true statement is expressed. Write "F" for each exercise in which a false statement is expressed.

1 If $\overline{LM} \cong \overline{NM}$ and $\overline{KJ} > \overline{LM}$, then $\overline{NM} > \overline{KJ}$. (43)

2 The sentence $17 + r \neq 32$ expresses a condition for inequality. (21)

3 The set whose members are all the natural numbers greater than 20 is a finite set. (23)

4 If U = N, then a true statement is obtained for each replacement of x in $x + 1 < 2x$. (24)

5 If U = N, then $5 + 7 = 12$ can be obtained from $x + x = 12$. (21)

6 $\angle RST$ includes ray ST and ray RS. (105)

7 $\{X \mid \overline{CX} \cong \overline{CD}\}$ is a standard description of a circle with center C and radius CD. (47)

What words or symbols best complete exercises 8 through 13?

8 The solution set of $x > 4 \wedge x < 8$ is the ~~~ of the solution sets of $x > 4$ and $x < 8$. (92)

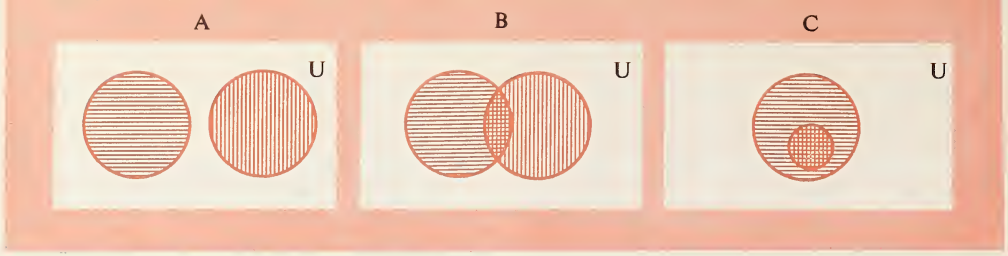
9 If ℓ_1 is parallel to ℓ_2, then $\ell_1 \cap \ell_2$ is ~~~. (101)

10 If points B and C are contained in plane BCD, then ~~~ is the intersection of $\overleftrightarrow{BC}$ and plane BCD. (101)

11 Point B is in the interior of $\triangle GHI$. Point C is in the exterior of $\triangle GHI$. There is exactly one member in $\overline{BC}$ ~~~ $\triangle GHI$. (114)

12 Point K is between points J and L. $\overline{JK} \cup \overline{KL} =$ ~~~. (87)

13 If plane RST and plane QRS are not the same plane, then plane RST $\cap$ plane QRS = ~~~. (100)



Test 40

Six sets are described below. Use these descriptions to decide which of Venn diagrams A, B, or C in the display above represents the set named in each of exercises 14 through 22. The universe is the set of all plane geometric figures. (84, 89)

Set D is the set of all simple closed curves.
Set E is the set of all simple polygons.
Set F is the set of all right triangles.
Set G is the set of all isosceles triangles.
Set H is the set of all equilateral triangles.
Set I is the set of all circles.

14 $D \cap E$ **15** $I \cap F$ **16** $G \cap E$ **17** $H \cap F$ **18** $E \cup H$ **19** $G \cap F$ **20** $E \cup I$ **21** $I \cap D$ **22** $G \cup H$

Test 41

For each of the conditions expressed in exercises 23 through 26, make a graph of the universe. Then encircle each dot that represents a point associated with a member of the solution set. $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

23 $5 + x = 9$. (52) **25** $b + 2 < 6$. (52)

24 $10 - w \neq 3$. (52) **26** $12 - x > 5$. (52)

For each compound condition tabulate the solution set. The universe for $(x, y) = N \times N$.

27 $x + 5 = 55 \wedge x - y = 42$. (147)

28 $x + y = 10 \wedge x + y < 10$. (147)

29 $x + y < 7 \wedge y < 2x$. (147)

30 $y < 3 + x \wedge x < 4$. (147)

Test 42

For each problem, write a sentence that expresses the condition, tabulate its solution set, and give the answer to the problem.

31 A committee in Highcrest School wrote a new school constitution that contained 2670 words. This was 875 more words than the old constitution contained. How many words did the old constitution contain? $U = N$. (68)

32 There are 5 children in the Hughes' family. How many boys and how many girls can be in this family? $U = N \times N$. (151)

33 The Pep Club had fewer than 32 pennants. After the club bought 18 pennants, it had more than 40. How many pennants could the club have had to begin with? $U = N$. (94)

34 Bill scored 21 points in a table tennis game. Fewer than 38 points were scored in the game. How many points could Bill's opponent have scored? $U = N$. (73)

35 Chuck and Will caddied 15 times in all at The Pines Golf Club. Will caddied more than 9 times. How many times could each of the boys have caddied at The Pines? $U = C \times C$. (153)

36 Fewer than 20 students are in Mrs. Turner's class. There are 8 more boys than girls in this class. How many boys and how many girls could be in this class? $U = C \times C$. (156)

Test 43

Multiply	Divide
37 64 by 19. (55)	**44** 216 by 24. (58)
38 507 by 98. (55)	**45** 962 by 13. (58)
39 813 by 45. (55)	**46** 1935 by 45. (58)
40 729 by 560. (55)	**47** 2584 by 76. (58)
41 609 by 304. (55)	**48** 57152 by 19. (58)
42 521 by 128. (55)	**49** 71120 by 508. (59)
43 937 by 467. (55)	**50** 89975 by 295. (59)

Unit 4

Conditions involving rate pairs

lesson page

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45 Per cent 186

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47 Problems involving compound conditions about rate pairs 195

48 Abstract problems involving rate pairs 199

Equivalent ordered pairs

In this unit you are going to use conditions to solve rate problems. First you must learn about ordered pairs that are related to each other in a special way.

If 3 pencils are sold for 12 cents, you can say that these pencils are sold at the rate of 3 pencils for 12 cents, or 3 pencils per 12 cents.

A You can represent this rate by using an ordered pair of numbers. Think of (3, 12) as representing the rate at which the pencils are sold.

3 pencils per 12 cents (3, 12)

Which component of (3, 12) refers to the number of pencils?

B Which component refers to the number of cents needed to buy the pencils? The black markers at the left represent the pencils. The red markers represent the cents.

C There are other ordered pairs that represent the rate at which these pencils are sold. The markers that represent the 3 pencils and the 12 cents have been separated into 3 equal groups.

1 pencil per 4 cents (1, 4)

Will the 4 cents in each group buy the 1 pencil in that group?

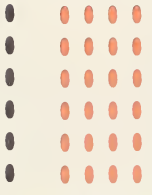
D Can you say that the pencils are sold at the rate of 1 pencil per 4 cents?

E Does (1, 4) also represent the rate at which the pencils are sold?

2 pencils per 8 cents (2, 8)

F Think of two of the groups just described. Does (2, 8) also represent the rate at which the pencils are sold?

G Suppose that you buy more than 3 pencils at the same rate. Can you buy 6 pencils for 24 cents?



H Can you represent the rate by using (6, 24)?

I Name five more ordered pairs that represent the rate at which the pencils are sold. Use the markers to help you.

J Set A, tabulated in D1, is a set of some of the ordered pairs that represent the rate at which the pencils are sold. Does set A contain all the ordered pairs that represent the rate at which the pencils are sold? Explain your answer.

Each member of set A represents the same rate. The exercises that follow will help you understand how a set like A differs from the other sets of ordered pairs that you have studied.

K Look at D2. First we will select any two members of A. Suppose that we select (1, 4) and (2, 8). D2 shows that you are to find the product of the first component of (1, 4) and the second component of (2, 8). What is this product? You also are to find the product of the first component of (2, 8) and the second component of (1, 4). What is this product?

L Is the product of 1 and 8 the same number as the product of 2 and 4? Is $1 \times 8 = 2 \times 4$ a true statement?

M Next we will select two other members of A. Suppose that we select (3, 12) and (4, 16). Look at D3. Is 3×16 the product of the first component of (3, 12) and the second component of (4, 16)? 4×12 is the product of which components of (3, 12) and (4, 16)?

$$A = \{(1, 4),\ (2, 8),\ (3, 12),\ (4, 16)\}.$$

D1

1×8

(1, 4) (2, 8)

2×4

$1 \times 8 = 2 \times 4.$

D2

3×16

(3, 12) (4, 16)

4×12

$3 \times 16 = 4 \times 12.$

D3

e quiv a lent ordered pairs (i kwiv′ə lənt). Ordered pairs that are related as follows: In (a, b) and (c, d) suppose that a, b, c, and d are replaced by any members of C. If a true statement is obtained from $a \times d = c \times b$, then (a, b) is equivalent to (c, d). Also, if (a, b) is equivalent to (c, d), then a true statement is obtained from $a \times d = c \times b$.

N Is the product of 3 and 16 the same number as the product of 4 and 12? Is $3 \times 16 = 4 \times 12$ a true statement?

O In exercises K through N, you first worked with (1, 4) and (2, 8) and then with (3, 12) and (4, 16). In each case, you found two products. Which components did you multiply to find each of these two products? Were these products equal? In any two ordered pairs, if these two products are equal, the ordered pairs are *equivalent ordered pairs*.

P Look again at D2 and D3. How do you know that (1, 4) and (2, 8) are equivalent ordered pairs? How do you know that (3, 12) and (4, 16) are equivalent ordered pairs?

Q Are (2, 8) and (3, 12) equivalent ordered pairs? Explain your answer.

Each exercise below names two ordered pairs. For each exercise, decide if the ordered pairs are equivalent.

R (55, 22), (5, 2)
S (9, 12), (54, 72)
T (5, 9), (35, 54)
U (2, 3), (20, 30)
V (1, 7), (7, 42)
W (63, 45), (7, 5)
X (27, 9), (9, 1)
Y (4, 13), (24, 78)

The work that follows will help you develop the definition of equivalent ordered pairs.

A Think of two ordered pairs of variables, (*a*, *b*) and (*c*, *d*). The universe for each of the variables is C. Can you tell if these ordered pairs are equivalent? Explain your answer.

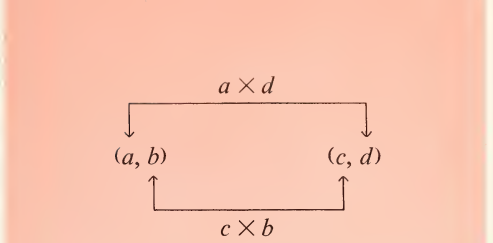
B Look at D4. Which components are used to obtain $a \times d$? How is $c \times b$ obtained?

C Look at D5. What is the universe for each of the variables? What replacements have been made for the variables in (*a*, *b*)? In (*c*, *d*)?

D How is the statement $2 \times 9 = 3 \times 6$ obtained? Is $2 \times 9 = 3 \times 6$ a true statement? How do you know that (2, 6) and (3, 9) are equivalent ordered pairs?

Any two ordered pairs, (*a*, *b*) and (*c*, *d*), are equivalent only if, upon replacement of each variable, a true statement is obtained from $a \times d = c \times b$. Notice that the two products must be equal before the two ordered pairs can be equivalent.

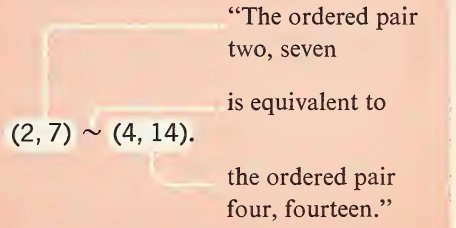
E What replacements must be made for *a*, *b*, *c*, and *d* to obtain (1, 4) and (4, 16) from (*a*, *b*) and (*c*, *d*)? With the same replacements, what statement is obtained from $a \times d = c \times b$? Is this a true statement? Is (1, 4) equivalent to (4, 16)?



D4

U = C.
(*a*, *b*) (*c*, *d*)
(2, 6) (3, 9)
$a \times d = c \times b$.
$2 \times 9 = 3 \times 6$.

D5



D6

F Is (2, 5) equivalent to (4, 7)? Explain your answer.

G Is (2, 7) equivalent to (10, 35)? Is (10, 35) equivalent to (2, 7)?

The symbol at the right below is used to express the idea of equivalence. D6 shows how to write and read a sentence that includes the symbol for equivalence.

$\sim$

Notice that "equivalence" and "equality" are not used in the same way. You know that (2, 7) is equivalent to (4, 14) because $2 \times 14 = 4 \times 7$. However, (2, 7) is not *equal* to (4, 14) be-

cause (2, 7) and (4, 14) are not the same ordered pair. They are equivalent ordered pairs.

Read each of the sentences in exercises H through K. Which sentences express true statements? Explain each answer by using the definition of equivalent ordered pairs.

H $(3, 9) \sim (4, 10)$.

J $(3, 9) \sim (1, 3)$.

I $(1, 7) \sim (9, 63)$.

K $(1, 7) \sim (7, 1)$.

L Look at D7. How many members are there in set B? You are going to decide if each member of B is equivalent to each of the other members of B.

M First decide if (1, 2) is equivalent to each of the other members of B. Is $(1, 2) \sim (2, 4)$? Is $(1, 2) \sim (3, 6)$? Is $(1, 2) \sim (4, 8)$? Use the definition of equivalent ordered pairs to explain your answers.

N Next decide if (2, 4) is equivalent to each of the other members of B. How do you know that (2, 4) is equivalent to (1, 2)? How do you know that (2, 4) is equivalent to (3, 6) and also to (4, 8)?

O How do you know that each of the ordered pairs (3, 6) and (4, 8) is equivalent to each of the other members of B?

P Is it correct to say that each member of set B is equivalent to each of the other members of set B?

Q Is each member of B equivalent to itself? Explain your answer.

In some sets of ordered pairs, each member is equivalent to each of the other members. Such a set of ordered pairs is a *proportional relation*. Set B, tabulated in D7, is a proportional relation.

pro por tion al re la tion (prə pôr′shən əl ri lā′shən). A set of ordered pairs in which each member is equivalent to each of the other members. {(3, 1), (6, 2), (9, 3)} is a proportional relation.

$B = \{(1, 2), (2, 4), (3, 6), (4, 8)\}$.

D7

$2 \times 6 = 12.$ $\quad$ $6 \div 2 = 3.$

$(6, 8) \sim (12, 16).$ $\quad$ $(6, 8) \sim (3, 4).$

$2 \times 8 = 16.$ $\quad$ $8 \div 2 = 4.$

D8

R Set $M = \{(7, 2), (14, 4), (21, 6), (28, 8)\}$. Is (7, 2) equivalent to (14, 4)? To (21, 6)? To (28, 8)?

S Is M a proportional relation? Explain your answer.

T $G = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Is G a proportional relation? Explain your answer.

You have learned how to use the definition of equivalent ordered pairs to decide if two ordered pairs are equivalent. Now you will learn one way to find ordered pairs equivalent to a given ordered pair.

A First you will see how (12, 16) and (3, 4) can be obtained from (6, 8). Is $(12, 16) \sim (6, 8)$? Is $(3, 4) \sim (6, 8)$?

B Look at D8. By what number can you multiply the first component of (6, 8) to obtain the first component of (12, 16)?

C How can the second component of (12, 16) be obtained from the second component of (6, 8)?

D By what number can you multiply each component of (6, 8) to obtain (12, 16)?

E Look again at D8. By what number can you divide the first component of (6, 8) to obtain the first component of (3, 4)?

F How can the second component of (3, 4) be obtained from the second component of (6, 8)?

G By what number can you divide each component of (6, 8) to obtain (3, 4)?

H Are (3, 4) and (12, 16) equivalent? Explain your answer.

Six ordered pairs that are equivalent to (6, 8) are named below. Tell how you can obtain each ordered pair from (6, 8).

I (18, 24) K (66, 88) M (12, 16)

J (42, 56) L (48, 64) N (72, 96)

O If you multiply each component of (6, 8) by the same counting number, will each ordered pair that you obtain be equivalent to (6, 8)?

Six ordered pairs that are equivalent to (120, 160) are named below. Tell how you can obtain each ordered pair from (120, 160).

P (12, 16) R (3, 4) T (30, 40)

Q (6, 8) S (15, 20) U (60, 80)

In the preceding exercises, you have been given ordered pairs and have used multiplication or division to find ordered pairs equivalent to them. Now you will be given an ordered pair and one component of an equivalent ordered pair. You will find the other component.

A Look at D9. The given ordered pair is (5, 8). The first component of an ordered pair equivalent to (5, 8) is 30. The second component of the pair is y. What is the universe for y?

B How can you obtain the first component of $(30, y)$ from the first component of (5, 8)?

C How can you obtain the replacement for y from the second component of (5, 8)? What is the replacement for y?

D Is $(5, 8) \sim (30, 48)$?

E Look at D10. The second component of an ordered pair equivalent to (10, 24) is 12. The first component of the pair is x.

F How can you obtain the second component of $(x, 12)$ from the second component of (10, 24)?

$U = C.$
$(5, 8) \sim (30, y).$
$6 \times 5 = 30.$
$6 \times 8 = y.$

D9

$U = C.$
$(10, 24) \sim (x, 12).$
$24 \div 2 = 12.$
$10 \div 2 = x.$

D10

G How can you obtain the replacement for x from the first component of (10, 24)? What is the replacement for x?

H Is $(10, 24) \sim (5, 12)$ a true statement?

Six ordered pairs are named below. For each ordered pair, what replacement should you make for the variable to obtain an ordered pair that is equivalent to (16, 24)? $U = C$.

I $(x, 6)$ K $(48, y)$ M $(8, y)$

J $(32, y)$ L $(x, 96)$ N $(x, 3)$

O For each ordered pair named in exercises I through N, how did you obtain the replacement for the variable?

In this lesson you have learned about equivalent ordered pairs and proportional relations. You have also learned how to find ordered pairs that are equivalent to a given ordered pair.

On your own

1 Suppose that you can buy 5 pieces of candy for 8 cents. Which ordered pairs named below represent the rate at which the candy is sold?

a (10, 8) c (10, 16) e (5, 8)

b (45, 72) d (15, 20) f (20, 32)

2 Explain why (5, 8) and (10, 16) are equivalent ordered pairs.

3 Explain why (10, 16) and (45, 72) are equivalent ordered pairs.

4 Explain why (5, 8) and (10, 8) are not equivalent ordered pairs.

Set A is tabulated below. For each of exercises 5 through 13, decide which member of A is equivalent to the ordered pair named. Then write a sentence using the symbol for equivalence.

A = {(1, 2), (2, 1), (2, 3), (3, 2), (3, 4), (4, 3), (4, 5), (5, 4)}.

5 (4, 6)
6 (6, 4)
7 (6, 3)
8 (3, 6)
9 (10, 8)
10 (8, 10)
11 (12, 9)
12 (9, 6)
13 (6, 8)

For each of exercises 14 through 19, tell whether or not the set tabulated is a proportional relation. Explain each of your answers.

14 {(60, 3), (3, 60)}

15 {(70, 5), (28, 2), (14, 1)}

16 {(7, 1), (63, 9), (35, 5)}

17 {(125, 70), (250, 140)}

18 {(100, 7), (200, 7)}

19 {(8, 24), (37, 121), (62, 185)}

20 How can you obtain (25, 15) from (5, 3)? Explain why (25, 15) and (5, 3) are equivalent ordered pairs.

21 How can you obtain (25, 15) from (50, 30)? Explain why (25, 15) and (50, 30) are equivalent ordered pairs.

For each ordered pair named in exercises 22 through 27, what replacement should you make for the variable to obtain an ordered pair that is equivalent to (2, 3)? U = C.

22 $(8, y)$
23 $(100, y)$
24 $(x, 15)$
25 $(14, y)$
26 $(x, 39)$
27 $(x, 51)$

For each ordered pair named in exercises 28 through 33, what replacement should you make for the variable to obtain an ordered pair that is equivalent to (40, 48)? U = C.

28 $(10, y)$
29 $(x, 6)$
30 $(x, 12)$
31 $(20, y)$
32 $(x, 24)$
33 $(5, y)$

KEEPING SKILFUL

For exercises 1 through 10, tell whether each sentence is an open sentence or a closed sentence.

1 $9 \neq 7 + 3$.
2 $s = s + 1$.
3 $d + 3 < 10$.
4 $13 = 26 \div 2$.
5 $r = 23(9)$.
6 $m > 4m$.
7 $11 = n + 4$.
8 $8(2) > 15$.
9 $c \neq 5 + 6$.
10 $15 - 7 = 8$.

11 For each closed sentence in exercises 1 through 10, tell which it expresses, an equation or an inequality.

12 For each open sentence in exercises 1 through 10, tell which it expresses, a condition for equality or a condition for inequality.

13 J = {0, 2, 4}. K = {1, 3, 5, 7}. Tabulate J × K.

For exercises 14 through 19, the universe is J × K, which you tabulated for exercise 13.

14 Tabulate the subset of J × K in which each member has a first component less than 3.

15 Tabulate the subset of J × K in which the sum of the components of each member is equal to 7.

16 Tabulate $\{(x,y) | x + y > 6\}$.

17 Tabulate $\{(x,y) | x < y\}$.

18 Tabulate $\{(x,y) | y > x + 3\}$.

19 Tabulate $\{(x,y) | x = y - 1\}$.

Tabulate each set described below. For exercises 20 through 27, U = N. For exercises 28 through 32, U = N × N.

20 $\{c | c = 29(14)\}$
21 $\{n | n < 18 + 13\}$
22 $\{r | 52 = 91 - r\}$
23 $\{s | 32 - s > 15\}$
24 $\{w | w > 109 - 11\}$
25 $\{t | t + 1 \neq t\}$
26 $\{m | 63 + m = 90\}$
27 $\{d | d < 42 - 29\}$
28 $\{(c, d) | c + d = 403 \wedge c + 72 = 213\}$
29 $\{(r, s) | 326 + s = 517 \wedge r + s < 200\}$
30 $\{(x, y) | x + y < 9 \wedge x + 2 = y\}$
31 $\{(m, n) | n - 49 = 172 \wedge m - n = 83\}$
32 $\{(r, s) | r > s \wedge r < 261 \div 87\}$

Rate pairs and ratios

In the last lesson you learned that a proportional relation is a set of ordered pairs in which each ordered pair is equivalent to each of the other ordered pairs in the set. In this lesson you will learn about special kinds of ordered pairs called rate pairs. You will use standard descriptions to describe proportional relations.

A Read the tabulation in D1. Is the set a proportional relation? Explain your answer.

You know that each member of a proportional relation is equivalent to each of the other members. Each ordered pair that is a member of a proportional relation is a *rate pair*.

B Are the members of the set tabulated in D1 rate pairs? How do you know?

Rate pairs are ordered pairs that are used to solve rate and comparison problems. There are various ways in which to express a rate pair. D2 shows four ways to express the same rate pair. You should read the first symbol as "one to two" because in this case (1, 2) expresses a rate pair. You also read the other three symbols "one to two."

Each name below expresses a rate pair. Read each name.

C 6:5 **D** 4/15 **E** $\frac{6}{9}$ **F** (100, 106) **G** 7/21

Although different kinds of symbols can be used in expressing a rate pair, it is convenient to use the same kind of symbol all the time. In this book, we will agree to use a symbol like the one at the right in expressing a rate pair.

1/2

{(1, 15), (2, 30), (3, 45), (4, 60), (5, 75)}

D1

(1, 2)	1:2	$\frac{1}{2}$	1/2

D2

rate pair (rāt). A member of a proportional relation. Each member of {(1, 2), (2, 4), (3, 6)} is a rate pair.

ra ti o (rā′shē o or rā′shō). A name of a rate pair. The symbols 1/4, 2/3, and 9/5 are ratios.

A name of a rate pair is a *ratio*.

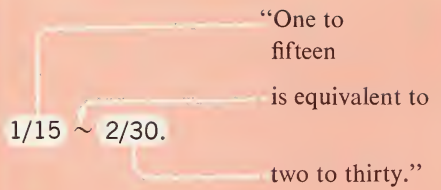
H In the ratio 1/2, where is the name of the first component of the rate pair placed? Where is the name of the second component of this rate pair placed?

Now that you have a special symbol for expressing rate pairs, use this symbol to express each member of a proportional relation.

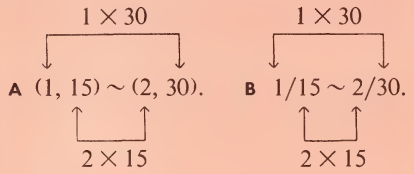
I Tabulate the set whose members are expressed below by writing the names of the members in the form 1/15, 2/30, and so on.

(1, 15), (2, 30), (3, 45), (4, 60), (5, 75)

In this book, rate pairs will be used to represent many kinds of rates. In problems about rates, you may see expressions like these: "20 persons *per* square mile," "4 cans *for* 89¢," "30 miles *per* hour," "100 yards *in* 10 seconds," "3 cups of sugar *to* 1 cup of water." But when you read a ratio that expresses a rate pair, we will agree that you can always use the word "to." By this agreement, you and your classmates will always read the mathematical symbol for a rate pair in the same way.



D3



D4

A $6/27 \sim 2/9$.
B $5/9 \sim 15/21$.
C $3/5 \sim 12/20$.
D $10/1 \sim 50/5$.
E $2/13 \sim 3/13$.
F $7/9 \sim 28/36$.

D5

J If you write the ratios in the sentence $(1, 15) \sim (2, 30)$ in the form 1/15 and 2/30, do you obtain the sentence in D3? To read this sentence, study D3.

K Look at D4. What do the arrows in A indicate?

L The arrows in B also indicate what numbers must be multiplied before you can tell if the sentence in B expresses a true statement. Are 1/15 and 2/30 equivalent rate pairs?

M Does the sentence $1/15 \sim 2/30$ express the same true statement as the sentence $(1, 15) \sim (2, 30)$?

N Read the sentences in D5. Tell what ratios each sentence contains.

O Which of the sentences in D5 express true statements? How do you know?

You already know how to tabulate finite proportional relations. Now you will learn how to express infinite proportional relations.

A Study D6. Are the rate pairs expressed equivalent rate pairs? How do you know?

B Are the rate pairs expressed in D6 all possible rate pairs that are equivalent to 4/7?

C Could you write all the ratios that express rate pairs equivalent to 4/7? Explain your answer.

D Study D7. It shows how to write and read the tabulation of a proportional relation that includes all rate pairs equivalent to 4/7.

E Is 16/28 a member of the proportional relation tabulated in D7? Is 24/42 a member? Is 200/350 a member? Name five other members.

F Is the set of all rate pairs equivalent to 4/7 a finite set? An infinite set? A set of rate pairs like the set tabulated in D7 is an *infinite proportional relation.*

G When you tabulate an infinite proportional relation, why is it unnecessary to list more than one member?

H Is $4/7 \sim 8/14$ a true statement? How do you know that {8/14, . . .} contains 4/7?

4/7, 8/14, 12/21, 16/28, 20/35

D6

D7

I Is each member of {8/14, . . .} equivalent to 4/7? Is {8/14, . . .} the same proportional relation as {4/7, . . .}? Is {100/175, . . .} the same proportional relation as {4/7, . . .}? Explain your answers.

Now you will use a standard description to express an infinite proportional relation.

J Study D8. Suppose that x is a variable for the first component of each rate pair that is equivalent to 4/7, and that y is a variable for the second component. Then (x, y) is an ordered pair of variables for each rate pair that is equivalent to 4/7. What is the universe for (x, y)?

K Is $x/y \sim 4/7$ satisfied by 4/7? By 8/14? By 60/105? Does each member of {4/7, . . .} satisfy $x/y \sim 4/7$?

L How do you know that the solution set of $x/y \sim 4/7$ is an infinite set? Is the solution set of $x/y \sim 4/7$ a proportional relation?

The proportional relation determined by $x/y \sim 4/7$ may be named by a standard description. D9 shows how to write and read this standard description.

M Is $\{(x, y) | x/y \sim 8/14\}$ the same proportional relation as $\{(x, y) | x/y \sim 4/7\}$?

Read each standard description given below.

N $\{(x, y) | x/y \sim 7/15\}$

O $\{(x, y) | x/y \sim 10/1\}$

P $\{(r, s) | r/s \sim 1/100\}$

Q $\{(m, n) | m/n \sim n/m\}$

D10 includes a graph of C × C and also of the proportional relation that is described in D10. Notice that a graph of C × C does not contain a dot for the point (0, 0). This is because 0 is not a member of C.

R Name the ordered pairs associated with points represented by encircled dots. Are these ordered pairs members of $\{(x, y) | x/y \sim 1/3\}$?

S Is the graph of $\{(x, y) | x/y \sim 1/3\}$ an incomplete graph? Name three other members of the proportional relation whose points are not represented by dots in the graph in D10.

Universe for $(x, y) = C \times C$.
$x/y \sim 4/7$.

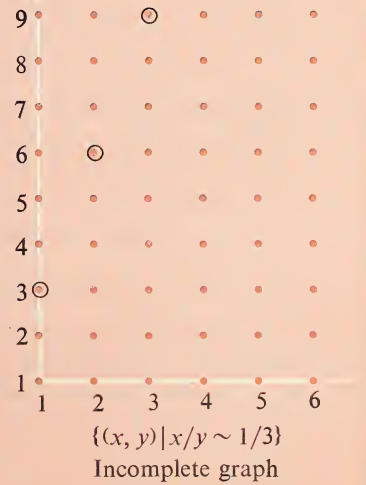
D8

"The set whose members are the ordered pairs x, y that satisfy the condition that x to y is equivalent to four to seven"

$\{ (x, y) \mid x/y \sim 4/7 \}$

D9

Universe for $(x, y) = C \times C$.



$\{(x, y) | x/y \sim 1/3\}$
Incomplete graph

D10

In this lesson you have learned that a rate pair is a member of a proportional relation and that a ratio is a name of a rate pair. You have also learned how to express an infinite proportional relation and how to make a graph of a proportional relation.

On your own

1 Write each of the following ratios in words:
a 1/6 **b** 2/12 **c** 3/1 **d** 8/5 **e** 100/1

Tabulate each proportional relation whose members are named in exercises 2, 3, and 4. Write the names of the members of each set in the form 1/3, 2/6, and so on.

2 (1, 1), (2, 2), (3, 3), (4, 4), (5, 5)

3 (9, 1), (18, 2), (27, 3), (36, 4)

4 (10, 3), (20, 6)

Which of the sentences in exercises 5 through 12 express true statements about rate pairs?

5 $1/2 \sim 6/8$.
6 $21/35 \sim 24/40$.
7 $9/1 \sim 99/11$.
8 $10/4 \sim 15/6$.
9 $101/1 \sim 202/2$.
10 $1/3 \sim 9/27$.
11 $7/5 \sim 35/30$.
12 $25/4 \sim 100/8$.

Each rate pair named in exercises 13 through 17 is a member of an infinite proportional relation. Write a tabulation and a standard description of the proportional relation to which the given rate pair belongs. $U = C \times C$.

13 3/8 **14** 5/25 **15** 1/4 **16** 2/2 **17** 4/17

18 For each proportional relation you described in exercises 13 through 17, name five members.

A standard description of a proportional relation is given in each of exercises 19 through 22. Make a graph of each proportional relation. You do not need to encircle more than three dots in each graph. The universe for $(x, y) = C \times C$.

19 $\{(x, y) | x/y \sim 3/2\}$
20 $\{(x, y) | x/y \sim 4/2\}$
21 $\{(x, y) | x/y \sim 2/5\}$
22 $\{(x, y) | x/y \sim 6/9\}$

Solution sets of conditions for equivalence

In this lesson you will use the definition of equivalent ordered pairs to determine missing components of rate pairs.

A Read the problem in D1. Sixteen plants cost how many dollars? Can you represent this rate by the rate pair 16/8? What does the first component of 16/8 refer to? What does the second component of 16/8 refer to?

B You do not yet know how many dollars 4 of the plants cost. First, you must find a rate pair that is equivalent to 16/8. How do you know that 4 is the first component of the rate pair you are to find?

C Does the open sentence in D2 express a condition for the problem? Notice that x is used as the second component of the rate pair that you are to find. For what is x a variable?

A condition like $16/8 \sim 4/x$ is a *condition for equivalence*. A condition for equivalence includes the idea of equivalence. You obtain a

Sixteen plants cost 8 dollars. At this rate, 4 of the plants cost how many dollars?

D1

$U = C$.
$16/8 \sim 4/x$.

D2

statement from a condition for equivalence by replacing the variable in the condition by a member of the universe. The condition $16/8 \sim 4/x$ requires you to find the replacement for x that makes $16/8$ and $4/x$ equivalent rate pairs.

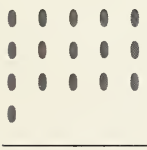
You can use the definition of equivalent ordered pairs to find a replacement for the variable that satisfies a condition like $16/8 \sim 4/x$. You are permitted to use the definition of equivalent ordered pairs when you work with rate pairs because rate pairs are ordered pairs.

D Study D3. What do the arrows indicate? How was the condition $16x = 32$ obtained from $16/8 \sim 4/x$?

E Will the replacement for x that satisfies $16x = 32$ also satisfy $16/8 \sim 4/x$?

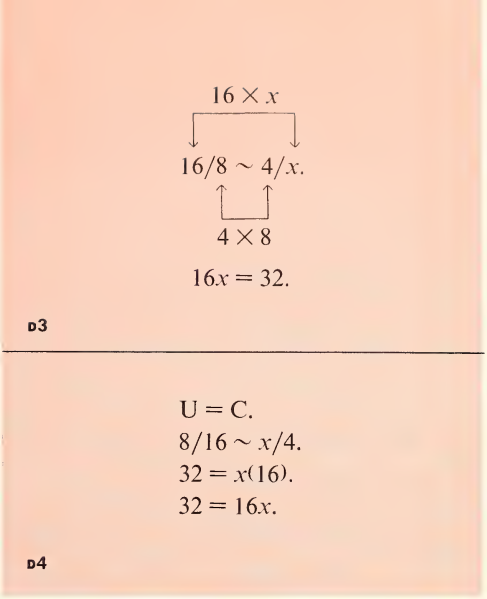
Now you must find the solution set of $16x = 32$. You can think of 16 times x, or $16x$, as 16 equal groups. Then you also must think of 32 as 16 equal groups. You need to find the number in each of the 16 groups. Markers will help you find this number.

These markers represent a group of 32 objects. The 32 markers are to be divided into 16 equal groups.



Notice how each new group has been started. One marker has been put in each of the 16 groups.

Another marker has been put in each of the groups. All 32 markers have been used. There are 2 markers in each group. To find the number of members in 16 equal groups, you can divide 32 by 16.



F To find the replacement for x that satisfies $16x = 32$, you divide 32 by what number?

G In $16x = 32$, replace x by 2. Do you obtain $16(2) = 32$? Is $16(2) = 32$?

H Is $16/8 \sim 4/2$ a true statement?

I Are $16/8$ and $4/2$ members of the same proportional relation?

Now you know that the rate pairs $16/8$ and $4/2$ represent the same rate. You also know that the rate pair $4/2$ represents the rate of 4 plants for 2 dollars. Therefore, you can say:

4 of the plants cost 2 dollars.

J Read the problem in D1 again. This time use the rate pair $8/16$ to represent the rate. What does the first component refer to? What does the second component refer to?

K To find how many dollars 4 of the plants cost, you need to find another rate pair that is equivalent to $8/16$. Why will you use 4 as the second component of the new rate pair?

L Look at D4. Does $8/16 \sim x/4$ express a condition for the problem? Why is x used as the first component of the new rate pair? For what is x a variable?

You obtain $32 = x(16)$ from $8/16 \sim x/4$ by finding the product of 8 and 4 and the product of x and 16. The product of 8 and 4 is 8×4, or 32.

M $x(16)$ is the product of x and 16. Is $16x$ also the product of x and 16?

In working with conditions for equivalence such as $a/b \sim c/d$, you find the product of a and d and the product of c and b. The product of a and d is ad or da. The product of c and b is cb or bc.

N Explain how you find the solution set of $32 = 16x$. Tabulate $\{x|32 = 16x\}$.

O First tabulate $\{x|8/16 \sim x/4\}$. Then tabu-
? late $\{x|16/8 \sim 4/x\}$. Does the sentence $\{x|8/16 \sim x/4\} = \{x|16/8 \sim 4/x\}$ express a true statement?

P If the first component of the first rate pair refers to a number of plants, then the first component of the second rate pair must refer to what?

Q If 16 plants cost 8 dollars, how many dollars did 4 of the plants cost?

R The solution set of $8/16 \sim x/4$ is the same as the solution set of $16/8 \sim 4/x$. May you use either of these conditions for the problem in D1?

The problem in D1 was solved by finding the solution set of a condition for equivalence. Now you will learn more about finding solution sets of conditions for equivalence.

A Look at D5. How is $480 = 15x$ obtained from $48/15 \sim x/10$?

The condition $15x = 480$ can also be obtained from $48/15 \sim x/10$. The solution set of $15x = 480$ is the same as the solution set of $480 = 15x$. First you will find the replacement for x that satisfies $15x = 480$.

B $15x = 480$ indicates that there are 15 groups, each containing a number represented by x. What is the total number in all 15 groups? What computation is needed to find the number in each group?

C What replacement for x satisfies $15x = 480$?

D Tabulate $\{x|15x = 480\}$.

E Tabulate $\{x|48/15 \sim x/10\}$. Why is this solution set the same as the solution set you tabulated for exercise D?

F Name the rate pair that has a second component of 10 and is equivalent to 48/15.

G Now name the rate pair that has a first com-
? ponent of 10 and is equivalent to 15/48. Write a sentence that expresses the statement that these two rate pairs are equivalent.

H Read the open sentence in D6. How do you obtain $17x = 4284$ from $x/63 \sim 68/17$? How do you find the solution set of $17x = 4284$?

I Tabulate $\{x|x/63 \sim 68/17\}$.

J Name the rate pair that has a second component of 63 and is equivalent to 68/17.

K Name the rate pair that has a first component of 63 and is equivalent to 17/68.

Sixteen plants cost 8 dollars. At this rate, 4 of the plants cost how many dollars?

D1

$U = C.$
$48/15 \sim x/10.$
$480 = 15x.$
$15x = 480.$

D5

$U = C.$
$x/63 \sim 68/17.$

D6

Now you know how to find the solution set of a condition for equivalence by using the definition of equivalent ordered pairs.

On your own

Use the definition of equivalent ordered pairs to help find the solution of each condition expressed below. Tabulate the solution set of each condition. U = C.

1 $25/10 \sim 15/x$.
2 $x/3 \sim 10/5$.
3 $7/21 \sim x/3$.
4 $9/15 \sim x/50$.
5 $1/2 \sim x/100$.
6 $13/26 \sim 5/x$.
7 $x/90 \sim 21/14$.
8 $x/1 \sim 55/11$.
9 $50/x \sim 1/5$.
10 $x/9 \sim 40/12$.
11 $1/3 \sim 8/x$.
12 $x/16 \sim 7/4$.
13 $13/9 \sim x/18$.
14 $x/25 \sim 40/100$.
15 $15/18 \sim 50/x$.
16 $4/9 \sim x/99$.
17 $x/26 \sim 6/39$.
18 $12/x \sim 15/40$.
19 $2/30 \sim x/75$.
20 $12/30 \sim 16/x$.
21 $14/x \sim 1/6$.
22 $x/85 \sim 12/10$.
23 $25/150 \sim 15/x$.
24 $72/x \sim 30/5$.

KEEPING SKILFUL

Find the sum.

1 112, 340, 5621, 569
2 3468, 1923, 4082, 601, 1977
3 12574, 81930, 64521
4 67689, 88677, 3594, 38624
5 36241, 132, 9876, 429, 29748

Find the difference.

6 9214, 5927
7 7345, 6216
8 2001, 798
9 82345, 17689
10 30026, 23472
11 92845, 41572

Multiply

12 32745 by 7.
13 49 by 68.
14 139 by 19.
15 451 by 367.
16 294 by 819.
17 635 by 708.

Divide

18 3684 by 38.
19 7805 by 45.
20 25572 by 36.
21 13592 by 321.
22 41965 by 235.
23 155480 by 598.

CHECKING UP

If you have trouble with these two tests, you should refer to lessons 41, 42, and 43.

Test 44

1 Write a ratio for a rate pair that represents the rate of 6 miles per hour.
2 Write a ratio for a rate pair that represents the rate of 2 apples for 7 cents.
3 Write a ratio for a rate pair that represents the rate of 3 eggs to 2 cups of milk.

Replacements for x, y, m, and n are named in exercises 4 through 7. For each exercise, write the sentence that expresses the statement you obtain when you use these replacements in $x/y \sim m/n$.

	x	y	m	n
4	3	7	12	24
5	2	9	4	18
6	5	8	10	80
7	4	6	16	24

8 Which of the sentences that you wrote for exercises 4 through 7 express true statements?
9 Now write the sentences that express the statements you obtain when you use the replacements named above in $xn = my$.
10 Which of the sentences that you wrote for exercise 9 express true statements?

Test 45

In each of exercises 11 through 14, one rate pair named is not equivalent to the other rate pairs named. Which pair is it?

11 2/5, 4/10, 6/12, 8/20
12 60/90, 30/60, 30/45, 2/3
13 8/14, 4/7, 12/28, 24/42, 16/28
14 6/4, 12/8, 9/6, 15/9, 3/2
15 Name the rate pair that is equivalent to 2/5 and has a first component of 6.
16 Name the rate pair that is equivalent to 15/9 and has a second component of 21.
17 Tabulate $\{x | x/12 \sim 132/22\}$. U = C.
18 Tabulate $\{x | 45/x \sim 72/168\}$. U = C.
19 Tabulate $\{x | x/40 \sim 15/100\}$. U = C.

Problems involving conditions for equivalence

You know that a rate pair is a member of a proportional relation. You also know how to find a rate pair that is equivalent to a given rate pair. You will use this knowledge to find solutions of conditions for problems.

Read the problem in D1. The exercises that follow will help you develop a compound condition for the problem. $U = C \times C$.

A For each 6 blocks that Jerry walks, Tim walks 8 blocks. Use 6 as the first component of a rate pair that represents this comparison. Use 8 as the second component. To what does the 8 refer?

6/8

B You know one rate pair that represents the comparison. You want to find rate pairs equivalent to 6/8. Use x as a variable for the number of blocks that Jerry walks. Why do you use x as a variable for the first component of each rate pair you are to find?

x

C Use y as a variable for the second component. What does y refer to?

y

D Why do you use x/y, and not y/x, to represent each rate pair equivalent to 6/8?

x/y

E Does the sentence at the right express a simple condition for the problem? What does the condition require about each rate pair you are to find?

$6/8 \sim x/y$.

Jerry and Tim compared their walking speeds. They found that Jerry walks 6 blocks while Tim walks 8 blocks. When Jerry has walked 24 blocks, how many blocks will Tim have walked?

D1

{3/4, . . .}

D2

F Study the tabulation of a proportional relation in D2. Is this set an infinite set? Name six rate pairs that are members of this set. Is each rate pair that you named a member of $\{(x, y) | 6/8 \sim x/y\}$? Explain your answer.

G What comparison does each member of the solution set of $6/8 \sim x/y$ represent?

H Read the problem in D1 again. What are you asked to find?

I How do you know that the rate pair you are to find must satisfy $x = 24$?

$x = 24$.

J Does the sentence at the right express a compound condition for the problem?

$6/8 \sim x/y \wedge x = 24$.

K Now you can find $\{(x, y) | 6/8 \sim x/y \wedge x = 24\}$. What is the first component of the rate pair that satisfies $6/8 \sim x/y \wedge x = 24$? Explain your answer.

REMINDER

Each solution of a compound condition that involves "and" must satisfy each of the simple conditions that make up the compound condition.

See lesson 24, page 90.

L What rate pair satisfies $6/8 \sim x/y$ and has 24 as a first component?

M Tabulate $\{(x, y) | 6/8 \sim x/y \wedge x = 24\}$.

N How many rate pairs are in the set you tabulated for exercise M? What does the first component of the member of the solution set refer to? What does the second component refer to?

Since you know the solution of the compound condition for the problem in D1, you can use the solution to get the answer to the problem.

When Jerry has walked 24 blocks, Tim will have walked 32 blocks.

O Verify the answer. You have found that when Jerry has walked 24 blocks, Tim will have walked 32 blocks. Does 24/32 represent the same comparison as 6/8?

To solve the problem in D1, you developed a compound condition for the problem. From the solution set of the simple condition $6/8 \sim x/y$, you obtained some of the rate pairs equivalent to 6/8. You used the other simple condition, $x = 24$, to select the rate pair that gave you the answer to the problem. There is another way to find the answer to the problem in D1. This way, in which you use a simple condition instead of a compound condition, is explained in exercises P through T.

P You know that 24 is the first component of the rate pair whose second component you are to find. If you replace x by 24 in $6/8 \sim x/y$, do you obtain $6/8 \sim 24/y$?

Q Is $6/8 \sim 24/y$ a condition for the problem in D1?

R How can the condition $6y = 192$ be obtained from $6/8 \sim 24/y$?

S What number satisfies $6y = 192$? The universe for y is C.

T Does the number that satisfies $6y = 192$ also satisfy $6/8 \sim 24/y$? Explain your answer.

At an automatic car wash, 20 cars can be washed in 60 minutes. At this rate, how many cars can be washed in 15 minutes?

D3

Now you can give the answer to the problem.

When Jerry has walked 24 blocks, Tim will have walked 32 blocks.

Problems like the one in D1 always give you the numbers that are the components of one rate pair and the number that is one component of an equivalent rate pair. You are to find the number that is the other component of this equivalent rate pair.

Read the problem in D3. You will develop a simple condition for this problem. U = C.

A Does the rate pair named below at the right represent the rate at which cars can be washed? What does the first component of the rate pair refer to? What does the second component refer to? 20/60

B What are you asked to find? How do you know that 15 is the second component of the rate pair whose first component you are to find? For what is w a variable? $w/15$

C Does the sentence below at the right express a simple condition for the problem? What does this condition tell you about $w/15$? $20/60 \sim w/15$.

D Explain how $300 = 60w$ can be obtained from $20/60 \sim w/15$.

E What number satisfies $300 = 60w$?

F Does the number that satisfies $300 = 60w$ also satisfy $20/60 \sim w/15$?

Now you can give the answer to the problem in D3.

5 cars can be washed in 15 minutes.

G Verify the answer. 5 cars can be washed in 15 minutes. Does 5/15 represent the same rate as 20/60?

Read the problem in D3 again. You can develop another condition for this problem. U = C.

H Why does the rate pair named below at the right also represent the rate at which cars can be washed? What does the first component of this rate pair refer to? What does the second component refer to? 60/20

I Is 15 the first component of the rate pair whose second component you are to find? Explain your answer. For what is w a variable? $15/w$

J Does the sentence at the right below express a condition for the problem? What does the condition tell you about $15/w$? $60/20 \sim 15/w$.

K Tabulate $\{w|60/20 \sim 15/w\}$. What does the member of the solution set refer to?

L Why can you use either $20/60 \sim w/15$ or $60/20 \sim 15/w$ as a condition for the problem in D3?

Read the problem in D4. You will develop a simple condition for this problem. U = C.

A Jim paid 18 cents for each package of paper. Does the rate pair named at the right below represent this rate? What does the first component of the rate pair refer to? What does the second component refer to? 18/1

B What are you asked to find in the problem? How do you know that 3 is the second component of the rate pair you are to find? For what is x a variable? $x/3$

At an automatic car wash, 20 cars can be washed in 60 minutes. At this rate, how many cars can be washed in 15 minutes?

D3

Jim bought 3 packages of notebook paper. He paid 18 cents for each package. How many cents in all did Jim pay for the 3 packages of paper?

D4

In an election for class secretary, Janet received 39 votes, and Sue received 13 votes. Janet received how many times as many votes as Sue?

D5

C Does the sentence below at the right express a simple condition for the problem? $18/1 \sim x/3$.

D Tabulate $\{x|18/1 \sim x/3\}$.

Now you can give the answer to the problem.

Jim paid 54 cents for the 3 packages of paper.

E Does 54/3 represent the same rate as 18/1?

Read the problem in D5. U = C.

F You know that Janet received 39 votes and that Sue received 13 votes. Does the rate pair named at the right represent this comparison? What does the first component of the rate pair refer to? What does the second component refer to? 39/13

G You are asked to find how many times as many votes Janet received as Sue. This means that you must find how many votes Janet

Two inches on a map represent a distance of 35 miles. How many miles are represented by 4 inches on this map?

$4/s \sim 2/35$.

D6

received for each vote that Sue received. Use x as a variable for the number of votes that Janet received. Why do you use $x/1$, and not $1/x$?

$x/1$

H Does the sentence below at the right express a condition for the problem? What does the condition tell you about $x/1$?

$39/13 \sim x/1$.

I Tabulate $\{x|39/13 \sim x/1\}$.

You can use the solution to get the answer to the problem.

Janet received 3 votes for each 1 vote that Sue received. Thus, Janet received three times as many votes as Sue.

Read the problem in D6. A sentence that expresses a condition for the problem is also given. U = C.

J In $4/s \sim 2/35$, what does the first component of 2/35 refer to? What does the second component of 2/35 refer to? For what is s a variable?

K Tabulate $\{s|4/s \sim 2/35\}$.

L Give the answer to the problem.

M For the problem in D6, you can also use the condition $s/4 \sim 35/2$, the condition $2/35 \sim 4/s$, or the condition $35/2 \sim s/4$. Explain why you can do this.

In this lesson you have learned to write sentences that express conditions for rate and comparison problems. You have also used solutions of conditions for equivalence to get the answers to problems.

On your own

For each problem, first write a sentence that expresses a condition for the problem. You may use any letter as a name for the variable. Then tabulate the solution set and give the answer to the problem. U = C.

1 Most of the known planets in our solar system have one or more satellites, or moons. Jupiter has 6 times as many satellites as Mars. Jupiter has 12 satellites. How many satellites does Mars have?

2 What is the cost of 5 cans of soup, if 3 cans of this soup cost 54¢?

3 Allen received 98 votes in a student council election. For each 2 votes that Allen received, Roger received 3 votes. How many votes did Roger receive in the election?

4 Last year the Gibson Auto Company sold 72 used cars. The company sold 8 used cars for each 6 new cars that they sold. How many new cars did the company sell last year?

5 Lois has 95 coins from England and 19 coins from France in her collection of foreign coins. She has how many times as many English coins as French coins?

6 Jerry was paid $6 for delivering 24 grocery orders. At this rate, how many dollars would he be paid for delivering 16 orders?

7 The heat energy of food is measured in calories. There are 4 times as many calories in a piece of mince pie as in a glass of milk. There are 170 calories in a glass of milk. How many calories are there in a piece of mince pie?

8 Mrs. Adams bought 18 ears of sweet corn priced at 6 ears for 29¢. How many cents did she pay for the sweet corn?

9 In the Canadian Senate in September, 1958, Ontario had 12 senators for every 2 that Prince Edward Island had. Ontario had 24 senators. How many did Prince Edward Island have?

10 Last year Bob scored 192 points during the basketball season. For each 8 points that Bob scored, John scored 7 points. How many points did John score?

11 The distance between two cities is represented by 9 inches on a road map. Two inches on the map represent an actual distance of 28 miles. What is the actual distance between the two cities?

12 The population density of a land area is the average number of persons per square mile. The population density of Canada is approximately 4 persons. The land area of Canada is 3,560,238 square miles. Approximately how many persons live in Canada?

13 The population density of Australia is about 3 persons. About 8,925,000 persons live in Australia. The land area of Australia is about how many square miles?

14 An office building that was 76 feet high above the ground was demolished. Another office building that was 228 feet high above the ground was built in its place. The height of the new building was how many times the height of the old building?

15 Alice saves $4 out of each $9 that she earns. If she continues to save at this rate, how many dollars will she have saved when she has earned $216?

16 There were 16 times as many motor vehicles in Canada in 1935 as there were in 1914. In 1914, there were 74,246 motor vehicles. How many motor vehicles were there in Canada in 1935?

17 A certain printing press can print 237,500 newspapers in 5 hours. At this rate, the press can print how many newspapers in 3 hours?

18 In 1956, the population density of Saskatchewan was 4 persons. About 880,000 persons lived in Saskatchewan in 1956. The area of Saskatchewan is about how many square miles?

APPLYING MATHEMATICS

The problems in this lesson include terms often used in business. You will use the solutions of conditions for equivalence to get the answers to these problems.

For each problem, write a sentence expressing a condition for the problem. Then tabulate the solution set of the condition and give the answer to the problem. U = C.

1 Persons who have checking accounts are usually charged a small amount for each check that they write. This amount is called a *service charge*. If a bank charges 8¢ per check, what is the service charge for 12 checks?

2 During one month Mrs. Harris wrote 13 checks on a local bank. The total service charge was 91¢. What was the service charge per check?

3 The City Bank charges $6 per year for each $100 that it lends. The charge for the use of money is called *interest*. If Mr. Peters borrows $700, how much interest must he pay the bank at the end of one year?

4 Miss Owens borrowed $980 from the Merchants' National Bank and paid $49 interest at the end of one year. How much interest per year did she pay for each $100 that she borrowed?

5 A store reduced the price of a sofa $3 for each $75 of the regular price. When a regular price is reduced, the amount of the reduction is called the *discount*. The regular price of the sofa was $375. The discount amounted to how many dollars?

6 Mrs. Carter received a $20 discount on a television set that was regularly priced at $250. How many dollars' discount did she receive for each $100 of the regular price?

7 Mr. Baker, a salesman, receives $16 for each $80 worth of merchandise he sells. The

amount he is paid is called his *commission*. How much commission would Mr. Baker be paid if he sold \$560 worth of merchandise?

8 Mr. Thompson receives a commission of \$24 for each \$240 worth of merchandise that he sells. At this rate, how many dollars' worth of merchandise does he have to sell to earn a commission of \$144?

9 After deductions for taxes and insurance are made from Mr. Hill's monthly salary, he receives a check for \$436. The amount of the check is his *take-home pay*. What is Mr. Hill's yearly take-home pay?

10 Mrs. Smith's take-home pay amounts to \$4 out of each \$5 she earns. The amount of her yearly take-home pay is \$5276. What is the amount of her yearly salary?

11 During the summer Tom works in a sporting goods store. One week he sold \$288 worth of merchandise and was paid a commission of \$16. At this rate, how many dollars' worth of merchandise did he have to sell to earn one dollar of commission?

12 Jay bought a camera on sale and received a discount of \$2 for each \$10 of the regular price. The amount of the discount he received was \$8. What was the regular price of the camera?

13 The Copper City National Bank charges \$5 interest per year for each \$100 it lends. Mr. Harper borrowed some money from this bank and paid \$33 interest at the end of one year. How many dollars had Mr. Harper borrowed?

14 Mr. Martin's yearly salary is \$8472. What is his monthly salary?

15 Mr. Johnson receives a commission of \$18 for each \$200 worth of merchandise that he sells. How many dollars' worth of merchandise did he sell if his commission was \$261?

16 Mr. Morgan borrowed \$1280 from his bank and paid \$64 interest at the end of one year. How much interest did his bank charge per year for each \$100 he borrowed?

17 Miss Moore's monthly salary is \$360. Her monthly take-home pay is \$288. Her take-home pay amounts to how many dollars for each \$10 of her salary?

18 Betty received a discount of \$20 on a phonograph that was regularly priced at \$200. For each \$10 of the regular price, how many dollars' discount did she receive?

19 Virginia works in a department store. She receives a commission of \$3 for each \$45 worth of merchandise she sells. At this rate, what will her commission be if she sells \$405 worth of merchandise?

KEEPING SKILFUL

For exercises 1 through 4, the universe for (m, n) is tabulated below. For each condition, tabulate the solution set and make graphs of the universe and the solution set.

Universe for $(m, n) = \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (1, 0), (1, 1), (1, 2), (1, 3), (1, 4), (2, 0), (2, 1), (2, 2), (2, 3), (2, 4), (3, 0), (3, 1), (3, 2), (3, 3), (3, 4)\}$.

1 $m + n > 3$.

2 $n + 1 = 4$.

3 $m + 1 = n$.

4 $m + 3 < 5$.

Tabulate the solution set of each condition expressed in exercises 5 through 12. U = N.

5 $28r = 1008$.

6 $16(205) = s$.

7 $507 \div 39 > t$.

8 $168w = 42672$.

9 $279(426) = h$.

10 $f = 23(1264)$.

11 $43g = 16598$.

12 $w > 76(92)$.

Tabulate the solution set of each condition expressed below. U = N × N.

13 $x + y < 9 \wedge 5x = 30$.

14 $2x > y \wedge x + y = 18$.

15 $x + 5 = y \wedge x + 13 < 22$.

16 $15x = 345 \wedge x - 18 > y$.

17 $x + 4 > y \wedge x + y = 12$.

Per cent

You know that rate pairs are used to solve rate and comparison problems. In this lesson you will learn about a special kind of rate pair that has 100 as a second component.

A During a money-raising drive, the students in five rooms in Morgan School decided upon their goals. The picture in D1 represents the goals and the results of the drive. For each room, the stack of silver dollars at the left represents the number of dollars the students in the room actually raised. The stack of silver dollars at the right represents the goal. How much money did each room raise? What was its goal?

B You can use a rate pair to represent the comparison of the money raised by each room with the goal for that room. The rate pairs named in D2 represent these comparisons. To what does the first component of each rate pair refer? To what does the second component refer? Use D1 to help you answer these questions.

Use the rate pairs named in D2 to help you answer the questions in exercises C, D, and E.

C Which room raised more money than its goal? Would you say that this room did better with respect to its goal than each of the other four rooms? Explain your answer.

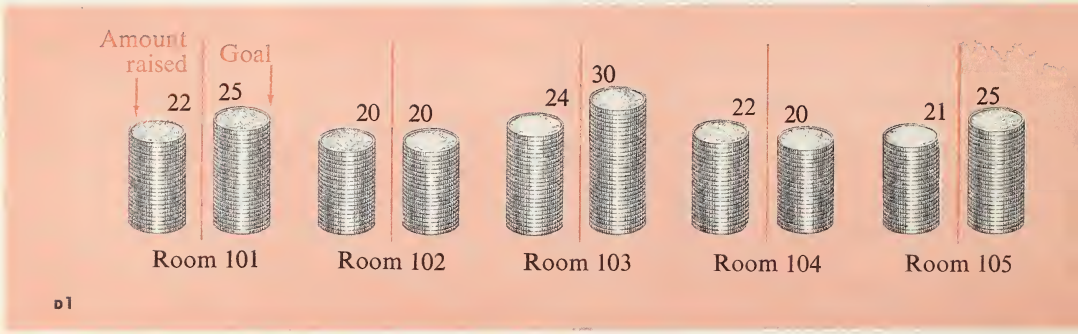
D How do you know that Room 101 did better than Room 105?

E Why is it hard to decide whether Room 103 or Room 105 did better?

It is easier to compare the performances of the different rooms when all the comparisons are represented by rate pairs that have the same second components. These second components can be any counting number. However, comparisons are often represented by rate pairs in which the second components are 100.

To help you compare the performances of the five rooms, you can use rate pairs that are equivalent to the rate pairs named in D2 and that have second components of 100.

F Room 101 raised \$22 of its goal of \$25. Does 22/25 represent this comparison? Imagine that the goal had been \$100 and that the room did just as well toward reaching this goal. You can use the first condition expressed in D3 to help you find how many dollars the room would have had to raise. For what is n a variable?



D1

Room 101	22/25
Room 102	20/20
Room 103	24/30
Room 104	22/20
Room 105	21/25

D2

U = C.

Room 101	$22/25 \sim n/100$.
Room 102	$20/20 \sim n/100$.
Room 103	$24/30 \sim n/100$.
Room 104	$22/20 \sim n/100$.
Room 105	$21/25 \sim n/100$.

D3

Room 101	$22/25 \sim 88/100$.
Room 102	$20/20 \sim 100/100$.
Room 103	$24/30 \sim 80/100$.
Room 104	$22/20 \sim 110/100$.
Room 105	$21/25 \sim 84/100$.

D4

G Suppose that each of the other rooms had set a goal of \$100 and had done as well in reaching the \$100 goal as it did in reaching the actual goal. Could you use the conditions expressed in D3 to help you find the amounts they would have had to raise? For what is *n* a variable in each condition?

H How can you find the replacement for *n* that satisfies each condition?

I Now look at D4. Does each of the sentences express a true statement?

J Does each statement expressed in D4 contain a rate pair whose second component is 100? Explain why it is now easier to decide which room did best, which room did second best, and so on.

K List the names of the five rooms in the order of their performance. Start with the name of the room that did best.

A rate pair that represents a comparison and has a second component of 100 is a *per cent*.

L Look again at D4. Is the rate pair 22/25 a per cent? Is the rate pair 88/100 a per cent?

M Is 88 per cent the same rate pair as 88/100?

The symbol at the right below is often used to express a rate pair that represents a comparison and that has a second component of 100. This symbol is the *per cent sign*. D5 shows how to write and read a ratio using the per cent sign.

%

N Read each ratio: 88%, 100%, 80%, 110%.

O Look at D6. For each room, is the rate pair named in column B equivalent to the rate pair named in column A?

P For each room, is the rate pair named in column C the same as the rate pair named in column B?

88 % — "Eighty-eight per cent"

D5

	A	B	C
Room 101	22/25	88/100	88%
Room 102	20/20	100/100	100%
Room 103	24/30	80/100	80%
Room 104	22/20	110/100	110%
Room 105	21/25	84/100	84%

D6

per cent (pər sent). A rate pair that represents a comparison and that has a second component of 100.

Q If you were to list the per cents expressed in column C in order, beginning with the name of the least, you would start with 80%. What would be the greatest per cent named in your list?

Notice how easy it is to compare per cents. Since the second component of each per cent is 100, you can use just the first components to decide if a given per cent is greater than or less than another per cent.

R What is the first component of 80%? Of 81%? Is 80% less than 81%? Is 81% greater than 80%?

S Look again at column C in D6. Which of the rate pairs named has a first component greater than 100?

T Sometimes it is convenient to say that a rate pair is "greater than 100%." This means that the first component of the rate pair is greater than 100, and the second component is 100. What does it mean to say that a rate pair is 100%?

For each rate pair expressed below, tell whether it is greater than 100%, less than 100%, or equal to 100%.

U 95% **W** 100% **Y** 115%
V 137% **X** 45% **Z** 53%

Now you will work with other rate pairs that can be thought of as per cents. Look at D7. The chart shows Susan's performance on five mathematics tests.

A Write a ratio that expresses the rate pair for each test listed in D7. The first component of each rate pair should refer to the number of questions Susan answered correctly. The second component should refer to the number of questions on each test.

B For each of the five rate pairs you named in exercise A, find an equivalent rate pair that has a second component of 100.

	A	B	C
Room 101	22/25	88/100	88%
Room 102	20/20	100/100	100%
Room 103	24/30	80/100	80%
Room 104	22/20	110/100	110%
Room 105	21/25	84/100	84%

D6

Number of questions on test ↓

Number of questions answered correctly ↓

Test 1	18	20
Test 2	12	15
Test 3	36	40
Test 4	28	28
Test 5	47	50

D7

C Now express each rate pair by using the per cent sign.

D You gave Susan's grade on each test as a per cent. Did Susan receive any grade greater than 100%? Would such a grade have been possible? Explain your answer.

E On which test did Susan do best? On which test did she do least well? Which rate pairs did you use to make these decisions? Which components did you use in making your decisions?

In this lesson you have learned that a per cent is a rate pair that represents a comparison and that has a second component of 100.

On your own

Express the rate pairs named in exercises 1 through 9 by using the per cent sign.

1 78/100	4 10/100	7 99/100
2 125/100	5 200/100	8 100/100
3 1/100	6 9/100	9 66/100

Write another name for each of the rate pairs expressed in exercises 10 through 18.

10 120%	13 71%	16 1000%
11 2%	14 5%	17 98%
12 500%	15 90%	18 43%

For each rate pair named in exercises 19 through 27, find an equivalent rate pair that has a second component of 100. Then express each rate pair by using the per cent sign.

19 4/5	22 90/30	25 10/1000
20 9/50	23 75/150	26 1000/10
21 32/64	24 96/32	27 5/4

At the beginning of the school year, several clubs in Logan School held membership drives. The chart in D8 shows how many new members each club hoped to have, and how many new members it actually did have at the end of the drive.

28 For each club listed in D8, write a ratio expressing the rate pair that compares the actual number of new members with the goal. The first component of the rate pair should refer to the number of new members obtained, and the second component should refer to the goal.

29 For each rate pair that you expressed for exercise 28, find an equivalent rate pair that has a second component of 100. Then express each rate pair by using the per cent sign.

30 Which per cents that you named for exercise 29 are greater than 100%? Less than 100%? Equal to 100%?

31 Which club did best in the membership drive? Which club did least well?

32 List the names of the clubs in the order of their performance. Start with the one that did best. Also express the per cent for each club.

33 If the Space Club had obtained 3 more members, would it have done better than the Pep Club? Explain your answer.

	New members	Membership goal
Camera Club	35	35
Drama Club	23	25
Glee Club	32	40
Mathematics Club	33	30
Pep Club	63	50
Space Club	18	15

D8

SPECIAL CHALLENGE

Each exercise below describes the intersection of two sets. If possible, make a sketch to represent the intersection. If a sketch cannot be made to represent the intersection, explain why it is impossible.

A The intersection of a segment and a triangle is the empty set.

B The intersection of a segment and a simple closed curve is a set with four members.

C The intersection of a line and a circle is a set with three members.

D The intersection of a triangle and a simple closed curve is a set with eight members.

E Points A and B are in the interior of a simple closed curve. The intersection of segment AB and the simple closed curve is a set with five members.

F The intersection of two planes is a set with one member.

G The intersection of a plane and a segment is an infinite set.

H The intersection of three planes is a set with one member.

I The intersection of a circle and a triangle is a set with five members.

Problems involving per cents

You have learned that per cents are rate pairs that represent comparisons and that have second components of 100. In this lesson you will solve comparison problems by using what you have learned about per cents. U = C.

Read the problem in D1. You will develop the condition for the problem.

A The rate pair named at the right below represents the comparison of the number of students who live more than five blocks from Lee School with the number of students enrolled. What does the first component of the rate pair refer to? What does the second component refer to?

540/720

B The question in the problem asks you to find what per cent 540 is of 720. This means that you are to find how many students out of each 100 students enrolled live more than five blocks from school. How do you know that 100 is the second component of the rate pair whose first component you are to find? This rate pair is expressed at the right. For what is x a variable?

$x/100$

C What does the condition expressed below tell you about the rate pair whose first component you are to find?

$540/720 \sim x/100.$

D How do you obtain $54000 = 720x$ from $540/720 \sim x/100$?

540 of the 720 students enrolled at Lee School live more than five blocks from school. What per cent of the students live more than five blocks from school?

D1

Jack took a mathematics test that contained 40 questions. He answered 85% of the questions correctly. How many of the 40 questions did he answer correctly?

D2

Last summer Doris saved 54% of her earnings. She saved $81. How many dollars did Doris earn last summer?

D3

E What number satisfies $54000 = 720x$?

F Tabulate $\{x \mid 540/720 \sim x/100\}$.

G Does the rate pair 75/100 tell you that, out of each 100 students enrolled at Lee School, 75 students live more than five blocks from school? Why is the rate pair 75/100 the same as 75%?

Now you can answer the question asked in the problem.

75% of the students enrolled at Lee School live more than five blocks from school.

H Read the problem in D2. What are you asked to find?

I You know that Jack answered 85% of the 40 questions correctly. This means that if Jack had done as well on a test that contained 100 questions, he would have answered 85 questions correctly.

J Is the rate pair named below at the right the same rate pair as 85%? What does the first component of this rate pair refer to? What does the second component refer to?

85/100

K The test contained 40 questions. How do you know that 40 is the second component of the rate pair whose first component you are to find? This rate pair is expressed at the right. For what is m a variable?

$m/40$

L What does the condition expressed below at the right tell you about the rate pair whose first component you are to find?

$85/100 \sim m/40.$

M Tabulate $\{m \mid 85/100 \sim m/40\}$.

Now you can use the solution of $85/100 \sim m/40$ to get the answer to the problem.

Jack answered 34 of the 40 questions correctly.

N Read the problem in D3. What are you asked to find?

O How do you know that Doris saved 54 dollars out of each 100 dollars she earned?

P Is 54/100 the same rate pair as 54%? To what does the first component of 54/100 refer? To what does the second component refer? What comparison is represented by 54/100?

54/100

Q You know that Doris saved 81 dollars. How do you know that 81 is the first component of the rate pair whose second component you are to find? For what is w a variable?

$81/w$

R What does the condition expressed below tell you about the rate pair whose second component you are to find?

$54/100 \sim 81/w.$

S Tabulate $\{w \mid 54/100 \sim 81/w\}$.

A Harold saves \$6 out of each \$10 that he earns. The money Harold saves is what per cent of the money he earns?

B 6 is what per cent of 10?

D4

---

6 is x% of 10.

D5

T Use the solution of $54/100 \sim 81/w$ to get the answer to the problem.

U Could the per cent used to compare the number of dollars saved with the number of dollars earned be greater than 100%? Explain your answer.

Now you will find the answers to other problems that involve per cents.

A Read problem A in D4. If you do not think about dollars, you can think of the question in problem A as it is written in problem B in D4. To find what per cent 6 is of 10, you must find the first component of a rate pair that has a second component of 100 and that is equivalent to 6/10.

B Is $6/10 \sim x/100$ the condition for problem A? Is it also the condition for problem B in D4?

C The condition expressed in D5 is also a condition for both problem A and problem B. How do you know that "6 is x% of 10" is the same condition as $6/10 \sim x/100$?

D What number satisfies $6/10 \sim x/100$? Does this number also satisfy "6 is x% of 10"? How do you know?

E Tabulate $\{x \mid 6/10 \sim x/100\}$.

F The money Harold saves is what per cent of the money he earns? 6 is what per cent of 10?

G Read problem C in D6. If you do not think about pages, you can think of the question in problem C as it is written in problem D in D6. To find 30% of 320, you must find the first component of a rate pair that has a second component of 320 and that is equivalent to 30/100.

H Is $30/100 \sim s/320$ the condition for problem C? Is it also the condition for problem D in D6?

I Is "30% of 320 is s" also the condition for problem C? For problem D?

J What number satisfies $30/100 \sim s/320$? Explain why this number also satisfies "30% of 320 is s."

K How many pages did Dolores read that evening? 30% of 320 is what number?

L Read problem E in D7. If you do not think about dollars, you can think of the question in problem E as it is written in problem F in D7. To find this number, you must find the second component of a rate pair that has a first component of 270 and is equivalent to 5/100.

M Is $5/100 \sim 270/n$ the condition for problem E? Is it also the condition for problem F in D7?

N Is "270 is 5% of n" the same condition as $5/100 \sim 270/n$?

O What number satisfies both $5/100 \sim 270/n$ and "270 is 5% of n"?

P Last year, Mr. Roberts' take-home pay was how many dollars? 270 is 5% of what number?

Q Read the problem in D8. Is $12/8 \sim x/100$ the condition for the problem?

R Why is 12/8, and not 8/12, used in the condition?

S Tabulate $\{x \mid 12/8 \sim x/100\}$.

T Answer the question asked in the problem.

U Read the problem in D9. 125 is which component of the rate pair that you are given?

C Dolores was reading a 320-page book. She read 30% of the pages in one evening. How many pages did she read that evening?

D 30% of 320 is what number?

D6

E Last year Mr. Roberts saved 5% of his take-home pay. He saved $270. What was his take-home pay?

F 270 is 5% of what number?

D7

June sold 8 tickets to the school play, and Martin sold 12 tickets. The number of tickets Martin sold is what per cent of the number of tickets June sold?

D8

125% of 32 is what number?

D9

V 32 is which component of the rate pair you are to find?

W Write a sentence that expresses the condition for the problem in D9.

X Tabulate the solution set of the condition.

Y Use the solution of the condition as the answer to the problem.

In this lesson you have learned to use per cents to solve problems.

On your own

For each problem, write a sentence that expresses the condition for the problem. You may use any letter to name the variable. Then

tabulate the solution set and answer the question asked in the problem. U = C.

1 A weather bureau reported that 40% of the days in April were cloudy. How many days in April were cloudy?

2 During a basketball game, David made 7 baskets out of 20 tries. The number of baskets David made was what per cent of the number of tries?

3 One year the Toronto Maple Leafs had 16 players on their hockey team. 25% of these players played centre ice. How many centre ice players did the Maple Leafs have that year?

4 20% of all the students at Long School are in the seventh grade. 92 students are enrolled in the seventh grade. How many students in all are enrolled at Long School?

5 Henry worked 15 mathematics problems and had correct answers for 100% of them. How many correct answers did he have?

6 During a fund-raising drive, the students in Cook School raised $52, and the students in Fisher School raised $65. The number of dollars raised by the students in Cook School was what per cent of the number of dollars raised by the students in Fisher School?

7 Use the information given in problem 6 to answer the following question: The number of dollars raised by the students in Fisher School was what per cent of the number of dollars raised by the students in Cook School?

8 Bill sold his bicycle for $39. This was 65% of the price he had paid for it. How many dollars had he paid for the bicycle?

9 During a sale, Marilyn paid $30 for a coat that was regularly priced at $40. The sale price of the coat is what per cent of the regular price?

10 In 1940, a British Columbia newspaper estimated that the province's population by 1960 would be 175% of its population in 1940. In 1940, the population was 800,000. What was the estimated population of British Columbia for 1960?

11 Mr. Becker's monthly take-home pay amounts to $340. His monthly salary amounts to $425. His monthly take-home pay is what per cent of his monthly salary?

12 5% of the science club members have paid their yearly dues of 50¢. Four members have paid their dues. How many members does the science club have?

13 The land area of Canada is about 3,560,000 square miles. 16% of this land is suitable for agriculture. About how many square miles of land are suitable for agriculture?

14 Mr. White sold his house for $17,250. The selling price was 138% of the price he originally paid for the house. How many dollars did Mr. White pay for his house?

15 An urban population is defined as the number of persons residing within the boundaries of cities, towns, and villages of 1,000 or more persons. In 1956 the total population of Canada was 16,080,791. The urban population was about 65% of the total population. What was the approximate urban population in 1956?

16 In 1956, the urban population of Manitoba was about 510,000. This was about 60% of the total population of the province. What was the approximate total population of Manitoba in 1956?

17 What number is 20% of 185?

18 150 is what per cent of 30?

19 What number is 16% of 425?

20 42 is 84% of what number?

21 174% of what number is 348?

22 47 is what per cent of 94?

23 What per cent of 47 is 94?

24 What number is 150% of 242?

25 1536 is 32% of what number?

APPLYING MATHEMATICS

In this lesson you will use per cents to solve problems that involve business arithmetic. You may want to refer to "Applying mathematics" on pages 184-185 for the meaning of some of the business terms.

For each problem, first write a sentence that expresses the condition for the problem. You may use any letter to name the variable. Then tabulate the solution set and answer the question asked in the problem. U = C.

1 Mr. Beck sold $2500 worth of merchandise. He received a commission of $200. His commission was what per cent of his sales?

2 In problem 1 you found what per cent the commission was of the total sales. This per cent of the total sales is the *rate of commission*. Mr. Monroe sold $5000 worth of merchandise. He received a commission of $450. What rate of commission did Mr. Monroe receive?

3 Mr. Martin's rate of commission is 4%. One month he sold $9650 worth of merchandise. What was the amount of his commission?

4 Miss Richardson gets a 5% commission on her total sales. One week her commission was $142. How many dollars' worth of merchandise had Miss Richardson sold that week?

5 Judy received a $5 discount on a dress that was regularly priced at $25. The discount was what per cent of the regular price?

6 In problem 5 you found what per cent the discount was of the regular price. This per cent of the regular price is the *rate of discount*. Mrs. Carey received a $45 discount on a television set that was regularly priced at $180. What was the rate of discount?

7 During a sale, Dobson's Music Store gave a discount of 20% on all merchandise. How much discount did Mr. Bailey receive on a phonograph that was regularly priced at $375?

8 A manufacturer gives a 2% discount to any buyer who pays for an order within 30 days. Mr. Jacobs paid for his order immediately and received a discount of $60. How many dollars' worth of merchandise had Mr. Jacobs bought?

9 Mr. Harvey borrowed $500 and paid $30 interest at the end of one year. The amount of money on which interest is paid is the *principal*. The interest that Mr. Harvey paid was what per cent of the principal?

10 In problem 9 you found what per cent the interest was of the principal. This per cent of the principal is the *rate of interest*. Unless stated otherwise, the rate of interest is given for one year. Mr. Evans paid $10 interest on $200 that he had borrowed. What rate of interest did he pay?

11 Mr. Howard borrowed $8500 and agreed to pay 4% interest. How many dollars of interest did he have to pay at the end of one year?

12 The First Federal Bank charged 5% interest on a loan to Mr. Taylor. After one year, Mr. Taylor paid the bank $69 interest. How many dollars had he borrowed?

13 Mr. Hanson's weekly take-home pay is 80% of his salary. His take-home pay amounts to $112. What is his weekly salary?

14 Mr. Adams bought a suit during a sale and received a discount of $9. The regular price of the suit was $75. What was the rate of discount?

15 Mr. Ross sells property for a real-estate company. He receives a 5% commission on all sales. He received $545 for selling a house. What was the selling price of this house?

16 Mr. Allen agreed to pay 4% interest on $600 he borrowed. How many dollars' interest did he owe at the end of one year?

17 Miss Harper's weekly salary is $80. Her take-home pay is $64. Her take-home pay is what per cent of her salary?

18 Mrs. Burton borrowed $1750 from a bank. At the end of one year, she owed $105 interest. What rate of interest had she been charged?

19 During a sale, a store advertised "All merchandise reduced 15%." The discount on a certain rug was $21. What was the price of the rug before the sale?

20 Miss Carter's monthly salary is $450. Her take-home pay is 78% of her monthly salary. Her monthly take-home pay amounts to how many dollars?

21 Mr. Howe sold a house and lot for $28,400. He received a commission of $852. What was his rate of commission?

22 A merchant was given a 32% discount on a coat that was priced at $175. He was given a discount of how many dollars?

23 Mr. Lund's yearly take-home pay is $5312. His take-home pay is 83% of his salary. What is his yearly salary?

24 Mr. Peters sold his house at an auction for $16,700. He paid the auctioneer a 5% commission on the total amount of the sale. How many dollars did he pay the auctioneer?

KEEPING SKILFUL

Tabulate the solution set of each condition expressed in exercises 1 through 24. U = C.

1 $5/3 \sim b/36$.
2 8 is $n\%$ of 10.
3 $4938 + 3967 = f$.
4 $21/w \sim 28/8$.
5 275 is 10% of x.
6 $34m = 14144$.
7 $b/45 \sim 32/20$.
8 f is 6% of 300.
9 $18/8 \sim 144/s$.
10 $806(150) = d$.
11 $325 - 178 > n$.
12 300 is 200% of p.
13 $3827 - k = 948$.
14 $50/x \sim 50/100$.
15 d is 16% of 2375.
16 $529 + h < 735$.
17 123 is $x\%$ of 82.
18 $138n = 31326$.
19 $h/90 \sim 250/100$.
20 420 is $m\%$ of 420.
21 $85/17 \sim g/187$.
22 $h > 42(73)$.
23 390 is $d\%$ of 78.
24 $486/324 \sim 12/s$.

Problems involving compound conditions about rate pairs

You have solved problems that involve simple conditions for equivalence. In this lesson you will learn how to solve problems by using compound conditions that contain rate pairs.

Read the problem in D1. First you will develop a compound condition for the problem. U = C × C.

A You know that Jane has 4 times as many records as Sally. This means that for each 4 records that Jane has, Sally has 1 record. Use 4 as the first component of the rate pair that represents this comparison. What number will you use as the second component of this rate pair? 4/1

B You know one rate pair that represents the comparison. You want to find other rate pairs equivalent to 4/1. Use x as a variable for the number of records Jane can have. Why is x a variable for the first component instead of for the second component of each rate pair? x/y

C Since y is used as a variable for the second component, to what does y refer?

Jane has 4 times as many phonograph records as Sally has. Jane has fewer than 25 phonograph records. How many records can each of the girls have?

D1

D Now you can express one simple condition for the problem. Does the sentence at the right express this simple condition?

$4/1 \sim x/y.$

E Why is the idea of "less than" used in another simple condition for the problem? Does the sentence at the right express this other simple condition? For what is x a variable?

$x < 25.$

F Does the sentence below at the right express a compound condition for the problem?

$4/1 \sim x/y \wedge x < 25.$

Now you can find the solution set of the compound condition.

G Is the proportional relation tabulated in D2 the solution set of $4/1 \sim x/y$? Is the solution set an infinite set?

> REMINDER
> $\{4/1, \ldots\}$ is the set whose members are the rate pair 4/1 and all rate pairs equivalent to 4/1.
> *See lesson 42, page 174.*

H Name five other rate pairs that are members of the proportional relation tabulated in D2.

I Each solution of the compound condition must satisfy both $4/1 \sim x/y$ and $x < 25$. Which members of $\{4/1, \ldots\}$ have first components that are less than 25? Can you use each of these members as a solution of $4/1 \sim x/y \wedge x < 25$?

J Tabulate $\{(x, y) | 4/1 \sim x/y \wedge x < 25\}$.

You can use each solution of the compound condition to get an answer to the problem.

If Jane has 4 records, Sally has 1 record.
If Jane has 8 records, Sally has 2 records.
If Jane has 12 records, Sally has 3 records.

Jane has 4 times as many phonograph records as Sally has. Jane has fewer than 25 phonograph records. How many records can each of the girls have?

D1

$\{4/1, \ldots\}$

D2

Mr. Taylor borrowed $260 from a bank. At the end of one year, he paid the bank $273. What was the rate of interest charged by the bank?

D3

Jim bought 2 pencils priced at 3 for 15¢ and 3 erasers priced at 4¢ each. How many cents in all did he spend for these articles?

D4

K What are the other answers to the problem in D1?

Read the problem in D3. You will develop the compound condition for this problem. $U = C \times C$.

L The problem asks you to find the rate of interest. To do this, you must find the number of dollars of interest that the bank charged for each $100 that it loaned. Use m as a variable for the number of dollars of interest charged per $100. Does the rate pair named at the right represent the rate of interest charged?

m

$m/100$

M Do you know how many dollars of interest Mr. Taylor paid for the use of \$260? Use n as a variable for the number of dollars of interest that he paid. n

N Why do you use $n/260$, and not $260/n$? Why do you use $n/260$, and not $n/273$? $n/260$

O Does the sentence at the right express a simple condition for the problem? $m/100 \sim n/260.$

P Why is n, and not m, used in another simple condition for the problem? Does the sentence at the right express this other simple condition? $273 - 260 = n.$

Q Does the sentence in red below express the compound condition for the problem?

$$m/100 \sim n/260 \wedge 273 - 260 = n.$$

Now you can find the solution of the compound condition.

R What replacement can you make for n that satisfies $273 - 260 = n$?

S If you replace n by 13 in $m/100 \sim n/260$, do you obtain $m/100 \sim 13/260$?

T What number satisfies $m/100 \sim 13/260$?

You know that (5, 13) satisfies $m/100 \sim n/260 \wedge 273 - 260 = n$. You can use (5, 13) to get the answer to the problem.

Since Mr. Taylor paid \$13 interest on \$260, the rate of interest charged by the bank was \$5 on each \$100, or 5%.

Read the problem in D4. The exercises that follow will help you develop a condition for the problem. The universe for each of the variables is C.

A Do you know how many cents Jim spent for pencils? For erasers? Use a as a variable for the number of cents Jim spent for pencils. Use b as a variable for the number of cents he spent for erasers. Why is $a + b$, and not $a - b$, used in the condition? $a + b$

B Do you know how many cents Jim spent in all? Use n as a variable for the total number of cents Jim spent. n

C Does the sentence at the right express a simple condition for the problem? $a + b = n.$

D You know that 3 pencils cost 15¢. Does the rate pair named below at the right represent this rate? What does the first component of the rate pair refer to? What does the second component refer to? $3/15$

E You must find how many cents Jim spent for 2 pencils. Remember that a refers to the number of cents he spent for pencils. Why do you use $2/a$, and not $a/2$? $2/a$

F Does the sentence at the right express another simple condition for the problem? $3/15 \sim 2/a.$

G You know that each eraser cost 4¢. Does the rate pair named below at the right represent this rate? What does the first component of the rate pair refer to? What does the second component refer to? $1/4$

H You must find how many cents Jim spent for 3 erasers. Why do you use $3/b$, and not $b/3$? $3/b$

I Does the sentence at the right express another simple condition for the problem? $1/4 \sim 3/b.$

J Does the sentence below express a compound condition for the problem in D4?

$$a + b = n \wedge 3/15 \sim 2/a \wedge 1/4 \sim 3/b.$$

Now you can find the answer to the problem. First you find the solutions of the simple conditions for equivalence. Then you use these solutions to find the solution of the condition for equality.

K What number satisfies $3/15 \sim 2/a$? What number satisfies $1/4 \sim 3/b$?

L If you replace a by 10 and b by 12 in $a + b = n$, do you obtain $10 + 12 = n$?

M What number satisfies $10 + 12 = n$?

N How many cents in all did Jim spend for the pencils and erasers?

Now you know how to solve problems by using compound conditions that involve rate pairs.

On your own

For each of problems 1 through 12, write a sentence that expresses a compound condition. Use any letters you wish to express the variables. Tabulate the solution set of the compound condition and use the solutions to get the answers to the problem. $U = C \times C$.

1 During a basketball game, Mike scored 4 points for each 3 points that Paul scored. Paul scored fewer than 14 points during the game. How many points could each of the boys have scored?

2 In a school election, Carl received 2 votes for each 5 votes that George received. If Carl had received 18 more votes than he did, both boys would have received the same number of votes. How many votes did each of the boys receive?

3 The price of an article after the discount has been deducted is the *net price*. Mrs. Mason received a 10% discount on a lamp that was regularly priced at \$20. The net price amounted to how many dollars?

4 A store gave a 15% discount on a clothes drier that was regularly priced at \$220. The net price of this drier amounted to how many dollars?

5 Larry had 274 stamps in his collection. He bought 5 packages of new stamps. Each package contained 24 stamps. How many stamps did Larry have after he bought the 5 packages?

6 Carol bought a loaf of bread priced at 27¢ and 6 oranges priced at 48¢ per dozen. How many cents in all did she spend for bread and oranges?

7 The number of stamps Jay has is 125% of the number of stamps Ed has. If Ed had 128 more stamps, he would have 364. How many stamps does each of the boys have?

8 Mr. Thomas borrowed \$300 from a credit union. He agreed to pay back \$303 at the end of one month. What was the rate of interest for one month on this loan?

9 Mr. Henderson borrowed \$1250 from his bank. At the end of one year, he paid the bank \$1325. What was the rate of interest?

10 Louise bought three 8-ounce bags of mints and a 12-ounce bag of caramels. How many ounces of candy in all did she buy?

11 The regular price of a chair was \$140. The net price of the chair was \$91. What was the rate of discount?

12 For each 4 lawns that Ed mowed last week, Frank mowed 3 lawns. Together, the boys mowed fewer than 15 lawns. How many lawns could each of the boys have mowed?

For problem 13, write a sentence that expresses a compound condition for the problem. Then give the answer to the problem. The universe for each of the variables is C.

13 Tickets for a school fair were priced at 25¢ for adults and 10¢ for children. Mr. Brown bought 3 tickets for adults and 2 tickets for children. How many cents in all did Mr. Brown spend for the tickets?

Abstract problems involving rate pairs

You have been using conditions for equivalence to solve problems about persons and things. In this lesson you will use conditions for equivalence to solve problems that concern numbers only. $U = C \times C$.

A Read the problem in D1. Use m as a variable for the first component of each rate pair you are to find. Use n as a variable for the second component of each rate pair. Is $m/n \sim 5/3$ one of the simple conditions for the problem? Is $n < 12$ the other simple condition?

B What is the compound condition for the problem?

Now you will find the solution set of the compound condition.

C Is the proportional relation tabulated in D2 the solution set of $m/n \sim 5/3$? Is this set an infinite set?

D Explain how you know that both 20/12 and 25/15 are members of the set tabulated in D2.

E You know that each solution of the compound condition must satisfy both $m/n \sim 5/3$ and $n < 12$. How do you know that 20/12 and 25/15 do not satisfy $m/n \sim 5/3 \wedge n < 12$?

F Which members of the set tabulated in D2 satisfy $n < 12$? Do these rate pairs satisfy the compound condition?

G Tabulate $\{(m, n) | m/n \sim 5/3 \wedge n < 12\}$. Is the set you tabulated an infinite set?

H Is each solution of the compound condition an answer to the problem in D1?

Certain rate pairs are equivalent to 5/3. The second component of each of these rate pairs is less than 12. What are the rate pairs?

D1

$\{5/3, \ldots\}$

D2

A certain rate pair is equivalent to 2/8. The sum of the first component and second component of this rate pair is 20. What is the rate pair?

D3

$\{1/4, \ldots\}$

D4

I Read the problem in D3. Use x as a variable for the first component of the rate pair you are to find. Use y as a variable for the second component of this rate pair. Is $x/y \sim 2/8$ one of the simple conditions for the problem? Is $x + y = 20$ the other simple condition?

J What is the compound condition for the problem?

Now you will find the solution set of the compound condition.

K Is the proportional relation tabulated in D4 the solution set of $x/y \sim 2/8$? Is $\{(x, y) | x/y \sim 2/8\}$ an infinite set?

L Is 5/20 a member of the set tabulated in D4? Is 6/24 a member? Is 8/32 a member? How do you know?

M How do you know that 5/20, 6/24, and 8/32 do not satisfy $x + y = 20$? Do any of these rate pairs satisfy the compound condition?

A certain rate pair is equivalent to 2/8. The sum of the first component and second component of this rate pair is 20. What is the rate pair?

D3

$\{1/4, \ldots\}$

D4

Certain rate pairs are equivalent to 6/8. The first component of each of these rate pairs is less than the second component. What are the rate pairs?

D5

$h/k \sim 6/8 \wedge h < k.$

D6

N Which member of the set tabulated in D4 satisfies $x + y = 20$? Does this rate pair also satisfy the compound condition?

O Tabulate $\{(x, y) | x/y \sim 2/8 \wedge x + y = 20\}$.

P Is 4/16 the answer to the problem in D3?

Q Read the problem in D5. Use h as a variable for the first component of each rate pair you are to find. Use k as a variable for the second component.

R Does the sentence in D6 express the compound condition for the problem?

S Tabulate $\{(h, k) | h/k \sim 6/8\}$.

T Which members of the set you tabulated for exercise S have a first component that is less than the second component?

U Tabulate $\{(h, k) | h/k \sim 6/8 \wedge h < k\}$. Is the set you tabulated an infinite set?

V Does each rate pair that satisfies $h/k \sim 6/8$ also satisfy the compound condition?

In this lesson you have learned to solve problems that ask questions about rate pairs. The conditions that you used for the problems were compound conditions.

On your own

For each problem, write a sentence that expresses the compound condition for the problem. You may use any letters you wish as names for the variables. Then tabulate the solution set of the compound condition. $U = C \times C$.

1 Certain rate pairs are equivalent to 4/6. The sum of the first component and second component of each of these rate pairs is less than 20. What are the rate pairs?

2 Certain rate pairs are equivalent to 3/7. The first component of each of these rate pairs is less than 25. What are the rate pairs?

3 A certain rate pair is equivalent to 4/9. The first component of this rate pair is twice the first component of 4/9. What is the rate pair?

4 A certain rate pair is equivalent to 15/18. The second component of this rate pair is 1 more than the first component. What is the rate pair?

5 Certain rate pairs are equivalent to 2/10. The second component of each of these rate pairs is less than 10 times the first component. What are the rate pairs?

6 Certain rate pairs are equivalent to 2/2. The sum of the first component and second component of each of these rate pairs is less than 15. What are the rate pairs?

7 A certain rate pair is equivalent to 7/8. The second component of this rate pair is 3 more than the first component. What is the rate pair?

8 Certain rate pairs are equivalent to 9/6. The first component of each of these rate pairs is less than the sum of the second component and 4. What are the rate pairs?

9 Certain rate pairs are equivalent to 2/3. The sum of the first component and second component of each of these rate pairs is less than 5. What are the rate pairs?

10 A certain rate pair is equivalent to 8/5. The sum of the first component and second component of this rate pair is 52. What is the rate pair?

APPLYING MATHEMATICS

In this lesson you will solve problems similar to those you have studied so far in this book. For each problem, first write a sentence that expresses a condition for the problem. Use any letters you wish as names for the variables. Then tabulate the solution set of the condition and give the answer to the problem. The universe is given for each problem.

1 In a recent year, the expenses of the Canadian railways were $37,220,149 less than the earnings. The expenses amounted to $1,232,387,115. The earnings of the railways amounted to how many dollars? U = N.

2 The second United States astronaut traveled at a rate of approximately 337 miles per minute. At this rate, approximately how many miles did he travel in 15 minutes? U = C.

3 Harriet had more than 20 concert tickets to sell. She sold 14 tickets and then had fewer than 9 left to sell. How many tickets could Harriet have had to begin with? U = N.

4 Joe's golf score at the end of nine holes was 54. His score at the end of eighteen holes was more than 113. What is the lowest score he could have had on the second nine holes? U = N.

5 During a sale, a store advertised "15% off on all merchandise." Mrs. Horton bought a fur jacket and received a discount of $72. What was the regular price of the jacket? U = C.

6 For a science project, Donna mounted a total of 6 moths and butterflies. How many moths and how many butterflies could she have mounted? U = C × C.

7 Mr. Carter planted fewer than 12 shrubs and trees. He planted the same number of trees as shrubs. How many trees and how many shrubs could he have planted? U = C × C.

8 During a sale, a dress shop gave a discount of 20% on all merchandise. The regular price of a dress that was on sale was $30. What was the net price of this dress? U = C × C.

9 The melting point of copper is 1083° centigrade, and the melting point of iron is 1533° centigrade. The melting point of iron is how many degrees higher than the melting point of copper? U = N.

10 Mr. Lee receives a 4% commission on his monthly sales. During October, he sold $7475 worth of merchandise. His October commission amounted to how many dollars? U = C.

11 An outfielder had 36 doubles during a recent baseball season. He had 9 doubles for each 47 hits that season. How many hits did he have that season? U = C.

12 Carol did not have enough money to buy a book that cost 57 cents. After her mother gave her 15 cents, she had more than 62 cents. How many cents could Carol have had before her mother gave her the 15 cents? U = N.

13 The regular price of a sofa was $348. During a sale, the net price of this sofa was less than $252. What is the least number of dollars the discount could have been? U = N.

14 Mr. Bates borrowed $750 from the First Federal Savings Bank. At the end of one year, the interest on his loan amounted to $45. What was the rate of interest? U = C.

15 Helen is 9 years older than Anita. In 4 years, Anita will be 17 years old. How old is each of the girls now? U = C × C.

16 The citizens of West Haven had a campaign to raise $7500 for a new playground. By the end of the campaign, they had raised $8625. The amount raised was what per cent of the goal? U = C.

17 Bill has 15 more arrowheads than Ken has. If Ken had 23 more, he would have 50 arrowheads. How many arrowheads does each of the boys have? U = N × N.

18 Tom bought 4 pieces of miniature railroad track for 13 cents each. He gave the clerk 75 cents. How many cents should he have received in change? U = C × C.

19 When Janet went to the store, she had a total of 11 dimes and nickels in her purse. She had 5 more dimes than nickels. How many nickels and how many dimes did she have? U = N × N.

20 36 is what per cent of 225? U = C.

21 The sum of two numbers is less than 15. When the first number is subtracted from the second number, the difference is 9. What can the numbers be? U = N × N.

22 What numbers greater than 386 can be added to 134 so that each sum is greater than 525? U = N.

23 When the first component of each of certain rate pairs equivalent to 8/12 is subtracted from the second component, the difference is less than 8. What are these rate pairs? U = C × C.

24 What number is 68% of 9350? U = C.

25 The sum of two numbers is 15. The second number is less than the quotient of 78 and 13. What can the numbers be? U = N × N.

26 The sum of the first and second components of each of certain rate pairs equivalent to 6/5 is less than 35. What are the rate pairs? U = C × C.

27 15 is 125% of what number? U = C.

28 28 is what per cent of 112? U = C.

29 What number is 11% of 200? U = C.

CHECKING UP

The small numeral within parentheses tells what page to turn to if you need help with an exercise.

Test 46

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

1 The second component of 47% is 100. (187)

2 $5/7 \sim 45/56$. (174)

3 4/3 is a member of {8/6, . . .}. (174)

4 If you multiply each component of (6, 18) by 3, you obtain (18, 36). (170)

5 (3, 8) and (42, 112) are equivalent ordered pairs. (168)

6 A proportional relation is a set of ordered pairs in which each member is equivalent to each of the other members. (170)

7 If you divide each component of (60, 132) by 12, you obtain (5, 11). (170)

8 There is exactly one member in {2/9, . . .}. (174)

9 {3/5, . . .} = {9/15, . . .}. (175)

10 When U = C, the solution set of $16x = 32$ is the same as the solution set of $8/x \sim 4/16$. (177)

Test 47

From the list given below each exercise, choose the expression that correctly completes the exercise.

11 {9/12, . . .} does not contain the rate pair ~~~. (174)

a 12/15 b 6/8 c 27/36 d 3/4

12 If $8/3 \sim 32/12$, then ~~~. (174)

a 32(8) = 12(3). c 8(12) = 32(3).

b 3(12) = 8(32). d 12(8) = 12(3).

13 When a regular price is reduced, the amount of the reduction is called the ~~~. (184)

a service charge c net price

b commission d discount

14 If you divide each component of (119, 301) by ~~~, you obtain the ordered pair (17, 43). (170)

a 3 **b** 7 **c** 11 **d** 13

15 A charge for the use of money is called ~~~. (184)

a discount **c** interest
b commission **d** principal

16 ~~~ does not express the rate pair 2 to 3. (173)

a 2/3 **b** (2, 3) **c** {2, 3} **d** 2:3

17 (10, 6) is not equivalent to ~~~. (187)

a 5/3 **b** 20/12 **c** 10% **d** (15, 9)

18 ~~~ is a member of {2/3, . . .}, and the sum of its components is 20. (195)

a 6/14 **b** 8/12 **c** 10/15 **d** 4/16

19 ~~~ represents the same comparison as 30%. (188)

a 6/30 **b** 30/100 **c** 30/50 **d** 3/100

Test 48

For exercises 20 through 25, decide for each ordered pair expressed what replacement should be made for m to obtain an ordered pair that is equivalent to (24, 36). U = C. (171)

20 $(12, m)$ **22** $(4, m)$ **24** $(72, m)$
21 $(m, 72)$ **23** $(m, 9)$ **25** $(m, 3)$

For exercises 26 through 29, U = C. For exercises 30, 31, and 32, U = C × C. Tabulate each set described.

26 $\{x \mid 8/10 \sim x/15\}$ (178)
27 $\{x \mid 48/18 \sim x/12\}$ (178)
28 $\{x \mid 16/x \sim 1/16\}$ (178)
29 $\{x \mid 20/x \sim 85/34\}$ (178)
30 $\{(x, y) \mid x/y \sim 3/5 \wedge y - x = 6\}$ (195)
31 $\{(x, y) \mid x/y \sim 4/5 \wedge x + y < 40\}$ (195)
32 $\{(x, y) \mid x < y + 5 \wedge x/y \sim 2/3\}$ (195)

Test 49

For each problem, write a sentence that expresses a condition, tabulate the solution set of the condition, and give the answer to the problem. The universe is given for each problem.

33 Six pencils cost 25¢. What is the cost of 18 of these pencils? U = C. (176)

34 For every 5 correct answers that Joan had on a test, Barbara had 4 correct answers. Joan had 45 correct answers. How many correct answers did Barbara have? U = C. (180)

35 15% of all the students enrolled in Westview School are in the school band. There are 54 students in the band. How many students are enrolled in Westview School? U = C. (190)

36 Mr. Hanson receives a commission of $70 for each $1000 worth of merchandise he sells. What rate of commission does he receive? U = C. (194)

37 During a sale, a clothing store sold all merchandise at a 20% discount. What was the discount on a suit that was regularly priced at $45? U = C. (190)

38 Mr. Bender bought a set of tools. The regular price of the set was $140. Mr. Bender received a 15% discount. How much did he pay for the set of tools? U = C × C. (195)

39 During the baseball season last year, Bob had 3 hits for each 2 hits that Larry had. In all, Bob had 8 more hits than Larry. How many hits did each boy have? U = C × C. (195)

40 What number is 60% of 25? U = C. (192)

41 48 is 200% of what number? U = C. (192)

42 13 is what per cent of 25? U = C. (191)

43 15% of 120 is what number? U = C. (192)

44 105 is what per cent of 75? U = C. (191)

45 A certain rate pair is equivalent to 5/6 and has a second component that is 4 more than the first component. What is the rate pair? U = C × C. (199)

46 Each of certain rate pairs equivalent to 2/3 also has a second component less than 25. What are the rate pairs? U = C × C. (199)

CHECKING UP

The small numeral within parentheses tells what page to turn to for help if you need it.

Test 50

For exercises 1 through 7, write "T" for each sentence that expresses a true statement. Write "F" for each sentence that expresses a false statement.

1 A graph is a set of dots. (52)

2 $\{1, 3, 5, 7, 9\} = \{9, 5, 1, 7, 3\}$. (12)

3 The surface of a ball suggests a plane. (31)

4 The supplement of a right angle is a right angle. (107)

5 Ray MN is the same as ray NM. (105)

6 If set R has 2 members and set S has 7 members, then $R \times S$ has 14 members. (127)

7 $\{(2, 10), (4, 20), (7, 35)\}$ is a proportional relation. (170)

Test 51

From the list below each exercise, choose the expression that correctly completes the exercise.

8 If U = N, then $\{2, 3, 4, 5\}$ is the solution set of ~~~. (92)

a $x > 1 \wedge x > 5$. c $x < 2 \wedge x > 6$.

b $x > 2 \wedge x < 6$. d $x > 1 \wedge x < 6$.

9 $A = \{0, 1, 2\}$, $B = \{2, 4\}$, and $C = \{0, 4\}$. $A \cap B \cap C =$ ~~~. (84)

a $\{0, 4\}$ b $\{0, 1, 2, 4\}$ c $\{\ \}$ d $\{0, 1, 2\}$

10 If $D = \{\ \}$, $E = \{0\}$, and $F = \{0, 2, 4\}$, then $D \cup E \cup F =$ ~~~. (88)

a $\{0\}$ b $\{\ \}$ c $\{0, 2\}$ d $\{0, 2, 4\}$

11 The intersection of two parallel planes ~~~. (99)

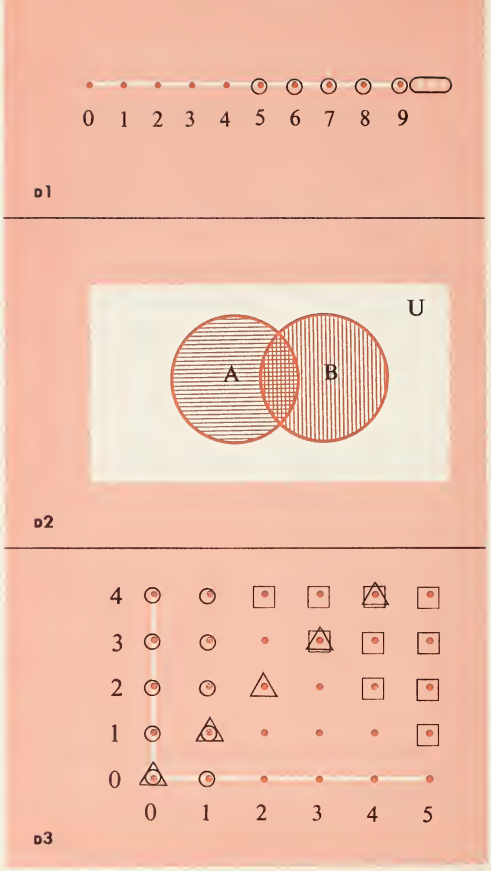
a contains one member c is a line

b is the empty set d is an angle

12 In D1, the encircled dots form a graph of the solution set of ~~~. U = N. (53)

a $x - 2 > 2$. c $x + 1 > 4$.

b $7 > x + 3$. d $7 - x > 4$.



13 Look at D2. The Venn diagram shows you that ~~~. (85)

a A is a subset of B. c A and B are disjoint.

b A and B meet. d B is a subset of A.

14 A ~~~ does not separate a plane. (115)

a line b ray c circle d triangle

15 ~~~ represents the same rate as 1 to 5. (174)

a (3, 10) b {5, 20} c 25% d 6/30

Test 52

Use the graph in D3 for exercises 16 through 24.

16 The set of dots in D3 is a graph of $R \times S$. $R \times S$ is the universe for (x,y). How many members are in $R \times S$? (131)

17 Tabulate R. (130)

18 Tabulate S. (130)

19 How are the dots in the graph of $\{(x,y) | x < 2\}$ indicated? (135)

20 How are the dots in the graph of the solution set of $x = y$ indicated? (132)

21 How are the dots in the graph of the solution set of $x + y > 5$ indicated? (132)

22 How are the dots in the graph of the solution set of $x < 2 \wedge x = y$ indicated? (147)

23 Tabulate $\{(x,y) | x = y \wedge x + y > 5\}$. (147)

24 Tabulate $\{(x,y) | x + y > 5 \wedge x < 2\}$. (147)

Test 53

For each of the following problems, first write a sentence that expresses a condition. Use any letters you wish as names for the variables. Then tabulate the solution set of the condition and give the answer to the problem. The universe is given for each problem.

25 Sally made 13 aprons. Martha made 5 fewer aprons than Sally. How many aprons did Martha make? U = N. (68)

26 Nancy sold 23 tickets to a school play. Nancy and Ruth sold fewer than 39 tickets in all. How many tickets could Ruth have sold? U = C. (73)

27 The combined score that two teams made during a football game was 18 points. The winning team scored more than 9 points. How many points could the losing team have scored? U = N × N. (153)

28 Carol's school was in session 180 days last year. Carol was absent on 5% of the days. How many days was she absent? U = C. (190)

29 In a volleyball game, Pete's team scored 3 points for every 2 points scored by Harold's team. If Pete's team scored fewer than 16 points, what is the greatest number of points that Harold's team could have scored? U = C × C. (195)

Unit 5

Numeration systems

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Tally, code, and grouping numeration systems

In your study of mathematics you have used many symbols to express ideas. You have used ratios to express rate pairs, for example, and you have used sentences to express statements. You have also learned that a numeral expresses, or names, a number.

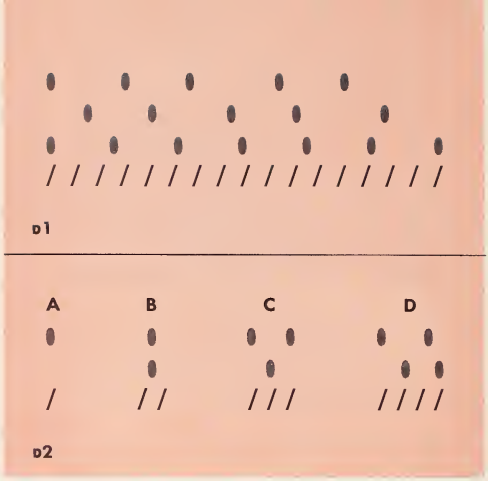
In earlier grades you learned how to express the natural numbers and fractions by using a decimal system for naming numbers. The decimal system you used included the symbols 0, 1, 2, and so on. In this unit you will learn more about the decimal numerals for the natural numbers. You will also study other systems for naming the natural numbers. By studying the properties of several different systems for naming numbers, you will better understand the properties of the familiar decimal system.

A Look at D1. When you look at the oval markers, you can think of any objects you wish. For example, you can think of model airplanes or pencils or books. Each stroke in D1 represents one of the objects. Are there as many strokes as there are objects?

The set of strokes is one way to name the number of objects. A numeral that is written as a set of one or more strokes is a *tally numeral.*

B How many different kinds of symbols are used in the tally numeral in D1?

C Think of each stroke in the tally numeral as expressing the number 1. By adding these numbers, you can find the number of objects represented in D1. What decimal numeral expresses the number of objects?



Notice that you can write a tally numeral to express a number. However, it is more convenient to use a decimal numeral. You know the words for our decimal numerals. Therefore, unless you are told otherwise, you should use these words or the decimal numerals when you talk about or express numbers. For example, you should use the word "seventeen" or the decimal numeral 17 to express the number of objects represented in D1.

D Look at D2. A tally numeral is shown for each set of objects represented. Does each tally numeral name the number of members in that set of objects?

E How does set B differ from set A? How does the tally numeral for set B differ from the tally numeral for set A?

F How does set C differ from set B? How does the tally numeral for set C differ from the tally numeral for set B?

G How does set D differ from set C? How does the tally numeral for set D differ from the tally numeral for set C?

You have seen how a *tally system* for naming numbers can be developed. A tally numeral

for each new number has one more stroke than the tally numeral for the number that immediately precedes it.

H Use a decimal numeral to express the number of members in each set of objects represented in D2.

Give a decimal numeral for each number named below.

I /

J //

K ///

L ////

M /////

N //////

O ///////

P ////////

Q /////////

R //////////

Since each stroke in a tally numeral expresses the number 1, and the tally numeral expresses the sum of the ones, a tally system has the property of *addition*. A tally system also has the property of *repetition*, because the same symbol, the stroke, may be used more than once in a tally numeral.

Now you will learn about another system for naming numbers.

The sets of objects represented in D3 are the same as the sets represented in D2. A new symbol is used to name the number of members in each set. Each of these symbols is a *code numeral*. You will use these symbols to develop a *code system* for naming numbers.

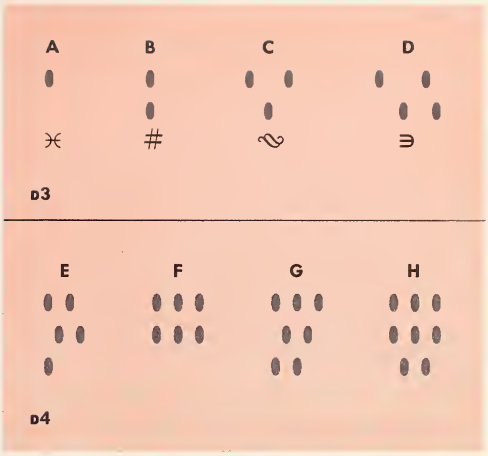
A In a tally system for naming numbers, each stroke expresses the number 1. What must you know before you can understand the symbols in a code system?

B In D3, what number is named by each of the code numerals?

Use one of the code numerals in D3 to express the number of members in each set tabulated below.

C {river, ocean, lake}

D {Africa, Europe}



E {green, yellow, blue, red}

F {Montreal}

In a code system for naming numbers, each numeral is made up of just one symbol. Notice that a different symbol must be used for each numeral.

G So far only the symbols shown in D3 have been used as numerals in this code system. Look at D4. Invent a new code numeral to name the number of members in each set of objects represented.

H You have learned that in a code system, no ? symbol is used more than once. You also have learned that a different symbol is used for each numeral. Does a code system have the property of repetition? Does a code system have the property of addition? Explain your answers.

I Think of a code system for naming the ? members of {1, 2, 3, . . .}. Do the different symbols used to make the numerals in this code system form an infinite set?

J Does a tally system for naming numbers ? have any advantage over a code system? Does a code system have any advantage over a tally system? Explain your answers.

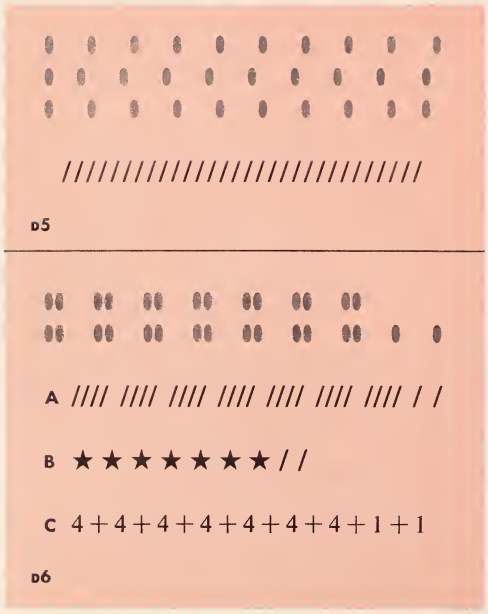
Next you will use the properties of a tally system together with code numerals to develop a new system for naming numbers.

A Look at D5. The tally numeral shown expresses the number of objects represented in the display. How many objects does each stroke represent?

B What must you know before you can decide what number is expressed by the tally numeral in D5? What decimal numeral expresses the number of objects?

C The same objects are represented in D6, but now they are grouped by 4's. How many objects were left over after the grouping was completed? In tally numeral A, how many strokes are needed to express the number of objects in each group of 4?

D Now suppose you use a new symbol, a star, in place of each group of four strokes. How many objects does each star represent?



E Why does numeral B in D6 contain seven stars?

F A stroke still represents one object. Why does numeral B also contain two strokes?

G Since each star in numeral B represents a group of four objects, numeral B is a *grouping numeral* for the number of objects represented in D6. You are developing a *grouping system* for naming numbers. Numeral B contains how many stars? How many strokes?

H If you count the number of stars and strokes in numeral B, do you obtain the number of objects represented in D6?

I What is the decimal numeral for the number expressed by the grouping numeral in D6? Use numeral C to help you.

Since a grouping numeral expresses the sum of the numbers expressed by the symbols in the numeral, a grouping system has the property of addition. A grouping system also has the property of repetition because a symbol, such as the star or stroke, may be used more than once in a grouping numeral.

J Study numerals D and E in D7. Each star expresses the number 4, and each stroke expresses the number 1. Do the two numerals name the same number? What is this number?

K Look at D8. It shows the same number of objects as D6, but in D8 four groups of 4 have been put together to form a new group of objects. How many objects are in this group?

L In numeral F in D8, a diamond has been used to represent the group of encircled objects. Does the new symbol express the number 4×4, or 16?

M If you count the number of diamonds, stars, and strokes in numeral F, do you obtain the number of objects represented in D8? How do you obtain this number? What is the number? Remember to use a decimal numeral to express your answer. Numeral G will help you.

N Does numeral F in D8 express the same number as numeral B in D6?

So far in our grouping system we have used a symbol, the star, to name the number 4. We have also used another symbol, the diamond, to name the number 4×4, or 16. If we were to continue with this grouping system, we would use a still different symbol to name the number $4 \times 4 \times 4$, or 64. When we group by 4's, we first think of groups of 4, then four groups of 4, then four groups of four groups of 4, and so on. We have used 4 as the *base* of this grouping system.

So far you have studied a tally system, a code system, and a grouping system for naming numbers. Such systems for naming numbers are *numeration systems.*

O One hundred six objects are represented in D9. Numeral H is a base-four grouping numeral that expresses the number of objects. Why are six diamonds used in numeral H? Does this numeral name the number 106?

P Now make a new group out of four of the groups of 16. Why do you use four of these groups of 16? Remember that you are using a base-four grouping system.

D ★ ★ ★ / /

E / ★ ★ ★ /

D7

F ◆ ★ ★ ★ / /

G $16+4+4+4+1+1$

D8

nu mer a tion sys tem (nü′mər ā′shən sis′təm). A method or system for naming numbers. A tally system, a code system, and a grouping system are numeration systems.

Q You need a new symbol to express the number $4 \times 4 \times 4$, or 64. Use an arrowhead as this new symbol. How many groups of $4 \times 4 \times 4$ objects are represented in D9?

R What number is expressed by numeral I in D9? Use numeral J to help you obtain the answer.

S Does numeral I express the same number as numeral H? Explain your answer.

H ◆ ◆ ◆ ◆ ◆ ◆ ★ ★ / /

I ◆ ◆ ★ ★ / /

J $64+16+16+4+4+1+1$

D9

K ◈ ★ /

L $100 + 10 + 1$

M ★ / ◈

D10

Any number greater than 1 may be used as a base of a grouping system. You have been using a base-four grouping system. You will use a base-ten grouping system next. In this base-ten grouping system, you will use the stroke for 1, the star for the base, the diamond for the base times the base, and so on.

T Look at D10. How many objects does a stroke represent? How many objects does a star represent?

U Does a diamond represent ten groups of 10 objects? How many objects does a diamond represent?

V Before you can obtain the number expressed ? by a grouping numeral, what must you know about the symbols in the numeral?

W Does numeral K express the number of objects represented in D10? What is this number? Use numeral L to help you find this number.

X Numeral M is also a base-ten numeral. Does this numeral express the same number as numeral K? In what way do numeral K and numeral M differ?

Now you know how tally, code, and grouping numeration systems can be developed.

On your own

Use grouping numeration systems for all the exercises in this section. For each system use only the symbols given in D11. Study D11 before you begin your work.

/ expresses the number 1.

★ expresses the number of objects equal to the base.

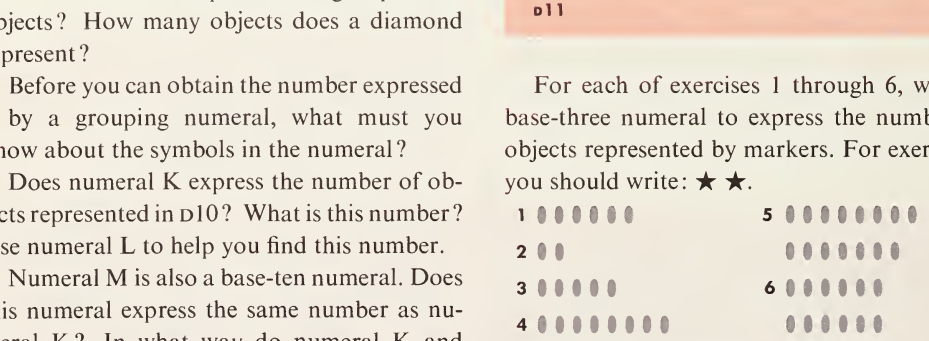
◈ expresses the number of objects equal to the base times the base.

◘ expresses the number of objects equal to the base times the base times the base.

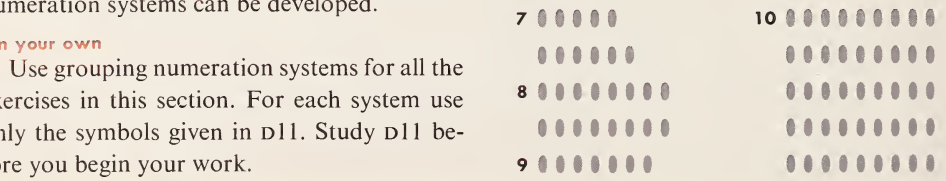
For example, if the base is 3, then / = 1, ★ = 3, ◈ = 9, and ◘ = 27.

D11

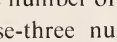
For each of exercises 1 through 6, write a base-three numeral to express the number of objects represented by markers. For exercise 1 you should write: ★ ★.

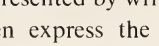


For each of exercises 7 through 10, express the number of objects represented by writing a base-eight numeral. For exercise 7 you should write: ★ / / /.



For each of exercises 11 and 12, first express the number of objects represented by writing a base-three numeral. Then express the same number by writing a base-eight numeral.

11 

12 

For each number named in exercises 13 through 20, express the number by writing a decimal numeral. In each case show how you obtained the numeral. For exercise 13 you should write: $9 + 9 + 3 + 1$, or 22.

13 ◈ ◈ ★ / (base three)
14 ◔ ◈ ★ / (base two)
15 ◈ ◈ ★ ★ ★ (base four)
16 ◔ ◈ ◈ ★ / / / / (base six)
17 ◈ ★ / / / / / / / / / / (base eleven)
18 ◔ ◔ / / / (base four)
19 ◈ ◈ ◈ ★ ◈ ◈ (base ten)
20 ★ / / / / / / / / / / (base twelve)

KEEPING SKILFUL

Find the sum.

1 3045, 796, 1368, 4201, 579, 607
2 9030, 1132, 475, 8671, 110, 89
3 32347, 98, 56451, 8925, 1003
4 13746, 92086, 47952, 88570, 75
5 273, 9645, 816, 49

Find the difference.

6 2134, 969
7 5000, 4296
8 8765, 6293
9 10073, 295
10 12356, 5974
11 30110, 12943
12 72842, 798
13 43022, 37496

Multiply

14 4862 by 9.
15 38 by 47.
16 105 by 62.
17 993 by 213.
18 800 by 560.
19 307 by 845.
20 864 by 705.

Divide

21 6230 by 63.
22 2886 by 74.
23 36363 by 51.
24 55436 by 134.
25 39123 by 189.
26 22140 by 205.
27 47293 by 65.

5 50 Exploring ideas

Numeration systems that have place value

In the last lesson you learned that numeration systems can have the properties of repetition, addition, and base. Now you will study another property of numeration systems.

A Look at D1. Each stroke in numeral A expresses the number 1. Each star expresses the number 3. Each diamond expresses the number 3×3, or 9. Each arrowhead expresses the number $3 \times 3 \times 3$, or 27. What is the base of the grouping system in which numeral A is written?

B What is the decimal numeral for the number named by numeral A? Use numeral B to help you obtain your answer.

C How do you know that numeral A has the property of repetition?

When we ask whether a numeral has a certain property, we really want to know whether or not the numeration system to which the numeral belongs has this property.

D How do you know that numeral A has the property of addition? Use numeral B to help you explain your answer.

Now you will learn how to develop a base-three system in which strokes are used in different positions to make the numerals.

A ◔ ◈ ◈ ★ ★ / /

B $1(3 \times 3 \times 3) + 2(3 \times 3) + 2(3) + 2(1)$

D1

E The base-three numeral in D1 is shown again in D2. Numeral C in D2 expresses the same number as numeral A. Notice the two strokes in numeral C that are below the two strokes in numeral A. Each of these strokes also expresses the number 1. In the space below the two stars, each stroke expresses the number 3. Why is it necessary to separate these strokes from the strokes in the space at the right?

F How do you know that the two strokes in the space below the stars express the number 2(3), or 6?

G Each stroke in the space below the diamonds expresses the number 3×3, or 9. Do these two strokes express the number $2(3 \times 3)$, or 2(9), or 18?

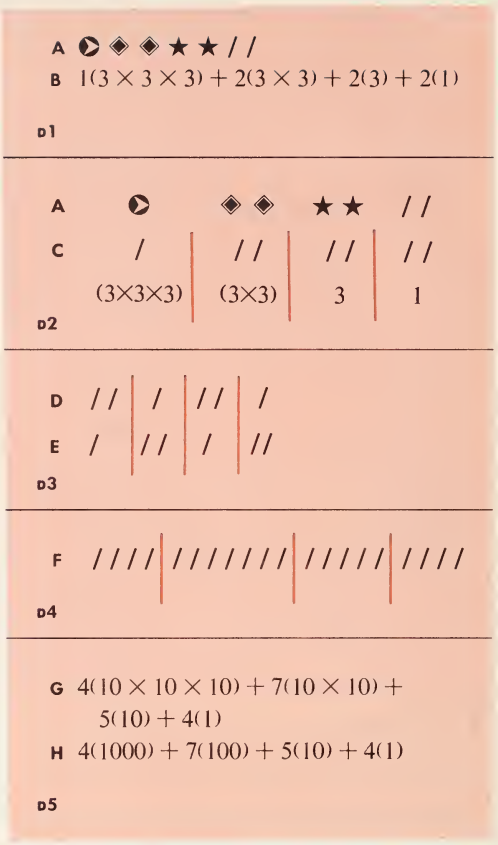
H The stroke in the space below the arrowhead expresses the number $1(3 \times 3 \times 3)$. Does this stroke express the number 27?

I What number is expressed by the strokes in numeral C? Use a decimal numeral to express your answer.

J Does numeral C have the property of repetition? Of addition? Explain your answers.

Before you can find the number expressed by the strokes in a grouping numeral like numeral C, you must know what number is expressed by each stroke. By finding the sum of these numbers, you can find the number expressed by the grouping numeral. For numeral C, you know that each stroke in the space farthest to the right expresses the number 1. In the second space from the right, each stroke expresses the number 3. Each stroke in the third space from the right expresses the number 3×3, or 9, and so on.

Notice that in numeral C, the position of a stroke determines the number expressed by the stroke. A numeration system like the one used for numeral C has the property of *place value*. It is called a *place-value numeration system*.



K Look at D3. Two different numbers are expressed by using a base-three numeration system that has the property of place value. What number is expressed by numeral D? What number is expressed by numeral E?

L Notice that numeral D contains six strokes. Does numeral E also contain six strokes? How is it possible to use six strokes to express two different numbers?

M In the place-value numeration system that we have been discussing, we have used a chart to show the different positions of the strokes. In what other ways might the different positions of the strokes be indicated?

N Look again at numeral A in D1. Does this numeral have the property of place value? How do you know?

Now you will work with other numeration systems that have the property of place value.

A Numeral F in D4 is a base-ten numeral. Each stroke in the space farthest to the right expresses the number 1. Each stroke in the second space from the right expresses the number 10. What number is expressed by each stroke in the third space from the right?

B What number is expressed by each stroke in the fourth space from the right?

C Do numerals G and H in D5 show how to obtain the decimal numeral for the number named in D4? What is the decimal numeral for the number named in D4?

D How does a base-ten system that has the property of place value differ from a base-three system that has the property of place value?

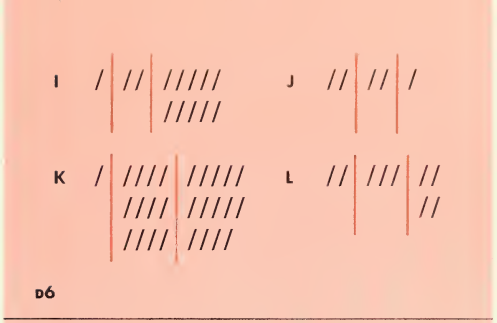
E Four numerals are shown in D6. First think of them as written in a base-three system with place value. Which numerals express the number twenty-five?

F Which of the numerals for the number twenty-five uses fewer strokes? What advantage is there in writing a numeral with as few strokes as possible?

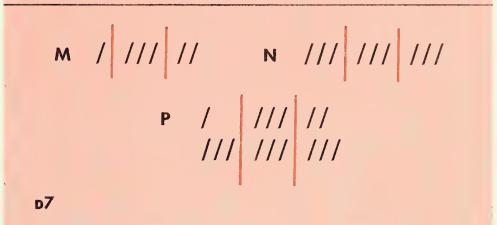
G Now think of the numerals in D6 as written in a base-ten system with place value. Which numerals express the number 234? Which of these numerals requires fewer strokes?

H Look again at numerals I and J in D6. When you write a numeral in a base-three numeration system, why is it never necessary to use more than two strokes in a space?

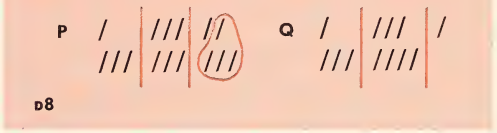
I When you use a base-ten numeration system, what is the greatest number of strokes needed in a space? Explain your answer.



D6



D7



D8

J Each numeral in D7 is a base-four numeral with place value. What number is expressed by numeral M?

K What number is expressed by numeral N?

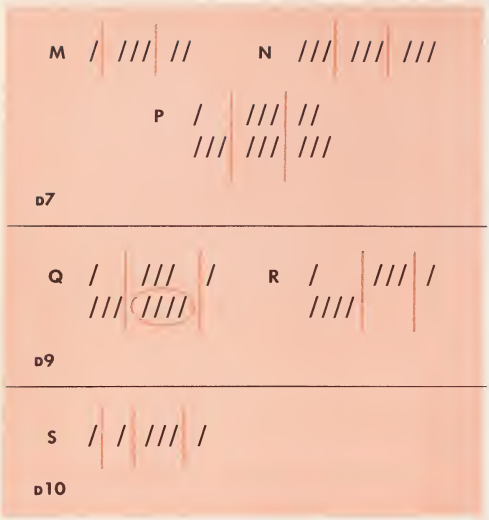
L Numeral P expresses the sum of the numbers named by numeral M and numeral N. Explain how numeral P was obtained from numerals M and N.

M What number is expressed by numeral P?

N When you are using a base-four numeration system, what is the greatest number of strokes needed in a space?

O Explain how you know that you can regroup the strokes in numeral P. This numeral is shown again in D8. Why should you replace four of the strokes in the space farthest to the right by one stroke in the second space from the right?

P Does numeral Q express the same number as numeral P?



Q Numeral Q is shown again in D9. Why can you replace four of the strokes in the second space from the right by one stroke in the third space from the right? Does numeral R express the same number as numeral Q?

R Explain how the strokes in the farthest space from the right in numeral R can be regrouped.

S The strokes in numeral R have been regrouped in D10. Does the stroke in the fourth space from the right express the number $4 \times 4 \times 4$, or 64?

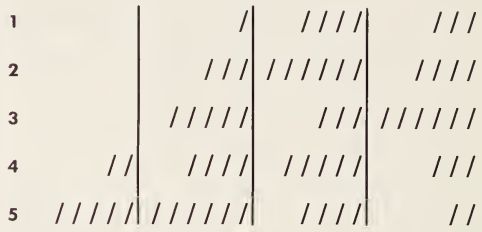
T What number is expressed by numeral S in D10? Do numeral R and numeral S express the same number? Do numeral S and numeral P in D7 express the same number?

U In a place-value numeration system like those you have been using, how can you use the base to determine the greatest number of strokes needed in a space?

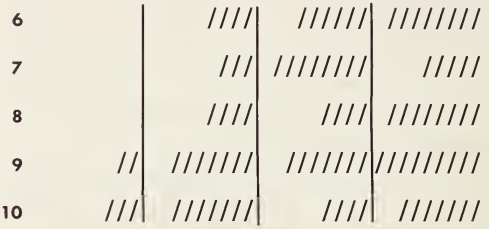
In this lesson you studied a few numeration systems that have the property of place value. You learned to use strokes in a numeration system that has place value.

On your own

The numerals in exercises 1 through 5 are written in a base-seven numeration system with the property of place value. Express each number named by writing a decimal numeral. In each case, show how you obtained the numeral. For exercise 1, for example, you should write: $1(7 \times 7) + 4(7) + 3(1)$, or 80.



The numerals in exercises 6 through 10 are written in a base-six numeration system that has the property of place value. For each exercise, make a chart that shows how to regroup the strokes. Remember to regroup in such a way that you use the fewest strokes possible in expressing the number in base six.



11 What number is named by each numeral that you made for exercises 6 through 10?

12 Make a chart and use strokes to express the sum of the numbers named in exercises 6 and 7. Remember that the numerals in these exercises are written in a base-six numeration system with the property of place value. Use as few strokes as possible in your chart.

5 51 Exploring ideas

Place value and digits in numeration systems

In lessons 49 and 50 you studied numeration systems that have the properties of addition, repetition, base, and place value. You used strokes to write the numerals in these numeration systems. In this lesson you will learn how to use symbols other than the stroke in these numeration systems.

A Numeral A in D1 is written in a base-four system with the property of place value. Do the strokes in the space farthest to the right express the number 3(1), or 3? Why is it necessary to separate these strokes from the other strokes in the numeral?

B What number does numeral A express? Use a decimal numeral to express the number.

C Numeral B in D1 shows how to use the familiar symbols 0, 1, 2, and 3 to write this base-four numeral. What symbol is in the space farthest to the right? Does this symbol tell you how many strokes there are in the space farthest to the right in numeral A? Does this symbol express the number 3(1), or 3?

D Notice the symbol 3 in the second space from the right. Does this symbol tell you how many strokes there are in the second space from the right in numeral A? Does this symbol express the number 3(4), or 12?

E The symbol 0 is used in the third space from the right in numeral B. Does this symbol tell you how many strokes there are in the third space from the right in numeral A? Does this symbol express the number $0(4 \times 4)$, or 0?

D1

	$(4\times4\times4)$	(4×4)	4	1
A	/		///	///
B	1	0	3	3

B 1033

C $1(4 \times 4 \times 4) + 0(4 \times 4) + 3(4) + 3(1)$

D2

F What number is expressed by the symbol 1 in numeral B?

G Does numeral B express the same number as numeral A?

H In numeral B, the symbol 3 is used instead of three strokes, and the symbol 1 instead of one stroke. For how many strokes is the symbol 0 used?

I For how many strokes would the symbol 2 be used in a numeral similar to numeral B?

The familiar symbols 0, 1, 2, and 3 are code symbols, called *digits*. Notice that a digit is exactly one symbol.

J Numeral B from D1 is shown again in D2. ? Notice that this time a chart has not been used. How do you know that the digit 1 expresses the number $1(4 \times 4 \times 4)$, and not the number $1(4 \times 4)$?

K Does the position of each digit in numeral B determine the number expressed by that digit?

L Does numeral B have the property of place value?

M What number does numeral B express? Remember to use a decimal numeral to express the number. Use numeral C to help you obtain your answer.

N Is the digit 3 used more than once in numeral B? Does numeral B have the property of repetition?

O Does numeral B in D2 have the property of addition? Use numeral C to help you explain your answer.

When you are working with numerals in bases other than ten, you need some way of naming them so that you will not become confused and also so that you can talk about them. D3 shows a good way to read and write numeral B in D2. Notice that the name of the base is written as a word and placed to the right and below the digits. Study D3.

P Look at the numerals in D4. What is the base of these numerals? Which numeral is read "two zero three one, base four"?

Q Read each of the other numerals in D4 in the way suggested in D3.

R Explain how you would obtain the decimal numeral for each of the numbers expressed by the numerals in D4. Then give a decimal numeral for each of the numbers.

You can also use digits in numeration systems that have bases other than four. Now you will study a base-ten numeration system that has place value.

A Numeral A in D5 is a base-ten numeral. What number do the five strokes in the space farthest to the right express?

B Do the strokes in the second space from the right express the number 9(10), or 90? How do you know?

C What number is expressed by the four strokes in the third space from the right? What number is expressed by the five strokes in the fourth space from the right?

D You know that each stroke in the fourth space from the right expresses the number $10 \times 10 \times 10$, or 1000. Since we are grouping by 10's, each stroke in the fifth space from the right expresses the number $10 \times 10 \times 10 \times 10$. Does each of these strokes express the number 10,000? What number is expressed by the two strokes?

E What number does numeral A express?

F Now look at numeral B in D5. It is also a base-ten numeral. Does the digit 5 in the space farthest to the right tell you how many strokes there are in the space farthest to the right in numeral A? Does the digit 5 express the number 5(1), or 5?

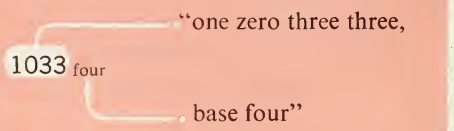
G For each digit in numeral B, tell what number it expresses.

H Does numeral B express the same number as numeral A?

B 1033

C $1(4 \times 4 \times 4) + 0(4 \times 4) + 3(4) + 3(1)$

D2



D3

D $2031_{four} = 2(64) + 0(16) + 3(4) + 1(1)$.

E 2301_{four}

F 2310_{four}

G 302_{four}

H 20_{four}

I 301_{four}

J 1000_{four}

D4

	(10×10×10×10)	(10×10×10)	(10×10)	10	1
A	//	/////	////	///// ////	/////
B	2	5	4	9	5

D5

I Why do you need just ten different digits to express any number in a base-ten system with place value? How many different digits do you need to express any number in a base-four numeration system with place value?

J Numeral B is shown again in D6. How do you know that the digit 5 in the fourth place from the right expresses the number 5(1000), and not the number 5(1)?

K Does the position of each digit in numeral B determine the number expressed by that digit?

L How do you know that numeral B has the property of place value? Of repetition? Of addition?

M Why is it not necessary to use a chart in writing numeral B? Why is it more convenient to use numeral B in D6 than to use numeral A in D5?

Since the base of numeral B is not expressed in D6, you know that the base is ten. Decimal numerals are written in a base-ten system with the property of place value. The word *decimal* comes from the Latin word for "ten." Consequently, numeral B is a decimal numeral.

You can read numeral B as "two five four nine five, base ten." But since you know that numeral B belongs to our familiar decimal system, you can say "twenty-five thousand four hundred ninety-five."

The digits from 0 through 9 can also be used in a base-twelve numeration system with place value.

A Numeral C in D7 is a base-twelve numeral. Do the strokes in the space farthest to the right express the number 10(1), or 10? How do you know?

B Do the strokes in the second space from the right express the number 4(12), or 48? How do you know?

B 25495

D6

	(12×12×12)	(12×12)	12	1
C	////	////// /////	////	///// /////
D	4	£	4	χ

D7

C Do the strokes in the third space from the right express the number 11(12 × 12), or 1584? What number is expressed by the strokes in the fourth space from the right?

D How can you determine that numeral C expresses the number eight thousand five hundred fifty-four?

E There are ten strokes in the space farthest to the right in numeral C. Can you use a familiar digit instead of the ten strokes? Explain your answer.

Since there is no familiar digit to use instead of the ten strokes, we must use a new digit. In this book, whenever we use the base-twelve numeration system, we will use the digit shown at the right instead of ten strokes. The new digit is read "dek." It was invented by a group of persons interested in the base-twelve system. For any numeration system whose base is greater than ten, we will use this digit for the number ten.

χ

F Look at numeral D in D7. What number does the digit χ express?

G What number is expressed by the digit in the second space from the right in numeral D?

H There are eleven strokes in the third space from the right in numeral C. Do you know of a digit that you can use instead of the eleven strokes?

	(12×12×12)	(12×12)	12	1
C	////	////// /////	////	///// /////
D	4	ℰ	4	𝒳

D7

D $4\mathcal{E}4\mathcal{X}_{\text{twelve}}$

D8

D9

The group of persons who invented the digit 𝒳 also invented a digit for the number eleven. We will use this digit, shown at the right, instead of eleven strokes. The digit is read "el." For any numeration system whose base is greater than eleven, we will use this digit for the number eleven.

ℰ

I Look again at numeral D in D7. Does the digit ℰ express the number $11(12 \times 12)$, or 1584?

J What number does the digit in the fourth space from the right express?

K Does numeral D express the same number as numeral C?

L Numeral D is shown again in D8. Why is it not necessary to use a chart in writing numeral D?

M What words would you say for numeral D?

You have learned that our decimal numeration system has the properties of addition, repetition, base, and place value. You know that decimal numerals are formed from ten digits. You have also seen that bases other than ten can be used to develop numeration systems with the same properties as our decimal system.

On your own

For each of the numeration systems named in exercises 1 through 4, tell what number is expressed by the numeral in D9. Remember to express each number by a decimal numeral.

1 Base four
2 Base six
3 Base seven
4 Base eleven

What number is expressed by each of the numerals below? Your answers should be expressed by decimal numerals.

5 16_{seven}
6 30_{seven}
7 300_{seven}
8 625_{seven}
9 560_{seven}
10 1502_{seven}
11 9_{twelve}
12 90_{twelve}
13 $1\mathcal{X}_{\text{twelve}}$
14 $\mathcal{X}1_{\text{twelve}}$
15 $\mathcal{E}03_{\text{twelve}}$
16 $12\mathcal{X}\mathcal{E}_{\text{twelve}}$
17 1111_{two}
18 1011_{two}
19 10111_{two}
20 11100_{two}
21 11001_{two}
22 10001_{two}

Express the first fifteen counting numbers in each of the following numeration systems.

23 Base five
24 Base six
25 Base nine
26 Base eleven

SPECIAL CHALLENGE

Give the answer to each problem.

A Mr. Ames sold 2 used cars for $528 each. He sold one of the cars for 20% less than he paid for it. He sold the other car for 20% more than he paid for it. Decide whether Mr. Ames made money or lost money on the sale of these 2 cars. Then tell how many dollars he made or lost.

B Jim has three samples of rock. He weighed the rocks two at a time so that each rock was weighed with each of the other rocks. The weights were 14 pounds, 15 pounds, and 19 pounds. How much does each rock weigh?

KEEPING SKILFUL

For each of exercises 1 through 5, first tabulate the solution set described. Then make a graph of each solution set. $U = N \times N$.

1 $\{(x, y) | x < 8 \wedge x + y = 7\}$

2 $\{(x, y) | y - 2 = 4 \wedge y - x > 1\}$

3 $\{(x, y) | x < 5 \wedge y = 4 + x\}$

4 $\{(x, y) | x + y = 9 \wedge x + 1 = y\}$

5 $\{(x, y) | 7 > 3 + x \wedge 8 - x > y\}$

For each of exercises 6 through 15, tabulate the solution set of the condition. U = C.

6 $27/54 \sim m/6$.

7 $r + 7 < 15$.

8 b is 3% of 1700.

9 78 is d% of 65.

10 $x + 6 < 93 \div 3$.

11 $9/7 \sim 207/s$.

12 $146 - t > 129$.

13 162 is 75% of n.

14 $3/4 \sim b/608$.

15 c is 130% of 150.

Make one sketch for exercises 16 through 20. Then use your sketch for exercises 21 through 26. The universe is the plane that includes circle G.

16 Make a sketch of circle G. Make a sketch of radius GH.

17 Make a sketch of chord HJ. Points G, H, and J are noncollinear.

18 Make a sketch of diameter JK.

19 Locate point M in the exterior of circle G. H is between G and M.

20 Locate point B between K and G. Locate C between H and J.

21 Write a standard description of circle G.

22 Write a standard description of the interior of circle G. Of the exterior of circle G.

23 Write a sentence that expresses a true statement comparing $\overline{GK}$ and $\overline{JG}$.

24 Write a sentence that expresses a true statement comparing $\overline{GM}$ and $\overline{GC}$.

25 Write a sentence that expresses a true statement comparing $\overline{HG}$ and $\overline{MG}$.

26 Write a sentence that expresses a true statement comparing $\overline{GH}$ and $\overline{JK}$.

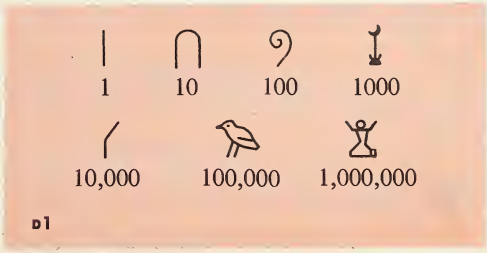
5 52 Exploring ideas

Egyptian, Roman, and Babylonian numeration systems

You know that a modern numeration system has the properties of addition, repetition, base, and place value. Many numeration systems of ancient times also had some, or all, of these properties. In fact, a modern numeration system was developed by combining the advantages of these ancient systems. In this lesson you will study the Egyptian, Roman, and Babylonian numeration systems. You will discover how each of these systems was helpful in the development of the numeration system that we use today.

A D1 shows some of the symbols in the numeration system of the ancient Egyptians. The Egyptians used a picture of a vertical staff to express the number 1. They used a symbol sometimes called a "heelbone" to express the number 10. The "heelbone" could be used instead of how many vertical staffs? The symbol for 100 could be used instead of how many "heelbones"?

B Study the symbols shown in D1. What was the base of the Egyptian numeration system?



D1

C D2 shows four numerals as the Egyptians might have written them. Numeral A expresses the number $100 + 100 + 10 + 10$, or 220. Do you obtain the number named in numeral A by counting the symbols in numeral A? Explain your answer.

D Now look at numeral B in D2. The Egyptian numeration system did not have the property of place value. What number does numeral B express? How did you obtain your answer?

E Are numeral A and numeral B names for the same number?

F What number does numeral C express? What number does numeral D express?

G Did the Egyptian numeration system have the property of addition? Of repetition? Explain your answers.

Next you will use the Egyptian numeration system to find the sum of several numbers.

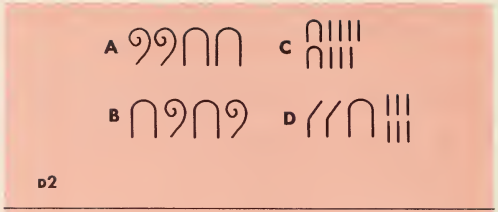
H What number is expressed by each numeral in D3?

I To find the sum of the numbers expressed in D3, you first collect the symbols in the numerals. Does numeral H in D4 show this collection?

J What number does numeral H express? How did you obtain your answer?

K How was numeral I obtained from numeral H? Does numeral I express the same number as numeral H?

L Why do you think our decimal numeration system is more convenient for naming numbers than the ancient Egyptian numeration system?



D2

D3

D4

I	V	X	L	C	D	M
1	5	10	50	100	500	1000

D5

You are already somewhat familiar with Roman numerals. Now you will compare the Roman numeration system with a modern numeration system.

A The seven letters commonly used as symbols in the Roman numeration system are shown in D5. The Roman symbol I expresses the number 1. What is the Roman symbol for the number 10? For the number 10×10? For the number $10 \times 10 \times 10$?

B The Roman numeration system also had a special symbol for the number 5(1). What is this symbol?

C What is the Roman symbol for the number 5(10)? For the number $5(10 \times 10)$?

A VI	E XI	G LX	J CX
B IV	F IX	H XL	K XC

D6

N XCIII
P CXIII

D7

Q XX
R LIII
S CXXXVII

D8

The Romans used groups of 10 in their numeration system. However, since there are special Roman symbols for 5 ones, for 5 groups of 10, and for 5 groups of 10×10, the Roman numeration system is really a "mixed" grouping system.

D Look at numeral A in D6. The Roman symbol for the number 1 is to the right of the symbol for the number 5. To find the number expressed by numeral VI, you add 1 to 5. What number does numeral A express?

E What is the Roman numeral for the number 7? For the number 8?

F How is numeral B in D6 like numeral A? How is numeral B different from numeral A?

To find the number expressed by the numeral IV, you subtract the number 1 from the number 5. If a symbol is written at the right of another symbol that expresses a greater number, you add the lesser number to the greater. If a symbol is written at the left of another symbol that expresses a greater number, you subtract the lesser number from the greater.

G To obtain the number expressed by numeral E, which do you do, add 1 to 10 or subtract 1 from 10? What number is expressed by numeral E?

H To obtain the number that numeral F expresses, which do you do, add 1 to 10 or subtract 1 from 10? What number is expressed by numeral F?

I Does the Roman numeration system have the property of addition? Is subtraction also used in the Roman system?

J What number is expressed by each of the numerals in D6?

K For which numbers that you obtained for exercise J did you use addition? For which numbers did you use subtraction?

L Study numerals N and P in D7. What number does each symbol in numeral N express? What number does each symbol in numeral P express?

M What number is expressed by numeral N? By numeral P?

N Does the position of a symbol in a Roman numeral always determine the number expressed by that symbol? Does the Roman numeration system have the property of place value?

O The same symbols are used in numeral N and numeral P. Explain why numeral N and numeral P do not name the same number.

P Does the Roman numeration system have the property of repetition? Explain your answer.

Now you will use the Roman numeration system to find the sum of several numbers.

Q What number is expressed by each numeral in D8?

R To find the sum of the numbers expressed in D8, you first collect the symbols in the numerals. You will have how many I's? You will have how many X's?

S Does numeral T in D9 show the collection of numerals? What Roman symbol can you use instead of the five I's? What Roman symbol can you use instead of the five X's?

T Now look at numeral U in D9. Does it name the same number as numeral T?

U How was numeral W obtained? What number does numeral W express?

V Are numeral T, numeral U, and numeral W names for the same number? Which of these numerals is the most convenient to use?

W When you add XCVII and XLV, why should your collection of numerals be XC + XL + VVII, and not CLXXVVII?

X Do you think that our decimal system is more convenient for naming numbers than the Roman system? Explain your answer.

The Babylonian numeration system used only two symbols and a base of sixty.

A D10 shows some Babylonian symbols. What number does each symbol in D10 express?

B Numeral A in D11 expresses the number 10 + 10 + 10 + 1 + 1 + 1, or 33. What number does numeral B express?

C Did the Babylonian numeration system have the property of addition? Of repetition?

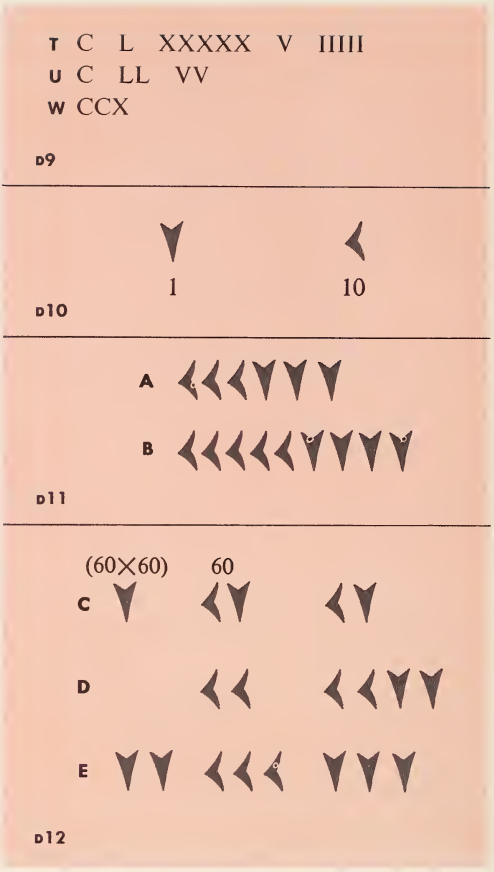
D To express numbers greater than sixty, the symbols in Babylonian numerals were arranged in a special way. Look at numeral C in D12. The symbols in the group farthest to the right express the number 10 + 1, or 11. Look at numeral D. What number is expressed by the symbols in the group farthest to the right?

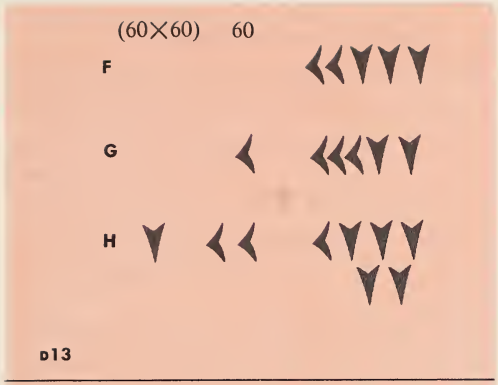
E To find the number expressed by the symbols in the group second from the right, you first find the number expressed by each symbol. Then you find the product of each of these numbers and 60. In numeral C, the symbols in the group second from the right express the number 10(60) + 1(60), or 660. In numeral D, what number is expressed by the symbols in the group second from the right?

F In numeral C, the symbol in the group third from the right expresses the number 1(60 × 60), or 3600. What number does numeral C express?

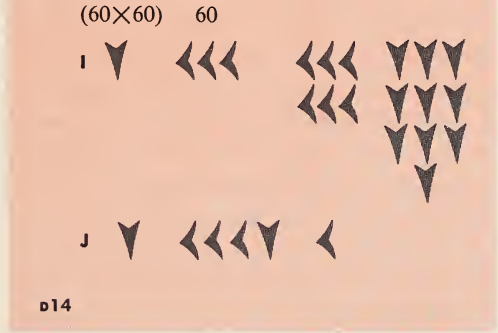
G To find the number expressed by numeral D, you find the sum of the numbers expressed by the symbols in the groups. What number does numeral D express?

H In numeral E, what number is expressed by the symbols in the group third from the right? What number does numeral E express?





D13



D14

I Did the Babylonian system have the property of place value? Explain your answer.

Now you will use the Babylonian numeration system to find the sum of several numbers.

J Look at D13. What number does each numeral express?

K To find the sum of the numbers expressed in D13, first collect the symbols in the numerals, keeping the groups of symbols separated. Does numeral I in D14 show this collection?

L What number does numeral I express?

M In the group farthest to the right in numeral I, how many times is the symbol for the number 1 used? What Babylonian symbol can you use instead of these ten symbols?

N In the group farthest to the right in numeral I, how many times is the symbol for the number 10 used? Why can you use a symbol for 1 in the group second from the right in place of these symbols for 10?

O Now look at numeral J in D14. Does numeral J express the same number as numeral I? Which of these numerals is more convenient to use?

P Do you think our decimal numeration system is a more convenient system for naming numbers than the Babylonian system? Explain your answer.

In this lesson you have compared the properties of some ancient numeration systems with the properties of a modern numeration system.

On your own

For each of exercises 1 through 9, write the decimal numeral for the number expressed. If you do not remember what numbers the symbols in the numerals express, look at D1 on page 219, D5 on page 220, and D10 on page 222.

1

2

3

4 MDCCLXXVI

5 DCXLIV

6 XCIX

7

8

9

The numerals in exercises 10 through 14 are decimal numerals. For each exercise, first use the Egyptian numeration system to write a numeral for the number named. Then write a Roman numeral for the number.

10 89 11 105 12 1416 13 3244 14 928

The numerals in exercises 15 through 20 are Roman numerals. Use the Egyptian numeration system to express each number named.

15 XLII 17 DLXXIV 19 CDL

16 CCCXXVI 18 MDXC 20 CMLVII

CHECKING UP

If you have trouble with these two tests, you can find help in lessons 49 through 52.

Test 54

For each numeration system described in exercises 1, 2, and 3, tell what number is expressed by the numeral in the display below. Give your answer as a decimal numeral.

1 Base six **2** Base nine **3** Base twelve

For each of exercises 4 through 12, what number is expressed by the Roman numeral?

4 VIII	**7** XLVI	**10** XXXIII
5 XIV	**8** XCI	**11** MCMLXIII
6 XIX	**9** CIX	**12** DXX

Test 55

A list of numeration systems is given below. For each of exercises 13 through 16, choose from the list the system that is described in the exercise.

a Tally system

b Grouping system without place value

c Grouping system with place value and digits

d Grouping system with place value and the stroke

e Code system

13 This numeration system does not have the property of repetition.

14 The greatest number of strokes needed in any space in a numeral is one less than the base.

15 The number of symbols needed to form the numerals is equal to the base of the numeration system.

16 The number of times a symbol is used in a numeral is the same number as the number expressed by that numeral.

APPLYING MATHEMATICS

The problems in this lesson include the various kinds of problems that you have studied so far. For each problem, write a sentence that expresses a condition. You may use any letters as names for the variables. Tabulate the solution set of the condition and give the answer to the problem. The universe is given for each problem.

1 One year, 8,174,779 tons of iron ore were mined in Newfoundland, and 8,872,948 were mined in Quebec. Altogether, how many tons of iron ore were mined in these two provinces that year? $U = N$.

2 Use the information given in problem 1 to answer the following question: How many fewer tons of iron ore were mined in Newfoundland than in Quebec that year? $U = N$.

3 During July and August, Jerry and Ken mowed 42 lawns. Ken mowed 14 lawns during July and 9 lawns during August. How many lawns did Jerry mow during these two months? $U = N \times N$.

4 In a recent year there were about four times as many factories in Montreal as there were in Vancouver. There were about 1200 in Vancouver. About how many factories were there in Montreal? $U = C$.

5 Mrs. Brown made more than 35 cupcakes. She sent 24 of them to a bake sale and had fewer than 17 cupcakes left. How many cupcakes could she have made? $U = N$.

6 Sue is younger than Ann. In 5 years, Ann will be 19 years old. What is Ann's age now? What can Sue's age be now? $U = C \times C$.

7 The Hayes Store for Men had 115 raincoats to sell at a 30% discount. By the end of the first day, there were fewer than 59 raincoats left to sell. How many raincoats could have been sold that day? $U = N$.

8 Kay bought some chocolate cookies at 5¢ each and some ginger cookies at 3¢ each. She bought fewer chocolate cookies than ginger cookies. She bought 7 cookies in all. How many cookies of each kind could she have bought? $U = C \times C$.

9 Over a five-year period, 39,138,000 tons of raw sugar were produced in the world. Approximately 34% of this total was produced in North America. About how many tons of raw sugar were produced in North America during this five-year period? $U = C$.

10 There are 71,768,000 telephones in Canada and the United States together. If there were 877,000 more telephones in Canada, the number of telephones in Canada would be 6,000,000. How many telephones are there in each of these two countries? $U = N \times N$.

11 Scientists believe that the Chubb Crater in the province of Quebec, was formed by the impact of a huge meteorite. The depth of this crater is 1800 feet. The comparison of its width to its depth is 64 feet to 10 feet. What is the width of the Chubb Crater? $U = C$.

12 The rate of interest at a certain bank is 6%. Mr. Horton borrowed $1350 and repaid his loan with interest at the end of one year. How much did he pay the bank at the end of the year? $U = C \times C$.

13 During a basketball game, Ted scored 7 more points than Bill. Together, they scored 21 points. How many points did each boy score during the game? $U = N \times N$.

14 Mrs. Jackson paid $172 for a freezer that was regularly priced at $215. What was the rate of discount? $U = C \times C$.

15 It is said that the earth's temperature rises 1° Fahrenheit for each 60 feet beneath the earth's surface. What is the earth's temperature 18,120 feet beneath the surface, if the surface temperature is 49° Fahrenheit? $U = C \times C$.

KEEPING SKILFUL

For exercises 1 through 8, round each number named to the next lesser thousand, to the next greater hundred, and to the next lesser ten.

1 1926 **3** 6203 **5** 35,165 **7** 2711
2 47,491 **4** 1999 **6** 9002 **8** 95,659

Find the sum.

9 8033, 6945, 9826, 9403
10 6858, 702, 73413, 553
11 14033, 34042, 50823, 41374, 56428
12 29639, 70130, 4775, 86562, 10075
13 711008, 647, 3246, 26054

Find the difference.

14 4938, 654
15 8023, 6042
16 3554, 2718
17 754110, 191786
18 10174, 2942
19 71857, 59759

Multiply

20 6293 by 82.
21 145 by 37.
22 208 by 145.
23 476 by 319.
24 194 by 284.
25 802 by 550.

Divide

26 17784 by 76.
27 14336 by 81.
28 5292 by 106.
29 41648 by 213.
30 50076 by 319.
31 38000 by 125.

For exercises 32 through 46, tabulate each set described. $U = C \times C$.

32 $\{(x, y) \mid x + y < 6\}$
33 $\{(m, n) \mid m/n \sim 3/4 \wedge n - m = 4\}$
34 $\{(r, s) \mid s - r = 3 \wedge r + 8 < 17\}$
35 $\{(c, d) \mid c/d \sim 15/10 \wedge c > d\}$
36 $\{(x, y) \mid x + y = 150 \wedge 68 - y = 17\}$
37 $\{(m, n) \mid 9/15 \sim m/n \wedge n - m < 9\}$
38 $\{(s, t) \mid s < t \wedge s + t < 9\}$
39 $\{(a, b) \mid a/b \sim 14/21 \wedge a + b < 30\}$
40 $\{(x, y) \mid 8/100 \sim 50/x \wedge x - 50 = y\}$
41 $\{(s, t) \mid t - 18 > s \wedge t = 972 \div 36\}$
42 $\{(m, n) \mid n/m \sim 5/6 \wedge 33 - m = n\}$
43 $\{(c, d) \mid c/47 \sim 432/282 \wedge c + d < 80\}$
44 $\{(a, b) \mid b + a < 25 \wedge b = 3a\}$
45 $\{(m, n) \mid m/2 \sim n/100 \wedge m = 11 - 8\}$
46 $\{(x, y) \mid x/y \sim 54/63 \wedge x + y < 13\}$

5 53 Exploring ideas

Factors, powers, and exponential notation

In the preceding lessons of this unit you studied the properties of a modern numeration system. In this lesson you will learn how to write numerals in a form that will make it easier for you to change numerals from one base to another. First you must learn more about products of numbers.

A Does each sentence in D1 express a true statement? Does each sentence suggest a different way of thinking about 16?

B You can think of 16 as the product of 2 and 8. Look at D1 again. Describe other ways to think of 16 as a product.

C The counting numbers used to obtain a product are *factors* of that product. Are 2 and 8 factors of 16? Are 16 and 1 factors of 16? Is 4 a factor of 16? Is 6 a factor of 16?

D Some numbers may be used as a factor more than once to obtain a product. 4 may be used twice as a factor to obtain 16. How many times may 2 be used as a factor to obtain 16?

E You know that $3 \times 4 = 12$ is a true statement. Is 3 a factor of 12? Is 4 a factor of 12?

F Is $6 \times 2 = 12$? Is 6 a factor of 12? Is 2 a factor of 12?

G Is $3 \times 2 \times 2 = 12$? Which factor is used more than once in $3 \times 2 \times 2 = 12$?

H The six statements expressed in D2 were obtained from $a \times b = c$. Is each statement true?

I What replacements were made for the variables in $a \times b = c$ to obtain $12 \times 1 = 12$? Are 12 and 1 factors of 12? Explain your answer.

J Are 7 and 2 factors of 14? Are 8 and 3 factors of 24?

K Which factors of 45 are expressed in D2?

If $a \times b = c$, then the replacements for a and b are both factors of the replacement for c. U = C.

L Is 1 a factor of 15? Is 1 a factor of 60?

M Is 15 a factor of 15? Is 60 a factor of 60?

N Suppose that b in $1 \times b = b$ is replaced by ? any member of {1, 2, 3, . . .}. Will the statement obtained be true?

O Think of any member of {1, 2, 3, . . .}. Is ? 1 a factor of the number? Is the number a factor of itself?

You know that the decimal numeration system has the property of place value. You know, for example, that the digit 3 in the decimal numeral 347 expresses the number $3(10 \times 10)$. Notice that 10 is used twice as a factor. There

A $2 \times 8 = 16$. C $4 \times 4 = 16$.
B $16 \times 1 = 16$. D $2 \times 2 \times 2 \times 2 = 16$.

D1

U = C.
$a \times b = c$.

E $12 \times 1 = 12$. H $3 \times 15 = 45$.
F $7 \times 2 = 14$. I $5 \times 9 = 45$.
G $8 \times 3 = 24$. J $45 \times 1 = 45$.

D2

fac tor (fak′tər). One of the numbers used to form a product. In $a \times b = c$, suppose that a, b, and c are replaced by members of {1, 2, 3, . . .}. If a true statement is obtained from $a \times b = c$, then the replacement for a is a factor of the replacement for c. The replacement for b is also a factor of the replacement for c.

is a shorter way to show how many times you are using the same number as a factor.

A D3 shows how to write and read a numeral that expresses the number obtained when 10 is used twice as a factor. Study D3.

B You can also read the numeral 10^2 as "ten squared." Is $10^2 = 10 \times 10$? How do you know that 10^2 and 100 are the same number?

C You say "the third power of 10" or "ten cubed" for the numeral 10^3. The numeral 10^3 expresses the number obtained when 10 is used three times as a factor. Is $10^3 = 10 \times 10 \times 10$? Are 10^3 and 1000 the same number?

When you write the numeral 10^2 to express the number 100 or the numeral 10^3 to express the number 1000, you are using *exponential notation* (eks′pō nen′shəl no tā′shən).

D Use exponential notation to express the number obtained when you use 10 as a factor four times. What is this number?

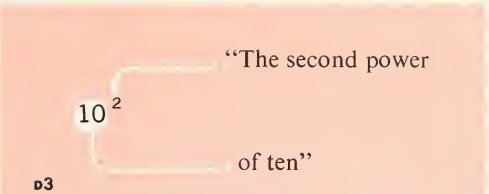
E The numeral 10^1 expresses the number 10. What number is expressed by the numeral 5^1?

F Numbers like 10^1, 10^2, 10^3, and 10^4 are *powers* of 10. The number 10^1 is the first power of 10. The number 10^2 is the second power of 10. What power of 10 is 10^4?

The number 10 is the *base* of the powers 10^1, 10^2, 10^3, and 10^4. The base of a power is the number that is used as a factor one or more times. The number 4 is the *exponent* of 10^4. The exponent is the number that tells how many times the base is used as a factor.

G What number is the exponent of 10^2? Of 10^3? Of 10^7? How many times is 10 used as a factor in 10^2? In 10^3? In 10^7?

H Look at D4. Exponential notation is used to express five numbers. Each of these numbers was obtained by replacing each variable in a^n by a member of {1, 2, 3, . . .}. What replacements were made in a^n to obtain 3^6? To obtain 7^1? To obtain 5^4?



D3

U = C.

a^n 3^6 7^1 5^4 11^2 10^7

D4

pow er (pou′ər). A number that is obtained by using the same factor one or more times. Suppose that, in a^n, a and n are both replaced by members of {1, 2, 3, . . .}. The number obtained from a^n is called the nth power of a. The replacement for the variable a is the **base** of a^n. The replacement for the variable n is the **exponent** (eks pō′nənt) of a^n.

I Is 3^6 a power of 3?

J What is the base of 3^6? What is the exponent of 3^6? How many times is 3 used as a factor in 3^6?

K For each of the other powers named in D4, name the number that is the base. Name the number that is the exponent.

L Express each number named in D4 without using exponential notation.

M Is $3^1 = 3$? Is $20^1 = 20$?

N Suppose that you replace a in $a^1 = a$ by any member of {1, 2, 3, . . .}. Will the statement you obtain be true? Is the first power of each counting number the same as that counting number?

What number is expressed by each of the numerals in exercises O through S?

O 1^1 **P** 1^3 **Q** 1^7 **R** 1^{25} **S** 1^{100}

T Suppose that you replace n in $1^n = 1$ by any member of {1, 2, 3, . . .}. Will the statement you obtain be true? Explain your answer.

A 4263

B $4(10\times10\times10)+2(10\times10)+6(10)+3(1)$

C $4(10^3)+2(10^2)+6(10^1)+3(1)$

D5

You have learned how to obtain the number expressed by a numeral written in exponential notation. Now you will learn to write numerals by using exponential notation.

A You can think of the number 16 as the product 4×4 or the product $2\times2\times2\times2$. Is $4^2=4\times4$? Is $2^4=2\times2\times2\times2$? Can 16 be expressed by either 4^2 or 2^4?

Each number named below is a power of 10. Use exponential notation to express each of these powers.

B 100 **C** 10,000 **D** 1000 **E** 100,000

F Look at D5. Numeral A is a base-ten, or decimal, numeral. Does numeral B show how the number 4263 was obtained?

G How was numeral C obtained? Does numeral C also show how the number 4263 was obtained? Numeral C is the *expanded form* of the decimal numeral 4263.

In the decimal numeral 4263, the digit 3 is associated with (1). The digit 3 expresses the number 3(1), or 3. The digit 6 is associated with the first power of ten. The digit 6 expresses the number $6(10^1)$, or 60.

H Which digit in the numeral 4263 is associated with the second power of ten? Which digit in the numeral is associated with the third power of ten?

I Which of the following is the expanded form of the decimal numeral 35,927?

$3(10^5)+5(10^4)+9(10^3)+2(10^2)+7(10^1)$

$3(10^4)+5(10^3)+9(10^2)+2(10^1)+7(1)$

Write each decimal numeral below in expanded form.

J 185 **K** 7330 **L** 29,537 **M** 663,501

In this lesson you have learned about factors, powers, exponents, and exponential notation. You also know how to write decimal numerals in expanded form.

On your own

Use set A, tabulated below, as you answer exercises 1 through 4.

A = {1, 2, 3, 5, 12, 18, 30, 36, 45, 54, 85, 99}.

1 List the names of the members of A that have 2 as a factor.

2 List the names of the members of A that have 3 as a factor.

3 List the names of the members that have 5 as a factor.

4 List the names of the members that have 7 as a factor.

Tabulate the set of all possible factors of each number expressed in exercises 5 through 10. For exercise 5 you should write: {1, 2, 4, 8}.

5 8 **6** 19 **7** 14 **8** 11 **9** 35 **10** 60

11 Is 1 a factor of each number expressed in exercises 5 through 10? Is each of these numbers a factor of itself?

For each of the powers expressed below, name the number that is the base. Name the number that is the exponent.

12 10^4 **13** 15^3 **14** 7^7 **15** 9^{10}

For each of exercises 16 through 27, express each number named without using exponential notation.

16 2^1 **17** 2^2 **18** 2^4 **19** 2^5 **20** 9^1 **21** 9^2 **22** 1^9 **23** 5^3 **24** 3^5 **25** 10^5 **26** 4^1 **27** 12^2

28 Use exponential notation to express 81 as a power of 3.

29 Use exponential notation to express 64 as a power of 2.

30 Use exponential notation to express 121 as a power of 11.

31 Use exponential notation to express 1,000,000 as a power of 10.

For each of sentences 32, 33, and 34, tell whether it expresses a true or a false statement.

32 $10^3 < 3^{10}$. 33 $1^7 = 7^1$. 34 $3^4 > 3^3$.

Each numeral in exercises 35 through 39 is a decimal numeral. Write each numeral in expanded form.

35 16 36 58 37 749 38 1007 39 262,844

KEEPING SKILFUL

For each of exercises 1, 2, and 3, first tabulate sets S and T. Next tabulate S $\cup$ T. Then tabulate S $\cap$ T. U = N $\times$ N.

1 $S = \{(m, n) | m + n < 10 \wedge m + 1 = n\}$.
$T = \{(m, n) | 4 > n \wedge n > m\}$.

2 $S = \{(r, s) | r < 6 \wedge 7 - r = s\}$.
$T = \{(r, s) | s > r \wedge s + 19 = 25\}$.

3 $S = \{(g, h) | g + 5 < 11 \wedge h + 2 = 9\}$.
$T = \{(g, h) | g + h = 9 \wedge h + 1 < 8\}$.

Use one sketch for exercises 4 through 11.

4 Show $\triangle ABC$.

5 Show circle B that intersects $\overline{BC}$ in point G and $\overline{AB}$ in point F.

6 Locate point J in the interior of circle B and in the exterior of $\triangle ABC$.

7 Locate point K in the interior of $\triangle ABC$ and in the interior of circle B.

8 Locate point M in the interior of $\angle CBA$ and in the exterior of $\angle CAB$.

9 Which points named are in the intersection of $\triangle ABC$ and the interior of circle B?

10 Which points named are in the union of circle B and the exterior of $\triangle ABC$?

11 Which points named are in the intersection of $\triangle ABC$ and the exterior of circle B?

5 54 **Exploring ideas**

Converting numerals from one base to another

In the last lesson you learned to write decimal numerals in expanded form. In this lesson you will learn to use the expanded form for numerals with bases other than ten. You will also learn more about changing numerals from one base to another.

A Suppose that 5623 is a decimal numeral. Which digit in the numeral expresses the number of ones?

B Which digit in the numeral 5623 is associated with the first power of ten? With the second power of ten? With the third power of ten?

C Write the decimal numeral 5623 in expanded form.

D Look at D1. What is the base of numeral A? Does numeral B show how to obtain the number expressed by numeral A?

REMINDER
Numerals that have bases other than ten are read and written in a special way.
See lesson 51, page 216.

D1

A 5623_{seven}

B $5(7 \times 7 \times 7) + 6(7 \times 7) + 2(7) + 3(1)$

C $5(7^3) + 6(7^2) + 2(7^1) + 3(1)$

You know that the digit 7 is not used in a base-seven system. Since the digit 7 is a decimal symbol for the number seven, numeral B in D1 is a decimal numeral.

E Explain how numeral C was obtained from numeral B.

F Does numeral C also show how to obtain the number expressed by numeral A?

Numeral C is the expanded form of the numeral 5623_{seven} Numeral C is also a decimal numeral. All of the expanded forms of numerals that you will work with in this book will be decimal numerals.

G Numeral C shows you that the digit 3 in numeral A expresses the number of ones. Is the digit 2 in numeral A associated with the first power of seven?

H Which digit in the numeral 5623_{seven} is associated with the second power of seven? With the third power of seven?

In lesson 49 you learned about the base of a grouping system. Now you will use what you know about powers to learn more about the base of a numeration system.

For exercise C, you wrote the decimal numeral 5623 in expanded form. Ten is the *base* of the numeration system to which the numeral 5623 belongs. The digits in the numeral 5623 are associated with powers of ten. D1 shows how to write the numeral 5623_{seven} in expanded form. In this case, seven is the base of the numeration system to which the numeral 5623_{seven} belongs. This time the digits in the numeral are associated with powers of seven.

I Look at D2. What is the base of numeral D?

J Numeral E in D2 is the expanded form of numeral D. Use numeral E to answer the following question: What is the number whose powers are associated with the digits in numeral D?

A 5623_{seven}
B $5(7 \times 7 \times 7) + 6(7 \times 7) + 2(7) + 3(1)$
C $5(7^3) + 6(7^2) + 2(7^1) + 3(1)$

D1

D 4817_{nine}
E $4(9^3) + 8(9^2) + 1(9^1) + 7(1)$

D2

A 310_{four}	$3(4^2) + 1(4^1) + 0(1)$
B 101_{two}	$1(2^2) + 0(2^1) + 1(1)$
C 452_{six}	$4(6^2) + 5(6^1) + 2(1)$
D $5\mathcal{E}9_{twelve}$	$5(12^2) + 11(12^1) + 9(1)$

D3

base of a place-value numeration system. The number whose powers are associated with the digits in the numerals in a place-value numeration system.

K Which digit in numeral D expresses the number of ones? With what power of the base is each of the other digits associated?

Read each of the numerals below. Then write each numeral in expanded form.

L 1302_{four} M 3542_{six} N 3277_{eight}

In previous lessons you changed, or converted, numerals with bases other than ten to base-ten numerals. Now you will use what you have learned about writing numerals in expanded form to convert numerals from one base to another.

A Look at D3. Read each numeral in the first column. What is the base of each of these numerals?

B In numeral A, how many ones does the digit 0 express? With what power of 4 is the digit 1

associated? With what power of 4 is the digit 3 associated?

C In numeral B, with what power of 2 is the digit 0 associated? In numeral C, with what power of 6 is the digit 4 associated? In numeral D, with what power of 12 is the digit £ associated?

D In the second column in D3, numerals A, B, C, and D have been written in expanded form. Does each expanded form show how to obtain the number expressed by the numeral in the first column?

E What number is expressed by each numeral in the first column?

Write each of the following numerals in expanded form:

F 30_{seven} **H** 210_{three} **J** 1314_{five}

G 25_{eight} **I** 435_{six} **K** 10101_{two}

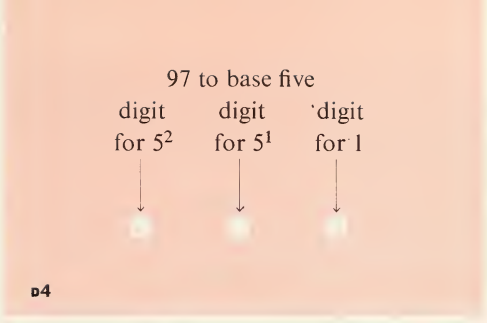
L Tell what number is expressed by each numeral in exercises F through K. Remember to use a decimal numeral to express each number.

You know how to convert a numeral with a base other than ten to a decimal numeral. In the following exercises you will learn to convert a decimal numeral to a numeral with a base other than ten.

A Suppose that you want to convert the decimal numeral 97 to a base-five numeral. To find a base-five numeral that expresses the number 97, first decide for what powers of 5 the numeral will have digits. If there is to be a digit for 5^2, for example, then 97 must be equal to or greater than 5^2. What number is the second power of 5? Will there be a digit for 5^2?

B If there is to be a digit for 5^3, then 97 must be equal to or greater than 5^3. Will there be a digit for 5^3? How do you know?

C Since there is to be a digit for 5^2, must there also be digits for 5^1 and 1? How many digits will there be in the base-five numeral for 97?



D4

D Study D4. To find the digit that belongs in the space associated with 5^2, you divide 97 by 5^2. Is this the same as dividing by 25?

E Study the computation at the right. How does it show that the digit 3 is associated with 5^2? In which space in D4 does the digit 3 belong?

$$\begin{array}{r|l} 25\overline{)97} & 3 \\ 75 & \\ \hline 22 & \end{array}$$

F When 97 is divided by 25, the remainder is 22. Divide the 22 by 5^1. Look at the computation. What digit is associated with 5^1? In which space in D4 does the digit 4 belong?

G How many ones remain? In which space does the digit 2 belong?

$$\begin{array}{r|l} 5\overline{)22} & 4 \\ 20 & \\ \hline 2 & \end{array}$$

H The numeral at the right expresses the same number as the numeral 97. Which digit in the numeral 342_{five} expresses the number of ones? With which power of 5 is each of the other digits in the numeral 342_{five} associated?

342_{five}

I Convert the numeral 342_{five} to a decimal numeral. Is $97 = 342_{\text{five}}$?

J Next you will convert the decimal numeral 132 to a base-eight numeral. In a base-eight numeral, the digits are associated with powers of what number?

K If there is to be a digit for 8^2, the number 132 must be equal to or greater than 8^2. What number is the second power of 8? Will there be a digit for 8^2?

L How do you know that there will not be a digit for 8^3? How many digits will there be in the base-eight numeral for 132?

M Study the computation at the right. Why do you divide by 64? What digit is associated with 8^2? What is the remainder?

$$\begin{array}{r|l} 64\overline{)132} & 2 \\ 128 & \\ \hline 4 & \end{array}$$

N Now divide the remainder, 4, by 8^1. How do you know that the digit associated with 8^1 is 0?

$$\begin{array}{r|l} 8\overline{)4} & 0 \\ 0 & \\ \hline 4 & \end{array}$$

O How many ones remain?

P The numeral at the right expresses the same number as the numeral 132. Which digit in the numeral 204_{eight} expresses the number of ones? With which power of 8 is each of the other digits associated?

204_{eight}

Q Convert the numeral 204_{eight} to a decimal numeral. Is $132 = 204_{\text{eight}}$?

Each numeral below is a decimal numeral. You are to convert each numeral to a numeral in another numeration system. The base of the other numeration system is expressed by the word within parentheses.

R 85 (six) S 123 (twelve) T 31 (two)

We can also convert a numeral that does not have ten as a base to another numeral that also does not have ten as a base. For example, let us convert the numeral 346_{nine} to a base-twelve numeral.

U First convert the numeral 346_{nine} to a decimal numeral. What is the decimal numeral for 346_{nine}?

V Now convert the decimal numeral you obtained for exercise U to a base-twelve numeral. What is the base-twelve numeral?

W Do the numeral 346_{nine} and the base-twelve numeral you obtained for exercise V express the same number?

In this lesson you have learned to use the expanded form for numerals with bases other than ten. You also have learned how to convert a numeral with any given base to a numeral with a different base.

On your own

Each numeral in exercises 1 through 4 is written in expanded form. First write the numeral from which each expanded form was obtained. Be sure to express the base of this number. Then write the decimal numeral for the number expressed.

1 $3(5^3) + 1(5^2) + 2(5^1) + 3(1)$
2 $6(7^2) + 4(7^1) + 5(1)$
3 $5(8^3) + 7(8^2) + 0(8^1) + 0(1)$
4 $2(12^2) + 11(12^1) + 8(1)$

For each of exercises 5 through 13, convert the numeral to a decimal numeral.

5 635_{seven}
6 1001_{two}
7 972_{twelve}
8 3405_{six}
9 1042_{five}
10 31022_{four}
11 1211021_{three}
12 495_{eleven}
13 2377_{eight}

For exercises 14 through 21, convert each decimal numeral to a numeral in the numeration system whose base is expressed within parentheses.

14 104 (base five)
15 64 (base twelve)
16 87 (base two)
17 648 (base five)
18 3257 (base seven)
19 43 (base three)
20 72961 (base eight)
21 213 (base nine)

For each of exercises 22 through 25, express the first fourteen counting numbers in the numeration system indicated in the exercise.

22 Base ten
23 Base two
24 Base twelve
25 Base eight

For exercises 26 through 33 convert each numeral to a base-twelve numeral.

26 346_{seven}
27 1011_{two}
28 437_{eight}
29 106
30 2102_{three}
31 5641_{seven}
32 156
33 43122_{five}

Computing in numeration systems other than the decimal system

Now that you have learned to express numbers using different numeration systems, you are prepared to compute in different systems. This practice in computation will help you to understand better the properties of the decimal numeration system.

A Look at D1. The first ten natural numbers are expressed as both decimal numerals and base-four numerals. Write decimal numerals for the next five natural numbers. Then write base-four numerals for the same five natural numbers.

B How many different digits are needed for a decimal numeration system? For a base-four numeration system?

When used alone, the ten digits of a decimal system express the natural numbers from zero through nine. You already know how to add any two of these natural numbers. You also know how to use a decimal numeral to express their sum.

When used alone, the four digits of a base-four system express the natural numbers from zero through three. The work that follows shows how to add any two of these numbers and how to use a base-four numeral to express the sum.

The table in D2 is part of an addition table for a base-four numeration system. When completed, the table will show how the sum of any two of the first four natural numbers can be expressed as a base-four numeral.

Decimal	0	1	2	3	4	5	6	7	8	9
Base four	0	1	2	3	10	11	12	13	20	21

D1

+	0	1	2	3
0	0	1	2	3
1	1	2	3	
2	2	3		
3	3	10	11	12

D2

C Study the table in D2. To find the numeral that expresses the sum of 1 and 3, first find the red numeral 1 in the top row. Then look down the column until you find the numeral that is opposite the red numeral 3. Notice that the numeral 10 ("one zero") is used to express the sum of 1 and 3. Why are no digits other than 0, 1, 2, and 3 used in a base-four numeration system?

D Suppose that you want to express the sum of 2 and 3 as a base-four numeral. Which should you use, the numeral 11 or the numeral 5? Explain your answer.

E What base-four numeral should you use to express the sum of 3 and 3? The sum of 2 and 2?

F Copy and complete the table in D2.

Next you will review how to express the sum of twenty-five and thirty-nine as a decimal numeral. Then you will learn how you can do the same computation with base-four numerals.

G Look at the work below. Add the ones in 25 and 39. Should you think of 14 as 1 ten and 4 ones? Where should you write a digit to express the 4 ones? You should remember the 1 ten to add to the other tens.

$$\begin{array}{r} 25 \\ 39 \\ \hline 4 \end{array}$$

H Add the tens in 25 and 39. Do not forget to include the 1 ten you obtained from adding the ones. How many tens in all are there? Where should you write a digit to express the total number of tens? What decimal numeral expresses the sum of 25 and 39?

$$\begin{array}{r} {\scriptstyle 1} \\ 25 \\ 39 \\ \hline 64 \end{array}$$

I To add 25 and 39 using base-four numerals, first convert each base-ten numeral to a base-four numeral. What number is expressed by the base-four numeral 121? What number does the base-four numeral 213 express?

121_{four}

213_{four}

J Add the ones first. From the addition table in D2, you see that $1 + 3 = 10$. (Read 10 as "one zero" and not as "ten.") Does the base-four numeral 10 express 1 four? Why should you write the digit 0 in ones' place? Remember 1 four to add to the other fours.

$$\begin{array}{r} {\scriptstyle 1} \\ 121 \\ 213 \\ \hline 0 \end{array}$$

K In the base-four numeral 121, how many fours does the digit 2 express? In the base-four numeral 213, how many fours does the digit 1 express? Add the fours. Remember to include the 1 four you obtained from adding the ones. How many fours are there in all? Why should you think of this sum as 10 fours or 1 sixteen? Why do you write the digit 0 in fours' place? What number do you remember?

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 121 \\ 213 \\ \hline 00 \end{array}$$

L In the base-four numeral 121, the digit 1 at the left expresses $1(4^2)$, or 1(16). What does the digit 2 in the base-four numeral 213 express? Add the sixteens. Remember to include the 1 sixteen you obtained from adding the fours. Is 10 the sum of the fours? Think of the 10 sixteens as 1 sixty-four. Why do you write the digit 0 in sixteens' place? Why do you write the digit 1 in sixty-fours' place?

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 121 \\ 213 \\ \hline 1000 \end{array}$$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	
2	2	3		
3	3	10	11	12

D2

Base four

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 233 \\ 212 \\ \hline 1111 \end{array}$$

D3

M Does the base-four numeral 1000 express the same number as the decimal numeral 64?

N Study D3. The numbers added are expressed as base-four numerals. So is their sum. You can verify the base-four computation by using the decimal system. Convert 233_{four} and 212_{four} to decimal numerals and add.

O Is $47 + 38 = 85$? Is $1111_{four} = 85$? Is the computation shown in D3 correct?

P Using the base-four system, add the numbers expressed by 203_{four} and 123_{four}. Verify your answer by using the decimal system.

In the following exercises you will use the base-twelve system to express numbers and their sums.

Q Make an addition table for the base-twelve numeration system. Your table should show how to express the sum of any two of the first twelve natural numbers in a base-twelve system. What digits will you use to express the numbers ten and eleven?

R D4 shows base-twelve numerals for two numbers and their sum. Is the sum correct? Use the addition table you made to help you.

Base twelve

$$\begin{array}{r} {\scriptstyle 1\ 1\ \ \ } \\ 1\,9\,\mathcal{E}\,9 \\ \mathcal{X}\,0\,\mathcal{E}\,\mathcal{X} \\ \hline \mathcal{E}\,\mathcal{X}\,\mathcal{E}\,7 \end{array}$$

D4

×	0	1	2	3
0	0	0		0
1	0	1	2	
2		2	10	
3	0		12	21

D5

S Use the decimal system to verify the computation shown in D4.

Using the base-twelve numeration system, add each pair of numbers expressed by the numerals below. Verify your answers by using the decimal system.

T $2\mathcal{E}\mathcal{X}_{\text{twelve}}$, 504_{twelve} U 137_{twelve}, $\mathcal{X}89_{\text{twelve}}$

Now you will learn to use various numeration systems to express the product of two numbers. The first numeration system you will use will be a base-four numeration system.

The table shown in D5 is part of a multiplication table for the base-four system. This table, when completed, will show how the product of any two of the first four natural numbers can be expressed as a base-four numeral.

A Which should you use, the numeral 10 or the numeral 4, to express the product of 2 and 2 as a base-four numeral? Why should you use the numeral 12, and not the numeral 6, to express the product of 2 and 3? Why should you use the numeral 21 to express the product of 3 and 3?

B Copy and complete the table in D5.

Now you are prepared to find the product of the two numbers expressed by the base-four numerals 1321 and 3. Use the base-four multiplication table you completed for exercise B.

C From your table, you see that $3 \times 1 = 3$. Why do you write the digit 3 in ones' place?

$$\begin{array}{r} 1321 \\ \times\ \ 3 \\ \hline 3 \end{array}$$

D Next you multiply the fours. Why can you think of the 2 in 1321 as 2 fours, or 20? $3 \times 20 = 120$. Why should you think of 120 as 1 sixteen and 2 fours? You write the digit 2 in fours' place and remember the 1 sixteen to add to the other sixteens.

$$\begin{array}{r} 1321 \\ \times\ \ 3 \\ \hline 23 \end{array}$$

E Now multiply the sixteens. Why can you think of the 3 sixteens as 300? $3 \times 300 = 2100$. Why should you think of 2100 as 2 sixty-fours and 1 sixteen? Why do you write the digit 2, and not the digit 1, in sixteens' place? Remember the 2 sixty-fours to add to the other sixty-fours.

$$\begin{array}{r} 1321 \\ \times\ \ 3 \\ \hline 223 \end{array}$$

F Finally, multiply the sixty-fours. How many sixty-fours are there? $3 \times 1000 = 3000$. How do you know that there are 11 ("one one") sixty-fours in all? Why should you write the digit 1 in sixty-fours' place? What digit is written in two hundred fifty-sixes' place?

$$\begin{array}{r} 1321 \\ \times\ \ 3 \\ \hline 11223 \end{array}$$

G To verify the base-four computation, convert 3_{four} and 1321_{four} to decimal numerals and multiply. Does the numeral 363 name the product you obtain? Then convert 11223_{four} to a decimal numeral. Your two products should be the same. Is $11223_{\text{four}} = 363$ a true statement?

Base four

$$\begin{array}{r} 3232 \\ \times\quad 3 \\ \hline 23022 \end{array}$$

D6

Base twelve

$$\begin{array}{r} 31\mathcal{E} \\ \times\quad 4 \\ \hline 1078 \end{array}$$

D7

H Look at D6. Two numbers and their product are expressed as base-four numerals. Verify the computation shown in D6. Is it correct?

I Next you will see how the base-twelve system can be used to express numbers and their products. Make a multiplication table for the base-twelve system.

J Look at D7. Two numbers and their product are expressed as base-twelve numerals. Why is the digit 8 written in ones' place? Why is the digit 7 written in twelves' place? In what place is the digit 0 written? The digit 1?

K Verify the computation shown in D7.

L 603 and 8 are base-twelve numerals. Using base-twelve numerals, show how you compute to find the product of the numbers expressed by these numerals. Verify your answer.

Now you will learn to compute with base-two numerals.

D8 shows an addition table and a multiplication table for a base-two numeration system.

M Why are the digits 0 and 1 the only digits used in the tables?

N To express the sum of 1 and 1 as a base-two numeral, why should you use the numeral 10, and not the numeral 2?

O Look at example A in D9. Two numbers and their sum are expressed as base-two numerals. Is the computation correct? Use the addition table in D8 to help you.

P Look at example B in D9. Two numbers and their product are expressed as base-two numerals. Is the computation correct? Use the multiplication table in D8 to help you.

Q Now use the decimal system to verify the computation shown in example A and example B in D9.

R Look at D10. The numerals are base-two numerals. Is the product correct? Verify the computation.

Now you know how to add and multiply using numeration systems other than the decimal system.

+	0	1
0	0	1
1	1	10

×	0	1
0	0	0
1	0	1

D8

Base two

A $\begin{array}{r} 101 \\ \times\quad 111 \\ \hline 1100 \end{array}$ B $\begin{array}{r} 101 \\ \times\quad 1 \\ \hline 101 \end{array}$

D9

Base two

$$\begin{array}{r} 111 \\ \times\quad 11 \\ \hline 111 \\ 1110 \\ \hline 10101 \end{array}$$

D10

On your own

For each of exercises 1 through 8, find the sum of the numbers expressed. Express your answer as a numeral in the same numeration system as the numerals in the exercise. Then verify your computation by using decimal numerals.

1 10_{four}, 10_{four}
2 21_{four}, 30_{four}
3 3303_{four}, 2032_{four}
4 1120_{four}, 1301_{four}, 3002_{four}
5 $15\mathcal{E}_{twelve}$, 32_{twelve}
6 19_{twelve}, $\mathcal{X}_{twelve}$
7 $1\mathcal{X}_{twelve}$, 13_{twelve}
8 23_{twelve}, $\mathcal{E}8_{twelve}$, 50_{twelve}

For each of exercises 9 through 16, find the product of the numbers expressed. Express your answer as a numeral in the same numeration system as the numerals in the exercise. Then verify the answer by using decimal numerals.

9 30_{four}, 2_{four}
10 22_{four}, 3_{four}
11 13_{four}, 2_{four}
12 1100_{four}, 3_{four}
13 592_{twelve}, 2_{twelve}
14 6301_{twelve}, 8_{twelve}
15 1550_{twelve}, $\mathcal{E}_{twelve}$
16 1008_{twelve}, $\mathcal{X}_{twelve}$

17 Make an addition table for a base-seven numeration system. Make your table so that it shows how to express the sum of any two of the first seven natural numbers as base-seven numerals.

18 Make a multiplication table for a base-seven numeration system. Your table should show how to express the product of any two of the first seven natural numbers as base-seven numerals.

For each of exercises 19 through 28, find the sum or product as directed and express the answer as a base-seven numeral.

Find the sum.

19 10_{seven}, 23_{seven}
20 236_{seven}, 25_{seven}
21 66_{seven}, 56_{seven}
22 606_{seven}, 536_{seven}
23 64_{seven}, 10_{seven}, 5_{seven}
24 64_{seven}, 43_{seven}, 21_{seven}

Find the product.

25 12_{seven}, 5_{seven}
26 303_{seven}, 4_{seven}
27 135_{seven}, 6_{seven}
28 546_{seven}, 3_{seven}

For each of exercises 29 through 36, find the sum or product as directed and express the answer as a base-two numeral.

Find the sum.

29 10_{two}, 1_{two}
30 11_{two}, 101_{two}
31 11_{two}, 10_{two}, 11_{two}
32 1001_{two}, 1101_{two}

Find the product.

33 110_{two}, 1_{two}
34 111_{two}, 101_{two}
35 1010_{two}, 11_{two}
36 1001_{two}, 11_{two}

A SPECIAL CHALLENGE

A sports writer gave his editor an article about Babe Ruth. The editor was confused for awhile but then smiled because he understood what the writer had done. The article is given below. Rewrite the article so that anyone can read it without confusion. Do not change the words or the order of the words.

"Babe Ruth was one of the greatest players in the history of baseball. He was born February 10, 12435, and died August 24, 13004, at the age of 125. He played on a major league baseball team from 12510 through 12550. He played with the New York Yankees from 12520 through 12542. Babe Ruth set many records during his lifetime of baseball. Some of these records are: 10054 strikeouts, 13304 bases on balls, 14121 runs batted in, and 10140 extra-base hits.

"Perhaps his most impressive record was set in 12531. In that year, Babe Ruth set a record for the number of home runs hit during one season in the major leagues. He hit 140 home runs in 414 games. This record has stood for 54 years!"

5 56 Exploring ideas

Decimal numerals for numbers greater than one thousand

In earlier grades you learned to use decimal numerals to express numbers greater than one thousand. In this lesson you will learn more about these decimal numerals.

Look at D1. Notice that the digits in the decimal numeral are written in groups. Counting from the right, each group contains three digits. Each group of digits in the numeral is called a *period*.

A Each period in a decimal numeral, except for the period farthest to the right, has a special name. These names are shown above the numeral in D1. Which digits are in the thousands' period? Which digits are in the millions' period? Which digits are in the billions' period?

The words you use to read the name of each period are shown in D2. You read the digits in each period in the same way that you read the digits in the period farthest to the right, and then you say or think the name of the period.

B How do you read the digits in the fourth period from the right?

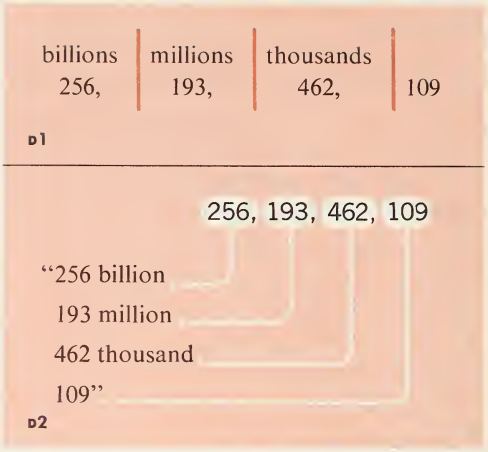
C How do you read the digits in the third period from the right?

D How do you read the digits in the second period from the right?

E How do you read the digits in the period farthest to the right? Remember that there is no special name for this period.

F Now read the numeral below.

69,840,020,002.



G Look at the numerals in exercises H through M. How are the periods in each numeral indicated?

H 721,635,599,483

I 20,840,500,080

J 87,692,110

K 660,030,900,002

L 1,005,000

M 2,470,000,002

Notice that the period farthest to the left is the only period that sometimes has only one or two digits in it.

N Read each of the numerals in exercises H through M.

O Which numeral in exercises H through M expresses the greatest number? Which numeral expresses the least number?

Sometimes you will use numerals for numbers greater than billions. The name of the next period after the billions' period is *trillions*. The name of the next period after the trillions' period is *quadrillions*. If you are curious to know the names of other periods, look up the word "numeration" in a dictionary or encyclopedia.

Read each of the numerals in exercises P through S.

P 4,672,792,844,631

Q 25,985,234,096,741

R 61,334,492,766,900

S 500,370,004,019

T Look at D3. Does the position of each digit in the numeral tell you the number expressed by that digit? Your answer depends upon what property of a decimal numeration system?

U What do the names above the periods in the numeral in D3 tell you?

Each of the positions within a period in a decimal numeral also has a special name. The names of the positions, or places, are shown below the digits in D3.

A Which digits form the thousands' period in the numeral in D3? Notice that in this period the digit 3 expresses the number three hundred thousand. The digit 1 expresses what number?

B Which digits form the millions' period? In this period, what number is expressed by the digit 7? By the digit 5? By the digit 4?

C Which digit is in ten thousands' place? In tens' place? In billions' place?

Many times it is convenient to express a number by using exponential notation.

D Look at D4. Are 10^2 and 100 the same number? What number is the same as 10^3? What number is the same as 10^4? Does each sentence in D4 express a true statement?

> REMINDER
> 10^2, 10^3, and 10^4 are powers of 10.
> A power is a number that is obtained by using the same factor one or more times.
> *See lesson 53, page 227.*

E Numeral D in D5 expresses the product of two numbers, 7 and 10^6. Is 7 a number less than 10? Is 10^6 a power of 10? Are 7×10^6 and 7,000,000 the same number?

Notice that 7 is one of the factors of the number 7×10^6. Notice also that 7 is one of the numbers from 1 to 10. The other factor, 10^6, is a power of 10. Any numeral that expresses the product of a number from 1 to 10 (but not including 10) and a power of 10 is in *scientific notation*. Numeral D is the scientific notation for the number 7,000,000.

F Numeral E is the scientific notation for what number? Numeral F is the scientific notation for what number?

D3

A $10^2 = 10 \times 10$.
B $10^3 = 10 \times 10 \times 10$.
C $10^4 = 10 \times 10 \times 10 \times 10$.

D4

D 7×10^6
E 5×10^5
F 4×10^3

D5

sci en tif ic no ta tion (sī′ən tif′ik nō tā′shən). A method of expressing a number as the product of a number from 1 to 10 and a power of 10.

The numeral 5×10^2 is the scientific notation for 500. The number 800,000,000 is expressed in scientific notation by the numeral 8×10^8.

G What number is expressed by the numeral ? 30×10^2? By the numeral 3×10^3? Which of these numerals is in scientific notation? How do you know?

Use scientific notation to express each of the numbers named below.

H 600
I 70,000
J 3,000,000
K 400,000,000
L 100,000
M 8,000,000,000

In this lesson you have learned more about reading and writing names for numbers greater than one thousand. You have also learned how to express some numbers by using scientific notation.

On your own

For each of exercises 1 through 6, write the words that you would say in reading the numeral. For exercise 1 you should write "eighty-five million ninety thousand eighty."

1 85,090,080
2 59,073
3 356,521
4 674,982,315
5 765,100,200
6 2,965,461,332

For each of exercises 7 through 16, use digits to write the numeral.

7 seventy thousand four
8 nine hundred thousand six
9 four hundred thousand seven hundred thirty
10 six hundred twenty-three thousand forty
11 seven million seven thousand seventy
12 forty-five million six hundred thousand
13 fifty-two million eighty-nine thousand fourteen
14 one billion twenty-four thousand three hundred nineteen
15 five hundred eighty million seven hundred thousand sixteen
16 three trillion four hundred million

For each of exercises 17 through 22, write the decimal numeral for the number expressed.

17 4×10^3
18 2×10^5
19 9×10^2
20 8×10^4
21 5×10^6
22 7×10^{10}

For each of exercises 23 through 32, use scientific notation to express the number named.

23 8000
24 200
25 700,000
26 40,000
27 3000
28 50,000,000
29 400,000,000
30 600,000,000
31 80,000,000
32 200,000,000

APPLYING MATHEMATICS

In this lesson you will solve problems similar to the ones that you have studied so far in this book. For each problem, first write a sentence that expresses a condition. Use any letters you wish as names for the variables. Then tabulate the solution set of the condition and give the answer to the problem. The universe is given for each problem.

1 Terry had 110 newspapers to sell. He sold all but 25 of them. How many newspapers did he sell? U = N.

2 A road crew can build 5 miles of road in 3 months. At this rate, how many months will it take the crew to build a road that is to be 115 miles long? U = C.

3 The area of the Pacific Ocean is 63,801,668 square miles. The area of the Atlantic Ocean is 31,839,306 square miles. What is the total area of the Atlantic and Pacific Oceans? U = N.

4 After Mrs. Smith bought 13 yards of drapery material, she had more than 41 yards. What is the least number of yards she could have had before she bought the 13 yards of material? U = N.

5 The citizens of Foster raised $9213 for park equipment. The amount they raised was $758 more than their goal. What was their goal? U = N.

6 Ken has 18 travel posters, and Dick has 13 travel posters. Carl has fewer posters than Ken and Dick have together, but more posters than Dick has. How many posters can Carl have? U = N.

7 Mr. Fisher borrowed $950 from a local bank. At the end of one year, he paid $57 interest. What rate of interest was he charged? U = C.

8 In Canada, there are over 30 national parks and at least 300 provincial parks. There are how many times as many provincial parks as national parks? U = C.

9 On May 1, Mr. Roberts had more than 63 cars to sell. During the month, he sold 17 of these cars and then had fewer than 54 cars left to sell. How many cars could Mr. Roberts have had to sell on May 1? U = N.

10 For each game played in some leagues, the home team is required to supply 48 baseballs. At the end of a certain game, there were fewer than 19 baseballs left. How many baseballs could have been used during this game? U = N.

11 Last year, Mr. Lang's take-home pay was $7395. He saved $2 out of each $17 of his take-home pay. How much money did he save last year? U = C.

12 The Second National Bank charges 6% interest on the money it lends. Miss Wilson borrowed $750 from this bank. She agreed to repay her loan with interest at the end of one year. How much did Miss Wilson owe the bank at the end of one year? U = C × C.

13 The area of Canada is 3,851,809 square miles. The area of Canada is about 17 times the area of France. The area of France is about how many square miles? U = C.

14 One day an appliance salesman sold more electric fans than steam irons. He sold fewer than 6 fans and irons in all. How many appliances of each kind could he have sold? U = C × C.

15 Mr. Davis received a commission of $260 for the month of April. He said that his March commission had been 125% of his April commission. How much commission did Mr. Davis receive for the month of March? U = C.

16 During a sale, Mrs. Jones paid $273 for a dresser that was regularly priced at $350. What rate of discount did she receive? U = C × C.

17 Certain rate pairs are equivalent to 2/3. The product of the first and second components of each of these rate pairs is less than 100. What are the rate pairs? U = C × C.

18 51 is 34% of what number? U = C.

19 What number is 96% of 7625? U = C.

20 660 is what per cent of 528? U = C.

SPECIAL CHALLENGE

Captain Nemo discovered a strange country where the inhabitants used a base-seven system of numeration with the property of place value. When the inhabitants counted, they used the words *re*, *mi*, *fa*, *sol*, *la*, *ti*, *re-do*, *re-re*, and so on. Use the word names of the numbers in this system of numeration for exercises A, B, and C.

A Write the names of the natural numbers from 1 through 25.

B Write the names of the natural numbers from 45 through 60.

C Write the names of the natural numbers from 340 through 354.

D Captain Nemo also discovered another country where the inhabitants used a base-nine place-value numeration system. Invent names for the numbers in this system of numeration. Then write the names of the natural numbers from 1 through 27, from 60 through 70, and from 510 through 521.

CHECKING UP

The small numeral within parentheses tells you what page to turn to for help if you have trouble with an exercise.

Test 56

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

1 A code numeration system has the property of addition. (207)

2 In the numeral 427,216, the first three digits are in the thousands' period. (239)

3 A tally numeration system has the property of repetition. (207)

4 The decimal numeral 60,000,000 is read "sixty billion." (238)

5 One billion is greater than one trillion. (238)

6 A place-value numeration system has the property of repetition. (218)

7 The digit 3 in the decimal numeral 34756 is associated with the fifth power of 10. (230)

8 $121_{three} = 21_{five}$ is a true statement. (232)

9 The decimal numeration system has the property of place value. (217)

10 The sum of the numbers expressed by the Roman numerals XVI and XIX is the number expressed by the numeral XXXV. (222)

Test 57

From the list given below each exercise, choose the expression that correctly completes the exercise.

11 The numbers 3^2, 3^3, and 3^4 are ~~~ of 3. (227)

a exponents b factors c powers d bases

12 The ~~~ of the numeral 742 is the numeral $7(10^2) + 4(10^1) + 2(1)$. (228)

a exponential notation c factored form
b expanded form d product

13 The decimal numeral 407,000,000,000 is read "~~~." (238)

a four hundred seven trillion
b forty million seven
c four hundred seven billion
d four hundred seven quadrillion
e forty-seven billion

14 The decimal numeral ~~~ names the same number as the numeral 25_{seven}. (215)

a 19 b 25 c 25(7) d 7

15 The set of factors of 48 does not contain ~~~. (226)

a 1 b 7 c 12 d 16

16 ~~~ is not a factor of 30. (226)

a 30 b 1 c 12 d 6

17 A ~~~ numeration system uses nine different digits. (217)

a base-twelve c base-nine
b base-ten d base-eight

18 The numeral 42031 cannot be a ~~~ numeral. (217)

a base-four c base-eight
b base-six d base-twelve

19 The numeral 10_{seven} expresses the ~~~ of the numeration system. (216)

a exponent c base
b sum d power

Test 58

20 What is the exponent of 5^3? (227)

21 What number is expressed by the numeral 1^{1000}? (227)

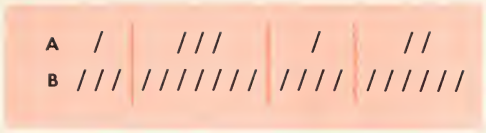
22 What base-two numeral expresses the sum of 111_{two} and 1101_{two}? (236)

23 What base-two numeral expresses the product of 1011_{two} and 111_{two}? (236)

24 What base-eight numeral expresses the sum of 476_{eight} and 1555_{eight}? (234)

25 What base-twelve numeral expresses the sum of $\mathcal{X}\mathcal{E}9_{twelve}$ and $3\mathcal{X}_{twelve}$? (234)

26 What base-twelve numeral expresses the product of $3\mathcal{E}_{twelve}$ and 5_{twelve}? (236)



Use the display for exercises 27 through 30.

27 Think of numeral A as a base-four numeral. What number is named by this numeral? (213)

28 Think of numeral A as a base-eight numeral. What number is named by this numeral? (213)

29 Think of numeral A as a base-ten numeral. What number is named by this numeral? (213)

30 Think of numeral B as a base-five numeral. Using a chart and strokes, write a base-five numeral that expresses the same number as numeral B. Use as few strokes as possible. (213)

Test 59

Write a decimal numeral for each number named in exercises 31 through 36. (215)

31 1011_{two}

32 340_{five}

33 592_{twelve}

34 555_{nine}

35 $6\mathcal{X}32_{\text{eleven}}$

36 110101_{two}

37 Convert the numeral 133 to a base-four numeral. (231)

38 Convert the numeral 29 to a base-two numeral. (231)

39 Convert the numeral 2000 to a base-twelve numeral. (231)

40 Convert the numeral 365_{seven} to a base-five numeral. (232)

Test 60

Write a decimal numeral for each number named in exercises 41 through 44. Do not use exponential notation. (239)

41 7×10^4

42 1×10^7

43 8×10^6

44 9×10^9

Use scientific notation to express each number named in exercises 45 through 48. (239)

45 2000

46 60,000

47 5,000,000

48 40,000,000

CHECKING UP

The small numeral within parentheses tells what page to turn to for help if you need it.

Test 61

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

1 A numeral is the same as a number. (5)

2 The quotient of 1200 and 150 is the same as the quotient of 40 and 5. (57)

3 The solution set of a condition in two variables is a subset of the universe for the variables. (129)

4 The sentence $x - 7 < y$ expresses an inequality. (21)

5 $\{x|1/x \sim 2/8\} = \{x|8/2 \sim x/1\}$. (178)

6 The exponent of the power 3^8 is the number 8. (227)

7 Infinitely many lines contain two given points. (33)

8 The numeral 651_{seven} expresses the same number as the numeral 2310_{five}. (232)

9 There is at least one point between any two different points. (37)

10 If ℓ_1 separates plane ABC into two half-planes, and the intersection of $\overline{BC}$ and ℓ_1 is the empty set, then point B and point C are in different half-planes. (41)

Test 62

What words or symbols best complete exercises 11 through 19?

11 The amount of money on which interest is paid is the ——. (194)

12 The price of an article after the discount has been deducted from the regular price is the ——. (198)

13 Each member of the solution set of a condition is a —— of the condition. (24)

14 A —— is the boundary of a half-space. (41)

15 The union of set A and set A is ——. (89)

16 If the union of two sets is ——, then each set is the empty set. (89)

17 $\{0, 1, 2\} \cap \{0, 3, 6\} \cap \{0, 5, 10\} =$ ——. (84)

18 The intersection of set B and set C is a —— of the union of sets B and C. (88)

19 A ratio is a name of a ——. (173)

Test 63

From the list given below each exercise, choose the expression that correctly completes the exercise.

20 The two axes used to make a graph of a Cartesian set are ——. (130)

a parallel
b perpendicular
c horizontal
d vertical

21 The interior of an angle is the intersection of two ——. (114)

a rays
b half-planes
c planes
d half-lines

22 If U = C, the union of $\{x|x/17 \sim 8/34\}$ and $\{x|34x = 8(17)\}$ contains ——. (177)

a no members
b one member
c two members
d infinitely many members

23 The set of counting numbers is ——. (73)

a {1, 3, 5, . . .}
b {0, 1, 2, . . .}
c {0, 2, 4, . . .}
d {1, 2, 3, . . .}

24 To change the numeral 392 to a base-seven numeral, you first divide the number 392 by ——. (231)

a seven
b twenty-one
c three hundred forty-three
d forty-nine

25 If U = C, the solution set of $x/y \sim 9/6 \wedge x + y < 22$ contains ——. (199)

a two members
b no members
c four members
d infinitely many members

26 A given line is included in ——. (35)

a exactly one plane
b exactly two planes
c exactly three planes
d infinitely many planes

27 Three noncollinear points are contained in ——. (35)

a exactly one plane
b exactly two planes
c exactly three planes
d infinitely many planes

28 The intersection of two disjoint sets is ——. (85)

a not a set
b a set with one member
c the universe
d the empty set

Test 64

For each problem, write a sentence that expresses a condition. Tabulate the solution set of the condition, and give the answer to the problem. The universe is given for each problem.

29 The regular price of a subscription to a certain magazine is $4. Fred was able to get this subscription for $3. What per cent of the regular price did Fred pay for the subscription? U = C. (190)

30 The basic price of one model of a car is $2450. The basic price of a different model is 90% of the price of the first model. What is the price of the cheaper model if it includes a heater that costs $69 extra? U = C × C. (195)

31 There are 231 more students enrolled in Scott School than in Brown School. If 78 more students enroll in Scott, the total enrollment will be 550. How many students are enrolled in each school? U = N × N. (153)

32 Mr. Lucas fills the radiator of his car with antifreeze at the rate of 1 quart of antifreeze to 2 quarts of water. If the radiator holds 21 quarts, how much antifreeze and how much water should he use? U = C × C. (195)

33 42 is what per cent of 70? U = C. (191)

34 65% of 180 is what number? U = C. (192)

35 24 is 30% of what number? U = C. (192)

36 391 is what per cent of 230? U = C. (191)

37 57 is 150% of what number? U = C. (192)

Unit 6

The natural-number system

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Equivalent sets

In units 1 through 5 you worked with the natural numbers only. You used natural numbers as replacements for variables, and you found answers to problems about natural numbers. In this unit you will think about some of these familiar ideas in a new way. You also will learn many new things about the natural numbers.

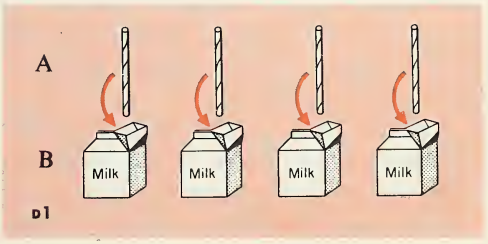
In this first lesson you will learn about sets that are called equivalent sets. Later you will learn how the natural numbers are associated with sets.

A The diagram in D1 represents one way to match the members of the set of straws with the members of the set of cartons of milk. Is each member of A matched with just one member of B? Is each member of B used only once in the matching?

Using the language of mathematics, we say that each straw is *mapped onto* a carton.

B Look again at D1. How many straws are mapped onto each carton?

The diagram in D1 represents a way in which each member of set A can be mapped onto just one member of set B. Each member of B is used only once. We say that set A is



D1

mapped onto set B. The mapping represented in D1 is a *one-to-one mapping*.

C Set A and set B are represented again in D2. Notice the direction in which the arrows point. The set of cartons is mapped onto what set?

D Each member of B is mapped onto how many members of A? Is each member of A used only once in the mapping? Is the mapping a one-to-one mapping? Explain your answer.

E Look at D3. Explain why the mapping represented in D3 is a one-to-one mapping of set L onto set K. Now imagine that the direction of arrows is reversed. Is the new mapping a one-to-one mapping of set K onto set L?

Whenever a one-to-one mapping of a first set onto a second set is possible, then a one-to-one mapping of the second set onto the first set is also possible. To emphasize that *two* such one-to-one mappings are possible, we say that the two sets can be put in *one-to-one correspondence*.

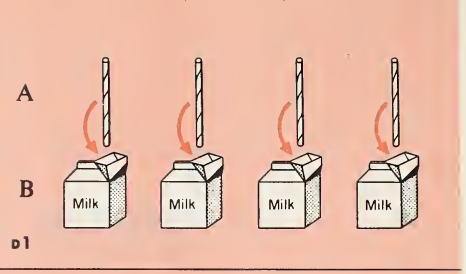
F Can the two sets represented in D1 be put in one-to-one correspondence?

Notice that in a one-to-one correspondence between two sets, each member of the first set is mapped onto exactly one member of the second set, and each member of the second set is mapped onto exactly one member of the first set.

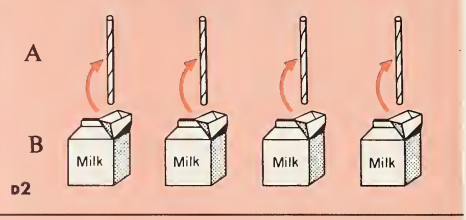
G Look at D4. Notice the double-headed arrows used to represent the mappings. For each member of X, tell which member of Y it is mapped onto. For each member of Y, tell which member of X it is mapped onto. How do you know that set X and set Y are in one-to-one correspondence?

H Now look at D5. The members of set X are named in a different order. For each member of X, tell which member of Y it is mapped onto. For each member of Y, tell which member of X it is mapped onto. Does the diagram in D5 represent another way to put sets X and Y in one-to-one correspondence?

I Can X and Y be put in one-to-one correspondence in more than two ways? Make a diagram like the one in D5 to represent a third way of putting set X and set Y in one-to-one correspondence.



D1



D2

$K = \{1, 3, 5\}$.

$L = \{7, 14, 21\}$.

D3

$X = \{6, 5, 9, 3, 1\}$.

$Y = \{17, 14, 21, 0, 6\}$.

D4

one-to-one correspondence (kôr′ə spon′dəns). Two one-to-one mappings that map a first set onto a second set and the second set onto the first set. {1, 2, 3, 4, 5} and {10, 20, 30, 40, 50} can be put in one-to-one correspondence. {5, 10} and {1, 2, 3} cannot be put in one-to-one correspondence.

Two sets that can be put in one-to-one correspondence are *equivalent sets*. Sets X and Y are equivalent sets.

J Read each tabulation in D6. Is each set tabulated a finite set? How do you know?

K Which other set tabulated in D6 is equivalent to set P? To set Q? How do you know?

L Is set R equivalent to any of the other sets tabulated in D6? Tabulate a subset of the set of natural numbers that is equivalent to set R.

M How do you know that a finite set cannot be equivalent to an infinite set?

N Are two equal sets always equivalent? Are two equivalent sets always equal? Explain your answers.

Next you will see how some infinite sets can be put in one-to-one correspondence.

A Look at D7. How do you know that each set tabulated in the display is an infinite set? Describe set N.

Set E is the set of even numbers. Every even number has 2 as a factor. Since $0 \times 2 = 0$, the number 2 is a factor of 0, and the number 0 is a member of the set of even numbers.

B First you will map set N onto set E. Is 3 a member of N? Is 2×3, or 6, a member of E? Is 17 a member of N? Is 2×17, or 34, a member of E? Is 28 a member of N? Is 2×28, or 56, a member of E?

C You will map 3 in N onto 2×3, or 6, in E. You will map 17 in N onto what member of E? You will map 28 in N onto what member of E? Is 2 times each member of N always a member of E?

Notice that you used a *rule* to find the members of E onto which you mapped the members of N. This rule is: Multiply each member of N by 2.

D Using this rule, onto what member of E will you map 5? 25? 500? 10,001?

$X = \{9, 3, 6, 5, 1\}$.

$Y = \{17, 14, 21, 0, 6\}$.

D5

$P = \{0, 1, 2, 3, 4\}$.
$Q = \{3, 6, 9, 12, 15, 18\}$.
$R = \{5, 10, 15\}$.
$S = \{20, 21, 22, 23, 24, 25\}$.
$T = \{10, 20, 30, 40, 50\}$.

D6

$N = \{0, 1, 2, 3, 4, \ldots\}$.
$E = \{0, 2, 4, 6, 8, \ldots\}$.

D7

e quiv a lent sets (i kwiv′ə lənt). Two sets that can be put in one-to-one correspondence. {1, 2, 3} and {1, 4, 9} are equivalent sets. {0, 1} and {11, 12, 13, 14} are not equivalent sets.

E Can you use the same rule to map each member of N onto just one member of E? Will each member of E be used only once in the mapping?

F Now suppose that x is a variable for any member of N. For each replacement of x, will $2x$ be a member of E? When you map x onto $2x$, you will obtain a one-to-one mapping of set N onto set E.

G Now you will map set E onto set N. Is 6 a member of E? Is $6 \div 2$, or 3, a member of N? You will map 6 onto $6 \div 2$, or 3.

H Is 36 a member of E? Is $36 \div 2$, or 18, a member of N? You will map 36 onto what member of N?

I What rule have you used to map members of E onto members of N? Using this rule, onto what member of N will you map 8? 18? 50? 806?

J Can you use the same rule to map each member of E onto just one member of N? Will each member of N be used only once in the mapping?

K Now suppose that y is a variable for any member of E. For each replacement of y, will $y \div 2$ be a member of N? If you map y onto $y \div 2$, will you obtain a one-to-one mapping of E onto N?

L What rules have you used to put the set of even numbers and the set of natural numbers in one-to-one correspondence?

M Make a diagram to represent a way in ? which set E and set N can be put in one-to-one correspondence.

N Are the set of even numbers and the set of natural numbers equivalent sets? How do you know?

O Make a diagram to represent a way in which ? the set of natural numbers and the set of odd numbers can be put in one-to-one correspondence. The odd numbers are 1, 3, 5, and so on.

P Are the set of natural numbers and the set of odd numbers equivalent sets? Explain your answer.

Now you know what is meant by one-to-one mapping, one-to-one correspondence, and equivalent sets.

On your own

In each of exercises 1 through 5, two sets, A and B, are tabulated. For each exercise, first tabulate a subset of A that can be mapped onto B by a one-to-one mapping. Then make a diagram to represent the mapping of that subset of A onto B. For example, if you use the subset

$N = \{0, 1, 2, 3, 4, \ldots\}$.
$E = \{0, 2, 4, 6, 8, \ldots\}$.

D7

$F = \{0\}$. $I = \{0, 1, 2, 3\}$.
$G = \{0, 1\}$. $J = \{0, 1, 2, 3, 4\}$.
$H = \{0, 1, 2\}$. $K = \{0, 1, 2, 3, 4, 5\}$.

D8

{Mary, Jill, Susan} for exercise 1, your diagram might look like this:

{Mary, Jill, Susan}
↓ ↓ ↓
{June, July, August}

1 $A = \{\text{Alice, Mary, Jill, Susan}\}$.
$B = \{\text{June, July, August}\}$.

2 $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$.
$B = \{\text{Sunday, Monday, Tuesday}\}$.

3 $A = \{\text{United States, Mexico, Chile}\}$.
$B = \{\text{Canada}\}$.

4 $A = \{0, 1, 2, 3, 4, 5, 6\}$.
$B = \{0, 1, 2, 3\}$.

5 $A = \{0, 1, 2, 3, 4, 5, 6\}$.
$B = \{(0, 1), (1, 0), (1, 1)\}$.

Two sets are described in each of exercises 6, 7, and 8. For each exercise, tabulate the sets described. Then make a diagram to represent one way in which the two solution sets can be put in one-to-one correspondence.

6 $\{x \mid x < 5\}$ $U = N$.
$\{a \mid a + 3 < 8\}$ $U = N$.

7 $\{n \mid n/4 \sim 3/1\}$ $U = C$.
$\{p \mid p/4 \sim 12/1\}$ $U = C$.

8 $\{(x, y) \mid x/y \sim 3/4 \wedge y < 16\}$ $U = C \times C$.
$\{(a, b) \mid 4/3 \sim a/b \wedge a < 16\}$ $U = C \times C$.

For each set named in exercises 9 through 14, find an equivalent set tabulated in D8. Make a diagram for each exercise.

9 {red, blue}
10 $\{x \mid x < 3\}$ $U = N$.
11 $\{x \mid 2x = 0\}$ $U = N$.
12 $\{(a, b), (c, d)\}$
13 $\{a, b, c, d\}$
14 $\{(x, y) \mid x/y \sim 5/6 \wedge x < 35\}$ $U = C \times C$.
15 Make a diagram to show that {10, 100, 1000, 10,000, . . .} is equivalent to the set of counting numbers.

KEEPING SKILFUL

Find the sum.

1 8650, 7584, 1360, 7662
2 679, 519, 306, 476, 493
3 2960, 9110, 1062, 9959
4 59467, 3488, 75176
5 99182, 60892, 89378, 56136
6 82347, 8341, 13654, 8119

Find the difference.

7 7674, 174
8 6805, 5058
9 4776, 3947
10 73039, 57186
11 40218, 16544
12 80124, 3563
13 51772, 708
14 154531, 82237

Tabulate the solution set of each condition expressed in exercises 15 through 26. U = N.

15 $174 + r = 312$.
16 $n < 27(16)$.
17 $x - 19 > 18$.
18 $b = 724(108)$.
19 $1023 - t = 469$.
20 $g + 95 < 111$.
21 $s - 728 = 8392$.
22 $43 + d > 102$.
23 $51x = 5304$.
24 $87 - k > 29$.
25 $c + 1 \neq 1$.
26 $38m = 3002$.

For each of exercises 27 through 32, first tabulate the solution set described. Then make a graph of each solution set. $U = N \times N$.

27 $\{(m, n) \mid m + 3 > n \wedge m + n = 9\}$
28 $\{(m, n) \mid n + 17 = 23 \wedge n - 3 = m\}$
29 $\{(m, n) \mid m > 4 \wedge m + n < 8\}$
30 $\{(m, n) \mid n + 19 < 22 \wedge 6 - n > m\}$
31 $\{(m, n) \mid m = n + 2 \wedge n < 5\}$
32 $\{(m, n) \mid m > n \wedge m + n = 7\}$

Natural numbers as properties of sets

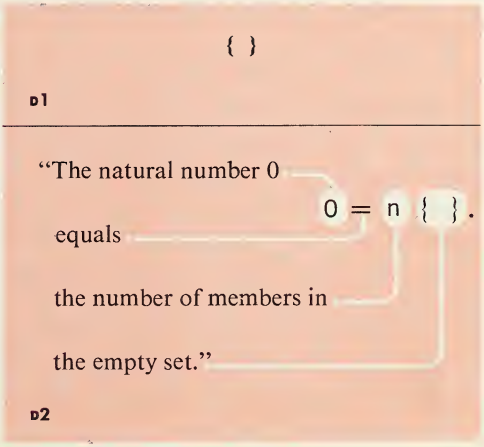
In the last lesson you learned that two sets are equivalent if they can be put in one-to-one correspondence. In this lesson you will use the idea of equivalent sets to associate natural numbers with sets.

A How many members are in the set tabulated in D1? What is a name of the set that has no members?

The natural number 0 is associated with the set that has no members. When we want to refer to the natural number associated with the empty set, we can use a special symbol. D2 shows how to write and read a sentence that has this symbol.

B Study D2. How does { } differ from n{ }?

So far, 0 is the only natural number that has been associated with a set. Now you will use



the set whose only member is 0 to help you understand the meaning of the natural number 1.

C Describe the set tabulated in D3. How many members are in the set?

D How does {0} differ from { }? How does {0} differ from n{ }?

E Look at the tabulations in D4. Is each set equivalent to each of the other sets? Is each set equivalent to {0}?

The natural number 1 is associated with {0} and with each set equivalent to {0}. The natural number 1 is a *common property* of all sets equivalent to {0}. The natural number 1 tells how many members there are in each of the sets.

F The symbol n{0} is read "the number of members in the set whose member is zero." Read each sentence in D5. Does each sentence express a true statement? How does {0} differ from n{0}?

Next you will use the natural numbers 0 and 1 to obtain a set associated with another natural number.

G Is each of the sets tabulated in D6 equivalent to each of the other sets?

H The natural number 2 is associated with {0, 1}. Why is 2 also associated with set G? With set H?

I Is 2 = n{0, 1}? What is a common property of the sets tabulated in D6?

J Look at the tabulations in D7. For each set tabulated, decide what natural number is associated with the set.

Notice that set A tabulated in D7 is the empty set and that B and C are sets of natural numbers. Each set tabulated in D7 is the *standard set* of the natural number that is associated with it. When you tabulate a standard set, it is convenient to write the numerals in the order 0, 1, 2, and so on.

{0}

D3

R = {Canada}.
S = {Mary}.
T = {a}.

D4

1 = n{0}.
1 = n{Canada}.
1 = n{Mary}.
1 = n{a}.

D5

F = {0, 1}.
G = {hat, coat}.
H = {table, chair}.

D6

A = { }.
B = {0}.
C = {0, 1}.

D7

3 = n{0, 1, 2}.

D8

K Read the sentence in D8. Does {0, 1, 2} contain those natural numbers, and only those natural numbers, that you have already associated with standard sets? {0, 1, 2} is the standard set of the natural number 3.

When you want to decide what natural number should be associated with a set, think of the standard set to which it is equivalent.

For example, if you are working with {Calgary, Regina, Toronto}, you know that {Calgary, Regina, Toronto} is equivalent to {0, 1, 2}. Since 3 is the natural number associated with {0, 1, 2}, you know that 3 is also associated with {Calgary, Regina, Toronto}.

L What natural number is associated with {tuba, drum, clarinet}? How do you know?

M What is a common property of {0, 1, 2} and each set equivalent to {0, 1, 2}?

You have learned that a natural number is a common property of a standard set and all sets equivalent to the standard set.

N What natural number is associated with each set tabulated in D9?

O Tabulate the set whose members are the days of the week. This set is equivalent to which set tabulated in D9? What natural number is associated with the set whose members are the days of the week? How do you know?

Tabulate the standard set of each of the natural numbers named below.

P 9 Q 14 R 19 S 20

T Suppose that m is a variable for any member of the set of counting numbers. Tabulate the standard set of m. Describe in words how you can form the standard set of any given natural number.

U Is the standard set of any natural number a finite set? Is the set of natural numbers a finite set? Explain your answers.

Now you know how to associate a natural number with a standard set and with each set equivalent to the standard set. You also know that the natural number associated with a finite set tells how many members are in the set.

On your own

For each of the sets described in exercises 1, 2, and 3, tabulate a subset that is equivalent to {0, 1, 2, 3, 4}.

$A = \{\ \}$.
$B = \{0\}$.
$C = \{0, 1\}$.
$D = \{0, 1, 2\}$.
$E = \{0, 1, 2, 3\}$.
$F = \{0, 1, 2, 3, 4\}$.
$G = \{0, 1, 2, 3, 4, 5\}$.
$H = \{0, 1, 2, 3, 4, 5, 6\}$.

D9

1 Set of vegetables
2 Set of furniture
3 Set of rivers
4 What is a common property of the sets that you tabulated for exercises 1, 2, and 3?

For each of exercises 5 through 10, write a sentence that expresses the standard set of the natural number named. For exercise 5 you should write the following sentence:

$10 = n\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

5 10 6 13 7 11 8 15 9 0 10 21

Each set named in exercises 11 through 15 is equivalent to a standard set. For each exercise, tabulate the standard set.

11 {elm, willow, oak, maple, sycamore, pine, walnut, poplar}
12 {bicycle}
13 { }
14 {=, ×, +, ≠, ∧, <, >}
15 {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21}

For each of exercises 16 through 19, write a sentence that expresses the natural number associated with the standard set tabulated. For exercise 16 you should write the following sentence:

$9 = n\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.

16 {0, 1, 2, 3, 4, 5, 6, 7, 8}
17 {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}
18 {0}
19 $\{0, 1, 2, \ldots, m - 1\}$

SPECIAL CHALLENGE

In lesson 57 you learned that infinite sets can be put in one-to-one correspondence.

A What rules do you use to put the set of even numbers and N in one-to-one correspondence? Make a diagram to help you answer the question.

B What rules do you use to put the set of odd numbers and N in one-to-one correspondence? Make a diagram to help you.

Sets that can be put in one-to-one correspondence with the set of natural numbers are *denumerable* (di nü′mər ə bəl) sets.

C Is the set of even numbers a denumerable set? Is the set of odd numbers a denumerable set? Explain your answers.

D Are all denumerable sets infinite? Explain your answer.

E Look at the display below. Set T is the set whose members are associated with the positions in a base-ten numeration system. Describe each of the other sets tabulated.

F Decide if each set tabulated in the display is a denumerable set. If the set is denumerable, make a diagram to represent a way in which the set and N can be put in one-to-one correspondence. Then see if you can think of a rule for pairing the members of the set with the members of N. If the set is not denumerable, explain why.

$T = \{1,\ 10^1,\ 10^2,\ 10^3,\ 10^4,\ \ldots\}.$
$S = \{0,\ 1,\ 4,\ 9,\ 16,\ \ldots\}.$
$V = \{3,\ 6,\ 9,\ 12,\ \ldots,\ 522\}.$
$C = \{1,\ 2,\ 3,\ 4,\ \ldots\}.$
$N \times N = \{(0, 0),\ (0, 1),\ (1, 0),\ (0, 2),\ (1, 1),\ (2, 0),\ (0, 3),\ (1, 2),\ (2, 1),\ (3, 0),\ (0, 4),\ \ldots\}.$

Ordering of the natural numbers

In your work with the natural numbers, you have often used the ideas of "less than" and "greater than." Now that you have studied standard sets, you will learn more about the meaning of these ideas.

A Read each sentence in D1. Is the standard set of 1 a subset of the standard set of 2? How do you know?

> REMINDER
> If each member of one set is also a member of a second set, then the first set is a subset of the second set.
> *See lesson 2, page 11.*

B You know that {0} is a subset of {0, 1}. Does {0} contain all the members of {0, 1}?

Since {0} does not contain all the members of {0, 1}, we say that {0} is a *proper subset* of {0, 1}. If a subset of a set does not contain all the members of the set, that subset is a proper subset of the set.

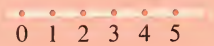
C Look again at D1. Is the standard set of 2 a subset of the standard set of 3? How do you know that the standard set of 2 is a proper subset of the standard set of 3?

D Is the standard set of 3 a proper subset of the standard set of 3? Explain your answer.

Now we can use the definition of proper subset to state exactly what we mean when we

$1 = n\{0\}$.
$2 = n\{0, 1\}$.
$3 = n\{0, 1, 2\}$.

D1



D2

proper subset of a set (prop′ər). A subset of a set such that the subset does not contain all the members of the set. A set is not a proper subset of itself. {4, 5} is a proper subset of {3, 4, 5}. {3, 4, 5} is not a proper subset of {3, 4, 5}.

"less than" for natural numbers. A natural number is less than another natural number if the standard set of the first number is a proper subset of the standard set of the second number.

say that a natural number is *less than* another natural number. A natural number is less than another natural number only when the standard set of the first number is a proper subset of the standard set of the second number.

E Is $1 < 2$? Is $2 < 3$? Explain your answers.

F Tabulate the standard set of 4. Is the standard set of 3 a proper subset of the standard set of 4? How do you know?

G Is $3 < 4$? How does your answer for exercise F help you answer the question?

By continuing in the same way, you can show that $4 < 5$, that $5 < 6$, that $6 < 7$, and so on. The idea of "less than" may be used to arrange the natural numbers in a special order. This arrangement is an *ordering* of the natural numbers.

H What is the standard set of the natural number 0?

You know that the empty set is the standard set of 0. Mathematicians have agreed to accept the empty set as a proper subset of every set except itself. Therefore, since 0 is the least natural number, it is the first number in an ordering of the natural numbers.

I Is $1 < 2$? Is $1 < 3$? Is $1 < 4$? Is $1 < 5$? Use standard sets to explain your answers.

J Is the number 1 less than each of the other natural numbers except 0?

K Tabulate the solution set of $2 < x$. The universe for x is N. Is 2 less than each of the other natural numbers except 0 and 1?

L For each replacement of a in $a < b$, are there infinitely many replacements for b that satisfy the condition? The universe for each of the variables, a and b, is N.

M The picture of a natural-number line in D2 will help you think about the order of the natural numbers. What point is represented by the dot farthest to the left?

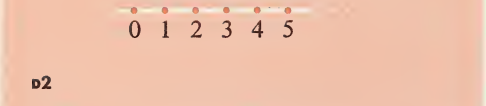
N Describe the location of each point associated with a number less than 5.

O If you obtain a true statement from $a < b$, how is the location of point a related to the location of point b in the natural-number line? The universe for a and b is N.

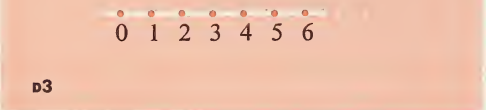
You have been studying a property of the set of natural numbers. Since the natural numbers can be arranged in a special order by using the idea of "less than," the set of natural numbers has the *property of order*.

P You know that $1 < 2$. The inequality $2 > 1$ gives you the same information about the order of the numbers 1 and 2 as the inequality $1 < 2$. In other words, $2 > 1$ means $1 < 2$. Does the inequality $3 > 2$ give you the same information as $2 < 3$?

Q What inequality involving the idea of "greater than" gives you the same information as $7 < 15$?



D2



D3

suc cess or of a natural number (sək ses′ər). The number that immediately follows a given natural number in the ordering of the natural numbers. The natural number 5 is the successor of the natural number 4. 126 is the successor of 125.

R Look again at D2. Is each point to the right of point 0 associated with a number greater than 0?

S If you obtain a true statement from $b > a$, how is the location of point b related to the location of point a in the number line? The universe for a and b is N.

Now you know what is meant by the idea of "less than" for natural numbers. You also know that the idea of "greater than" can be explained by using the idea of "less than."

Next you will study another property of the set of natural numbers.

A Look at D3. Does 4 immediately follow 3? What number immediately follows 5?

B For each natural number, is there another natural number that follows immediately? If so, how can you find this number for any given natural number?

A natural number that immediately follows a given natural number is the *successor* of the given number.

C What is the successor of 7? Of 19? Of 510? Does each natural number have a successor?

D Suppose that x is a variable for any natural number. What is the successor of x?

Since each natural number has a successor, the set of natural numbers has the *successor property*.

E Suppose that a set of natural numbers contains 0 and the successor of each of its members. Will it contain all the natural numbers?

F Is every natural number the successor of another natural number? Explain your answer.

G Look again at D3. Which points are between point 2 and point 6? Does each number associated with a point that you named satisfy $2 < x \wedge x < 6$? U = N.

H Now tabulate the set of natural numbers associated with points that are between point 8 and point 16. Is this set an infinite set?

I The set that you tabulated for exercise H is the solution set of a compound condition. Write a sentence to express this condition.

J Is there a natural number that is greater than 7 and less than 8? Tabulate the solution set of $7 < x \wedge x < 8$. U = N.

Suppose that the natural numbers have been ordered by using the idea of "less than." For each replacement of the variables, if $a < x \wedge x < b$, then x is *between* a and b. The universe for each of the variables is N.

K Tabulate the set of natural numbers between 4 and 11. Between 16 and 17. Between 472 and 480.

L How many natural numbers are in each set that you tabulated for exercise K?

Notice that it is possible to determine how many natural numbers are between any two given natural numbers.

M Think about a set whose members are the natural numbers that are between any two given natural numbers. Is this set a finite set? Explain your answer.

In this lesson you learned that the set of natural numbers has an order property and a successor property. You also learned that you can determine how many natural numbers there are between any two given natural numbers.

On your own

For each of exercises 1 through 4, tabulate four proper subsets of the set named. U = N.

1 $\{0, 1, 2\}$ **3** $\{x|5x < 50\}$

2 $\{5, 10, 15, 20, 25\}$ **4** $\{x|17 - x > 13\}$

For each ordered pair named in exercises 5 through 10, tell whether or not the standard set of the first component is a proper subset of the standard set of the second component.

5 (6, 9) **7** (15, 10) **9** (2, 11)

6 (1, 7) **8** (11, 9) **10** (10, 4)

11 Tell which ordered pairs named in exercises 5 through 10 are members of $\{(x, y)|x < y\}$. U = N × N.

12 Does each ordered pair that you named for exercise 11 satisfy $\{(x, y)|y > x\}$? How do you know? U = N × N.

13 Are $\{(x, y)|x < y\}$ and $\{(x, y)|y > x\}$ the same set? How do you know? U = N × N.

For each number expressed below, name its successor.

14 2043 **15** 101 **16** 599 **17** 10009

18 What is the successor of the successor of 7?

19 Suppose that the natural numbers are arranged in order by using the idea of "less than." Tabulate the set of natural numbers between 9 and 21.

20 How many members are in the set of natural numbers between 0 and 5? Between 14 and 15?

21 Tabulate $\{x|0 < x \wedge x < 14\}$. U = N.

22 Tabulate $\{x|35 < x \wedge x < 36\}$. U = N.

23 Tabulate $\{x|13 > x \wedge x > 11\}$. U = N.

24 Tabulate $\{x|21 > x \wedge x > 15\}$. U = N.

CHECKING UP

If you have trouble with these tests, you can find help in lessons 57, 58, and 59.

Test 65

For each of exercises 1 through 5, tabulate the set described. Then tell whether or not the set you tabulated is equivalent to {0, 1, 2, 3}.

1 $\{x|52 + x < 54\}$ U = N.

2 $\{x|x + 2 < 9 \wedge x - 1 > 1\}$ U = N.

3 $\{(x, y)|x + y < 3\}$ U = N × N.

4 $\{(x, y)|3/4 \sim x/y \wedge x + y < 40\}$ U = C × C.

5 $\{(x, y)|x + 7 < y \wedge x + y = 15\}$ U = N × N.

Tabulate the standard set of the natural number named in each exercise below.

6 1 **7** 8 **8** 12 **9** 2 **10** 14 **11** 17

12 Which sets that you tabulated for exercises 6 through 10 are proper subsets of the set that you tabulated for exercise 11?

13 Is the standard set of the number 17 a proper subset of itself?

Test 66

A standard set is tabulated in each of exercises 14 through 18. For each exercise, name the number associated with the set; then name the successor of this number. U = N.

14 { }

15 {0, 1, 2, 3, 4, 5}

16 {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}

17 {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

18 {0, 1, 2, 3, 4, 5, 6}

Tabulate each set described in exercises 19, 20, and 21. U = N.

19 $\{x|0 < x \wedge x < 6\}$

20 $\{x|3 < x \wedge x < 10\}$

21 $\{x|7 > x \wedge x > 1\}$

22 Make a diagram to represent a one-to-one mapping of {5, 10, 15, 20, 25} onto {3/7, 6/14, 9/21, 12/28, 15/35}.

APPLYING MATHEMATICS

For each problem in this lesson, write a sentence that expresses a condition. You may use any letters as names for the variables. Then tabulate the solution set of the condition and give the answer to the problem. The universe is given for each problem.

1 The manager of a warehouse has 176 window shades of a certain type on hand. He received an order for 320 shades of this type. How many more shades of this type does he need before he can fill this order? U = N.

2 Mr. Robinson drove 60 mi. in 75 min. At this rate, how many miles did he drive in 15 min.? U = C.

3 Kay had more than 27 in. of ribbon. She used 13 in. of this ribbon and then had less than 19 in. left. How many inches of ribbon could she have had before she used the 13 in.? U = N.

4 Twelve students entered a science contest. More boys than girls entered the contest. How many boys and how many girls could have entered the contest? U = C × C.

5 A company printed more than 1245 copies of its annual report. After 1195 copies had been distributed, fewer than 66 copies were left. How many copies of the annual report could the company have printed? U = N.

6 Bill went fishing and caught fewer than 15 perch and sunfish. He caught 5 more perch than sunfish. How many perch and how many sunfish could he have caught? U = C × C.

7 During a sale, Mrs. Morton bought a chair for $147. The regular price of this chair was $210. What rate of discount did she receive? U = C × C.

8 Together, Sally and Linda picked fewer than 40 pt. of raspberries. Sally picked 7 pt. for each 6 pt. that Linda picked. How many pints of raspberries could each of the girls have picked? U = C × C.

9 What number is 58% of 2300? U = C.

10 69 is 75% of what number? U = C.

11 What numbers greater than 52 can be added to 689 so that each sum is less than 756? U = N.

12 The sum of two numbers is 13. When the second number is subtracted from the first number, the difference is 4. What are the numbers? U = N × N.

13 Certain rate pairs are equivalent to 6/9. The sum of the first and second components of each of these rate pairs is less than 32. What are the rate pairs? U = C × C.

The closure property of addition

So far in this unit you have learned that natural numbers are associated with sets. You have also ordered the natural numbers. The set of natural numbers is a part of a system called the natural-number system. In this lesson you will begin to study the natural-number system.

A Are the sets tabulated in D1 disjoint sets? What number is associated with each set?

REMINDER

Two sets that have no common members are disjoint sets.

See lesson 22, page 84.

$A = \{1, 3\}$.
$B = \{0, 2, 4\}$.

D1

"The number of members in A" — n (A)

D2

$2 = n(A)$.

D3

"The number of members in the union of A and B" — n (A ∪ B)

D4

E = {Prince Edward Island}.
F = {British Columbia, Ontario, Alberta, Saskatchewan, Manitoba}.

D5

sum of two natural numbers (sum). The number associated with the union of two disjoint sets. Suppose that s is the number associated with set S and that t is the number associated with set T. The universe for s and t is N. If S and T are disjoint sets, the sum $s + t$ is the number associated with S ∪ T.

B D2 shows how to write and read a symbol for the number of members in set A. Are n(A) and n{1, 3} the same number? How do you know?

C Are n(B) and n{0, 2, 4} the same number? How do you know?

D Now read the sentence in D3. Does it express a true statement about set A?

E Is 3 = n(B) a true statement about set B?

F Look again at D1. Tabulate the union of {1, 3} and {0, 2, 4}.

REMINDER

The union of two sets is the set that contains those objects, and only those objects, that belong to either set or to both sets.

See lesson 23, page 87.

D4 shows how to write and read a symbol for the number of members in the union of A and B.

You know that, for the disjoint sets A and B, the number 2 is associated with set A, and the number 3 is associated with set B. The natural number associated with the union of A and B is the *sum* of the number 2 and the number 3. The sum is the number of members in the union of A and B. The number 2 + 3 is the sum of 2 and 3.

G Is 2 + 3 = n(A ∪ B)?

H Look at D5. How do you know that E and F are disjoint sets? Tabulate E ∪ F.

I Is 1 = n(E)? Is 5 = n(F)? Is 1 + 5 = n(E ∪ F)?

J Suppose that S and T are any disjoint sets. Suppose also that s is the number associated with set S and that t is the number associated with set T. With what set is the number $s + t$ associated? The universe for s and t is N.

K Suppose that Y and Z are disjoint sets, and that 5 = n(Y) and 9 = n(Z). With what set is the sum 5 + 9 associated? Is 5 + 9 = n(Y ∪ Z) a true statement?

L Suppose that A = {16, 17, 18, 19} and B = {18, 19, 20, 21, 22}. Is 4 + 5 the number associated with A ∪ B? Explain your answer.

When you write a symbol to express the sum of 3 and 4, you probably write the numeral 7. But you can also write the symbol 3 + 4, because this symbol shows that you are thinking of the number 3 + 4 as the sum of the two natural numbers 3 and 4. The statement 3 + 4 = 7 means that every set associated with 3 + 4 is equivalent to the standard set of 7.

Now you will learn something new about addition. You will see that addition is a special kind of mapping of one set onto another set.

A Look at the tabulation in D6. Set M is a set of ordered pairs. Are the first and second components of each ordered pair natural numbers?

B Look at D7. The diagram represents a mapping of each of the ordered pairs in set M onto the sum of its components. Which ordered pair is mapped onto the sum 9 + 8, or 17?

C Is (7, 9) mapped onto the sum of 7 and 9, or 16? Is 16 a natural number? Onto what natural number is (9, 10) mapped? Onto what natural number is (10, 9) mapped?

D Make a diagram that represents a mapping of each of the ordered pairs named below onto the sum of its components.

(4, 3) (3, 4) (6, 1) (1, 6)

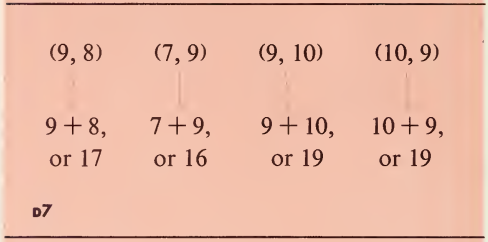
E Look at D8. Is each ordered pair named in D8 mapped onto the natural number that is the sum of its components?

F Why are (7, 1), (1, 7), and (6, 2) each mapped onto 8? Name three other ordered pairs of natural numbers that can be mapped onto 8.

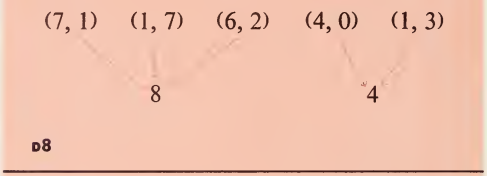
G Why are both (4, 0) and (1, 3) mapped onto 4? Name three other ordered pairs of natural numbers that can be mapped onto 4.

M = {(9, 8), (7, 9), (9, 10), (10, 9)}.

D6



D7



D8

(5, 1) (1, 5) (9, 3) (3, 9)

D9

H Is the mapping represented in D8 a one-to-one mapping?

You see that more than three ordered pairs are mapped onto 8, and more than two ordered pairs are mapped onto 4. The mappings represented in D8 are *many-to-one mappings.*

When an ordered pair is mapped onto the sum of its components, the mapping is an example of the *operation of addition.* The operation of addition of natural numbers is a special many-to-one mapping of the set of ordered pairs of natural numbers onto the set of natural numbers.

I Write a sentence that expresses a true statement you can obtain from $a + b = c$ by replacing each of the variables by a member of N. Now, using the same replacements for a and b, write the name of the ordered pair obtained from (a, b). Can this ordered pair be mapped

by addition onto the replacement that you used for c?

You may say that, when the operation of addition is used, if c is the sum of a and b, then (a, b) maps onto c. The universe for each of the variables is N.

You have seen that you can use the components of an ordered pair of natural numbers to obtain a third natural number that is their sum. You can use the components of (3, 5) to obtain $3 + 5$, or 8. You can use the components of (9, 20) to obtain $9 + 20$, or 29. Therefore, when you use the operation of addition, you must always work with an ordered pair of numbers. For this reason, addition is called a *binary* (bī′nə ri) operation. The word *binary* comes from the Latin word that means "two at a time," or "two."

J Look at D9. What is the sum of the components of each ordered pair of natural numbers named in the display?

K For each ordered pair of natural numbers, ? is there a natural number that is the sum of the components of the pair?

The sum of any two members of N is also a member of N. To express this in a shorter way, we say that N is *closed under the operation of addition.* N is closed under addition because the sum of any two natural numbers is always a natural number. This *closure* (klō′zhər) *property of addition of the set of natural numbers* is expressed below.

The universe for a *and* b *is N. For each replacement of* a *and* b, *there is a number* a + b *that is a member of N.*

You have used ideas about sets to learn about sums of natural numbers and about the operation of addition. Now you will use what you know about sums and the operation of addition to learn more about subtraction.

A What is the solution of $8 - 3 = x$? Is this also the solution of $8 = 3 + x$? The universe for x is N.

B What number do you obtain when you subtract 16 from 28? What number do you add to 16 to obtain 28?

Notice that subtraction is related to addition. You can use this relationship to explain the meaning of the difference of two natural numbers.

C You know that $8 = 2 + 6$. Is $8 - 2 = 6$? The difference of 8 and 2 is 6. How is this difference obtained from $8 = 2 + 6$?

D Write a sentence that expresses a true statement you can obtain from $a - b = c$ by replacing each of the variables by a member of N. Now, using the same replacements, write the sentence that expresses the statement you obtain from $a = b + c$. Does this sentence express a true statement?

You may say that, if a is the sum of b and c, then c is the difference of a and b. The universe for the variables is N.

E You know that 9 is the sum of 2 and 7. How do you know that 7 is the difference of 9 and 2? Use the explanation given above to answer this question.

F You also know that 9 is the sum of 7 and 2. From this, do you know the difference of 9 and 7? What is the difference?

G Look again at D9. Use subtraction to find the difference of the first component and the second component of (5, 1). What number is the difference of 5 and 1? What number must you add to 1 to get 5?

H When you are asked to find the difference of the components of an ordered pair, it means that you are to find the difference of the first component and the second component. What is the difference of the components of (9, 3)? What number must you add to 3 to get 9?

(5, 1) (1, 5) (9, 3) (3, 9)

D9

I Look again at D9. To find the difference of the components of (1, 5), you must find a number that you can add to 5 to get 1. Is the difference of 1 and 5 a natural number? Explain your answer.

J Is the difference of the components of (3, 9) a natural number? How do you know?

K Name three ordered pairs of natural numbers in which the first component of each pair is greater than the second component. For each pair that you named, what number is the difference of the components?

L Name three ordered pairs of natural numbers in which the first component of each pair is equal to the second component. For each pair that you named, what number is the difference of the components?

M Name three ordered pairs of natural numbers in which the first component of each pair is less than the second component. Is the difference of the components of any one of the pairs that you named a natural number? Explain your answer.

N For each ordered pair of natural numbers, is there always a natural number that is the difference of the components? Explain your answer.

O Suppose that the difference of a and b is a natural number. Then the replacement for a is greater than or equal to what number? The universe for the variables is N.

The difference of two natural numbers is *not* always a natural number. Therefore, N is *not* closed under subtraction.

Now you know how the sum of two natural numbers and the operation of addition may be explained by using sets. You know how subtraction is related to addition. You also know that the set of natural numbers is closed under addition but not under subtraction.

On your own

For exercises 1 through 9, suppose that sets E, F, G, and H are disjoint sets. Suppose also that $8 = n(E)$, $20 = n(F)$, $19 = n(G)$, and $70 = n(H)$. Tell what natural number is associated with the set named in each exercise.

1 $E \cup F$	4 $F \cup H$	7 $H \cup G$
2 $G \cup E$	5 $F \cup E$	8 $G \cup F$
3 $E \cup H$	6 $F \cup G$	9 $E \cup G$

For each ordered pair named below, name the natural number onto which you can map the ordered pair by addition.

10 (70, 50)	12 (36, 105)	14 (63, 18)
11 (29, 54)	13 (99, 2)	15 (118, 189)

For each ordered pair named below, tell what natural number must be added to the second component to obtain the first component.

16 (50, 25)	18 (62, 28)	20 (87, 19)
17 (78, 40)	19 (100, 28)	21 (54, 38)

22 $M = \{3, 6, 9, \ldots\}$. Name the next five members of M that follow 9.

23 Is $\{3, 6, 9, \ldots\}$ closed under addition? Is this set closed under subtraction? Explain your answers.

For each set described in exercises 24 through 31, first tell whether or not the set is closed under addition. Then tell whether or not the set is closed under subtraction.

24 The set of even numbers

25 The set of odd numbers

26 The set of odd numbers less than 27

27 The set of even numbers less than 14

28 The set of even numbers greater than 20

29 The set of natural numbers greater than 50

30 The set of natural numbers less than 50

31 The set of odd numbers greater than 99

KEEPING SKILFUL

Multiply

1 6512 by 9.
2 88 by 12.
3 397 by 34.
4 650 by 27.
5 914 by 56.
6 226 by 189.
7 139 by 417.
8 611 by 840.
9 428 by 501.
10 709 by 605.

Divide

11 5694 by 59.
12 2408 by 28.
13 26864 by 73.
14 15888 by 26.
15 17138 by 19.
16 14451 by 127.
17 38290 by 415.
18 12691 by 370.
19 24723 by 123.
20 129804 by 174.

Tabulate the solution set of each condition expressed in exercises 21 through 40. U = C.

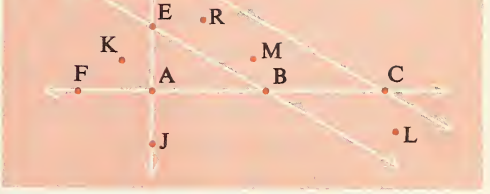
21 $c/3 \sim 15/15$.
22 $2/7 \sim d/42$.
23 12 is $n\%$ of 24.
24 $15/24 \sim 5/x$.
25 4% of 200 is g.
26 25% of x is 9.
27 $48/72 \sim b/3$.
28 72 is $g\%$ of 96.
29 $3/7 \sim m/147$.
30 $9/c \sim 100/100$.
31 125% of 64 is f.
32 $d/17 \sim 76/68$.
33 91 is $m\%$ of 52.
34 $21/23 \sim 168/s$.
35 150% of h is 135.
36 $9/d \sim 387/215$.
37 $1/20 \sim x/200$.
38 24% of 75 is y.
39 80% of m is 156.
40 $42/x \sim 26/91$.

For each of exercises 41 through 46, first tabulate sets G and H. Next tabulate G $\cup$ H. Then tabulate G $\cap$ H. U = C $\times$ C.

41 $G = \{(g, h) | 6/8 \sim g/h \wedge g + h < 24\}$.
$H = \{(g, h) | 3/4 \sim g/h \wedge 21 - g = h\}$.
42 $G = \{(g, h) | 9/15 \sim g/h \wedge g + h > 7\}$.
$H = \{(g, h) | 6/10 \sim g/h \wedge g + h < 35\}$.
43 $G = \{(g, h) | 8/7 \sim g/h \wedge g + h = 30\}$.
$H = \{(g, h) | g + h < 32 \wedge 7/8 \sim g/h\}$.
44 $G = \{(g, h) | 12/18 \sim g/h \wedge h + g > 4\}$.
$H = \{(g, h) | h + 3 > g \wedge 6/9 \sim g/h\}$.
45 $G = \{(g, h) | 2/5 \sim g/h \wedge g + h = 21\}$.
$H = \{(g, h) | g + 9 = h \wedge g/h \sim 4/10\}$.
46 $G = \{(g, h) | 6/4 \sim g/h \wedge g - h < 2\}$.
$H = \{(g, h) | g/h \sim 9/6 \wedge g + h = 6\}$.

Use the display below to help you answer exercises 47 through 63. Line DJ is perpendicular to line FC. If you need help with these exercises, review lessons 26 through 29.

47 Point A is the endpoint of four rays represented in the display. Name the four rays.
48 Describe the union of $\overrightarrow{BA}$ and $\overrightarrow{BE}$.
49 Name the sides and vertex of $\angle ADC$.
50 Name the sides and vertex of $\angle DEB$.
51 Name the four linear pairs of angles formed by $\overleftrightarrow{DJ}$ and $\overleftrightarrow{FC}$.
52 Name a supplement of $\angle DAB$. Of $\angle FAJ$.
53 $\overleftrightarrow{DC}$ and $\overleftrightarrow{EB}$ are parallel lines. Describe $\overleftrightarrow{DC} \cap \overleftrightarrow{EB}$.
54 Name the right angles represented.
55 $\triangle ABE$ is the union of what segments?
56 Name the sides and vertices of $\triangle ADC$.
57 Name the sides and vertices of polygon EBCD.
58 Describe the intersection of polygon EBCD and $\triangle ABE$.
59 Describe the intersection of $\triangle ABE$ and $\triangle ADC$.
60 Which points named are in the interior of $\triangle ADC$ and the exterior of $\triangle ABE$?
61 Which points named are in the exterior of $\angle DEB$ and the exterior of $\angle EBC$?
62 Which points named are in the interior of $\angle DCF$ and the exterior of $\angle JAC$?
63 Which points named are in the exterior of $\triangle ABE$ and the interior of $\angle DCA$?



The closure property of multiplication

In this lesson you will learn more about the meaning of multiplication, and you will see how division is related to multiplication.

A Are sets D and E, tabulated in D1, disjoint sets? Is F the Cartesian set of D and E?

> REMINDER
> The Cartesian set of two sets is the set of all ordered pairs formed by matching each member of the first set in turn with each member of the second set.
> *See lesson 31, page 127.*

B Is $2 = n(D)$ a true statement about set D? Is $3 = n(E)$ a true statement about set E?

C D2 shows how to write and read a symbol for the number of members in $D \times E$. Remember that the members of $D \times E$ are ordered pairs. Are $n(D \times E)$ and n(F) the same number? How do you know?

Since set D has 2 members and set E has 3 members, you know that the number 2 is associated with set D, and the number 3 is associated with set E. The natural number that is associated with the Cartesian set of D and E is the *product* of the number 2 and the number 3. The product 2×3 is the number of members in the Cartesian set $D \times E$. The number 2×3 is the product of 2 and 3.

D Is $2 \times 3 = n(D \times E)$?

$D = \{0, 1\}$.
$E = \{0, 1, 2\}$.
$F = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$.

D1

"The number of members $n(D \times E)$ in D cross E"

D2

$G = \{0, 1, 2, 3\}$.
$H = \{0, 1, 2, 3, 4, 5\}$.

D3

product (prod′əkt). The number associated with the Cartesian set of two sets. Suppose that s is the number associated with set S and that t is the number associated with set T. The universe for s and t is N. The product $s \times t$ is the number associated with the Cartesian set $S \times T$.

E Sets G and H are tabulated in D3. Tabulate $G \times H$.

F What natural number is associated with set G? With set H?

G Is $4 \times 6 = n(G \times H)$? Explain your answer.

H Suppose that S and T are two given sets. ? Suppose also that s is the number associated with set S and that t is the number associated with set T. With what set is the product $s \times t$ associated? The universe for s and t is N.

I Tabulate a set A for which $5 = n(A)$. Tabulate a set B for which $2 = n(B)$. Now tabulate $A \times B$.

J Look at the tabulations you made for exercise I. With which set is the product 5×2 associated?

K Suppose that Q is the empty set and R is {2, 3, 4, 5}. What natural number is associated with Q? With R?

L What set is the Cartesian set of { } and {2, 3, 4, 5}? How do you know?

M Is 0 the product of 0 and 4? Use the empty set and {2, 3, 4, 5} to explain your answer.

N When you work with sums, you work with two disjoint sets. When you work with products, must the sets be disjoint?

Now you will learn that multiplication is a special kind of mapping.

O Look at D4. Is (2, 3) mapped onto the product of 2 and 3? Is 6 a natural number? Is each ordered pair of natural numbers named in D4 mapped onto a natural number that is the product of the components of the ordered pair?

P Onto what product can you map (8, 2)? (4, 4)? (1, 16)? Name the two other ordered pairs of natural numbers that are mapped onto the product 16 by multiplication.

When an ordered pair is mapped onto the product of its components, the mapping is an example of the *operation of multiplication*. The operation of multiplication of natural numbers is a special many-to-one mapping of the set of ordered pairs of natural numbers onto the set of natural numbers.

Q When the operation of multiplication is used, if c is the product of a and b, then (a, b) maps onto what number? The universe for each of the variables is N.

R You have learned that addition is a binary operation because you work with an ordered pair of numbers to obtain a sum. Is multiplication also a binary operation? Explain your answer.

S Three ordered pairs are named in D5. Are the components of each of these pairs natural numbers?

(2, 3)	(4, 6)	(8, 0)	(6, 4)
2×3, or 6	4×6, or 24	8×0, or 0	6×4, or 24

D4

(12, 2)	(3, 4)	(11, 4)

D5

T What is the product of the components of each ordered pair named in D5? Is each product a natural number?

U A natural number can be associated with any set. Can a natural number also be associated with the Cartesian set of any two sets?

V For each ordered pair of natural numbers, is there a natural number that is the product of the components?

You have learned that the product of any two members of N is also a member of N. We say that N is *closed under the operation of multiplication* because the product of any two natural numbers is always a natural number. This property is the *closure property of multiplication of natural numbers*. The property is expressed below.

The universe for a *and* b *is N. For each replacement of* a *and* b, *there is a number* a × b *that is a member of N.*

What you have learned about products and the operation of multiplication will give you a better understanding of quotients and division.

A What is the solution of $30 \div 10 = y$? Is this number also the solution of $30 = y \times 10$? The universe for y is N.

B What number do you get when you divide 12 by 4? What number do you get when you multiply 4 by 3?

You know that when you divide one number by another number, you obtain a quotient. $10 \div 2 = 5$ means that $10 = 5 \times 2$. The number 5, by which 2 is multiplied to get a product of 10, is the *quotient* of 10 and 2. You use *division* to find the quotient of two numbers. Notice that you have thought of division as it is related to multiplication.

C You know that $21 = 3 \times 7$ is a true statement. How does this statement help you to find the quotient of 21 and 7?

D Since the product of 5 and 3 is 15, the quotient of 15 and 3 is what number?

When you are asked to find the quotient of the components of an ordered pair of numbers, you find a natural number by which you can multiply the second component to obtain the first component.

E Find the quotient of the components of (32, 8). Is this the quotient of 32 and 8?

F Find the quotient of the components of (12, 12). By what number must you multiply 12 to get a product of 12?

G To find the quotient of the components of (3, 4), you must find a natural number by which you can multiply 4 to get a product of 3. How do you know that the quotient of 3 and 4 is not a natural number?

H Is there a natural number by which you can multiply 4 to get a product of 11? Is the quotient of the components of (11, 4) a natural number?

I Is the quotient of any two natural numbers always a natural number?

Since the quotient of two natural numbers is *not* always a natural number, N is *not* closed under division.

Now you know what is meant by the product of two natural numbers and the operation of multiplication. You have seen how division is related to multiplication. You also know that the set of natural numbers is closed under multiplication, but not under division.

On your own

For each of exercises 1 through 9, suppose $5 = n(E)$, $6 = n(F)$, $0 = n(G)$, and $100 = n(H)$. Tell what natural number is associated with the set named in each exercise.

1 $E \times F$	**4** $H \times F$	**7** $F \times G$
2 $F \times E$	**5** $H \times H$	**8** $F \times F$
3 $E \times E$	**6** $E \times H$	**9** $H \times G$

For each ordered pair named below, name the natural number onto which you can map the ordered pair by multiplication.

10 (70, 35)	**14** (1, 4)	**18** (15, 6)
11 (50, 3)	**15** (10, 8)	**19** (9, 1)
12 (20, 3)	**16** (39, 5)	**20** (13, 2)
13 (3, 3)	**17** (12, 0)	**21** (14, 7)

For each ordered pair named below, tell by what number you can multiply the second component to obtain the first component.

22 (50, 25)	**26** (4, 4)	**30** (6, 6)
23 (35, 7)	**27** (75, 15)	**31** (48, 12)
24 (18, 3)	**28** (27, 9)	**32** (28, 4)
25 (10, 1)	**29** (32, 16)	**33** (63, 7)

Tell whether or not the quotient of the components of each ordered pair named below is a natural number.

34 (45, 9)	**36** (5, 10)	**38** (72, 12)
35 (16, 3)	**37** (10, 5)	**39** (13, 13)

Tell whether or not each set described below is closed under multiplication. If it is not, give an example to explain your answer.

40 The set of even numbers

41 The set of odd numbers

42 The set of odd numbers greater than 9

43 The set of even numbers less than 16

44 The set of natural numbers greater than 20

45 The set of natural numbers less than 20

46 The set of natural numbers from 1051 through 1060

$0+1=1+0.$ $\quad 1+0=0+1.$
$0+2=2+0.$ $\quad 1+2=2+1.$
$0+3=3+0.$ $\quad 1+3=3+1.$
⋮ ⋮ ⋮ ⋮ ⋮ ⋮

$117+26=26+117.$
$117+27=27+117.$
⋮ ⋮ ⋮

D1

$C=\{1,\ 3,\ 5,\ 7\}.$
$D=\{12,\ 14,\ 16\}.$

D2

$4+3=n(C\cup D).$
$3+4=n(D\cup C).$

D3

6 62 Exploring ideas

The commutative properties

You have already studied the operations of addition and multiplication of natural numbers. In this lesson you will study an important property of each of these operations.

A Look at D1. Does each sentence express a true statement? The dots tell you that each list of sentences is endless.

The true statements expressed in D1 involve the natural numbers and the operation of addition. You have already used sets to help you think about addition of natural numbers. Now you will use sets to help you decide whether or not every statement of the kind expressed in D1 is true.

B Look at the tabulations in D2. Are sets C and D finite sets?

C Are sets C and D disjoint sets? How do you know?

D Tabulate $C\cup D$.

E Tabulate $D\cup C$.

F Is $C\cup D$ the same set as $D\cup C$? Remember that the order in which you list the names of the members of a set does not change the membership of the set.

G Look again at D2. How do you know that the number 4 is associated with set C? How do you know that the number 3 is associated with set D?

H Read each sentence in D3. Does each sentence in D3 express a true statement? Explain your answer.

I Is the sum $4+3$ associated with the same set as the sum $3+4$?

J Is $4+3$ the same number as $3+4$? Is $4+3=3+4$?

K Suppose that A and B are any two disjoint sets and that $a=n(A)$ and $b=n(B)$. With what set is $a+b$ associated? With what set is $b+a$ associated? Are $a+b$ and $b+a$ associated with the same set? Are $a+b$ and $b+a$ the same number? The universe for a and b is N.

You have been studying an important property that concerns addition of natural numbers. You have learned that the order in which you add any two given natural numbers does not affect the sum. This property is the *commutative* (kə mū′tə tiv) *property of addition of natural numbers.* The property is expressed below.

The universe for a *and* b *is N. For each replacement of* a *and* b, *a true statement is obtained from* $a+b=b+a$.

Universe for a and b = N.

$a + b = b + a$.

A $6 + 9 = 9 + 6$.

B $52 + 30 = 30 + 52$.

C $89 + 102 = 102 + 89$.

D4

$0 \times 1 = 1 \times 0$.
$0 \times 2 = 2 \times 0$.
$\vdots \quad \vdots \quad \vdots$

$5 \times 0 = 0 \times 5$.
$5 \times 1 = 1 \times 5$.
$\vdots \quad \vdots \quad \vdots$

$56 \times 79 = 79 \times 56$.
$56 \times 80 = 80 \times 56$.
$\vdots \quad \vdots \quad \vdots$

D5

$R = \{13, 14\}$.
$S = \{0, 1, 2\}$.

D6

L Three statements are expressed in D4. For each statement, what replacements were made for the variables in $a + b = b + a$? Is each statement true?

Think about the condition expressed in D4. You know that, for each replacement of the variables, you obtain a true statement from this condition. Therefore, we say that $a + b = b + a$ is a *property*. Any condition from which you obtain a true statement for each replacement of the variables is a property.

M Now look at D5. Does each sentence express a true statement?

The true statements expressed in D5 involve the natural numbers and the operation of multiplication. You will use sets to help you decide whether or not every statement of the kind expressed in D5 is true.

N Look at D6. Are sets R and S, tabulated in D6, finite sets?

O Tabulate $R \times S$. The first components of the members of $R \times S$ are members of what set? The second components are members of what set?

P Tabulate $S \times R$. The first components of the members of $S \times R$ are members of what set? The second components are members of what set?

Q Are the sets you tabulated for exercise O and for exercise P the same set? Explain your answer.

R Look at D7. For each member of $R \times S$, tell which member of $S \times R$ it is mapped onto. For each member of $S \times R$, tell which member of $R \times S$ it is mapped onto.

S Are sets $R \times S$ and $S \times R$ in one-to-one correspondence? Is $R \times S$ equivalent to $S \times R$?

REMINDER

A one-to-one correspondence consists of two one-to-one mappings that map a first set onto a second set and the second set onto the first set. Equivalent sets are two sets that can be put in one-to-one correspondence.

See lesson 57, pages 246 and 247.

T Does $R \times S$ have the same number of members as $S \times R$? Is $n(R \times S) = n(S \times R)$? Explain your answers.

U Is $2 \times 3 = n(R \times S)$? Is $3 \times 2 = n(S \times R)$? Is $2 \times 3 = 3 \times 2$?

V Suppose that A and B are any two sets and that $a = n(A)$ and $b = n(B)$. With what set is $a \times b$ associated? With what set is $b \times a$ associated? Are $a \times b$ and $b \times a$ associated with equivalent sets? Are $a \times b$ and $b \times a$ the same number? The universe for a and b is N.

$$R \times S = \{(13, 0), (13, 1), (13, 2), (14, 0), (14, 1), (14, 2)\}.$$

$$S \times R = \{(0, 13), (0, 14), (1, 13), (1, 14), (2, 13), (2, 14)\}.$$

D7

Universe for a and b = N.

$ab = ba$.

D $7 \cdot 8 = 8 \cdot 7$.

E $29 \cdot 16 = 16 \cdot 29$.

F $115 \cdot 21 = 21 \cdot 115$.

D8

Universe for a and b = N.

$a - b = b - a$.

G $1 - 0 = 0 - 1$.

H $52 - 8 = 8 - 52$.

I $120 - 97 = 97 - 120$.

D9

Now you know that the order in which you multiply any two natural numbers does not affect the product. This property is the *commutative property of multiplication of natural numbers.* The property is expressed below.

The universe for a *and* b *is N. For each replacement of* a *and* b, *a true statement is obtained from* $a \times b = b \times a$.

Notice that we used the familiar symbol for multiplication ($\times$) in the sentence $a \times b = b \times a$. However, mathematicians often express a product by using placeholders and omitting the symbol for the operation. Thus, ab means the same as $a \times b$. You can, therefore, use the sentence $ab = ba$ instead of the sentence $a \times b = b \times a$.

W You know that the symbol $2m$ expresses the product of 2 and the replacement for the variable m. Suppose you replace m by 4. Then you obtain 2×4, or 8. You already know that you can express 2×4 by the numeral 2(4). Why do you need to use some symbol for multiplication to express a product like 2(4)?

When mathematicians need a symbol for multiplication, they often use a raised dot, as shown in D8.

X If sentence D in D8 did not contain symbols for multiplication, you might mistake the product $7 \cdot 8$ for 78. Does the numeral $7 \cdot 8$ express the same number as the numeral 78? How do you know?

Y For each statement expressed in D8, what replacements were made for the variables in $ab = ba$? Is each statement true? Upon what property does your answer depend?

You have learned that addition and multiplication of natural numbers have the commutative property. Next you will decide if subtraction of natural numbers has this property.

A Look at D9. For each statement expressed in D9, what replacements were made for the variables in $a - b = b - a$?

Now you must decide whether or not each statement expressed in D9 is true.

B Look again at sentence G. You know that $1 - 0$ is the number 1. Is $0 - 1$ also the number 1? Explain your answer.

$0 - 1$ is not the number 1. Later you will learn that $0 - 1$ is a number, although it is not a natural number. $8 - 52$ and $97 - 120$ are not natural numbers, either.

Universe for a and b = N.

$a - b = b - a$.

G $1 - 0 = 0 - 1$.

H $52 - 8 = 8 - 52$.

I $120 - 97 = 97 - 120$.

D9

J $12 \div 3 = 3 \div 12$.

K $96 \div 16 = 16 \div 96$.

L $144 \div 24 = 24 \div 144$.

D10

C Does each sentence in D9 express a false statement?

D Does subtraction of natural numbers have the commutative property? Remember that, if subtraction has the commutative property, then you must obtain a true statement from $a - b = b - a$ for each replacement of a and b. The universe for a and b is N.

E How many false statements do you need to obtain before you can decide that subtraction of natural numbers does not have the commutative property? Explain your answer.

Next you will decide if division of natural numbers has the commutative property.

F Look at D10. Does each sentence express a true statement? A false statement?

G Does division of natural numbers have the commutative property? How do you know?

Now you know that both addition and multiplication of natural numbers have the commutative property. You also know that subtraction and division of natural numbers do not have the commutative property.

On your own

For each of exercises 1 through 10, tell whether the sentence expresses a true statement or a false statement. Make your decision without using computation.

1 $119 + 45 = 45 + 119$.

2 $186 \div 3 = 3 \div 186$.

3 $42 \cdot 59 = 59 \cdot 42$.

4 $23 - 18 = 18 - 23$.

5 $74 \cdot 16 > 16 \cdot 74$.

6 $26 + 0 = 0 + 26$.

7 $116 + 9 < 9 + 116$.

8 $14 \cdot 0 = 0 \cdot 14$.

9 $14 + 1001 = 1001 + 14$.

10 $95 \cdot 20 < 20 \cdot 95$.

Tabulate the solution set of each condition expressed in exercises 11 through 18. You can use what you have learned about the commutative properties to find the solutions without computing. U = N.

11 $36 + b = 15 + 36$.

12 $a \cdot 17 = 17 \cdot a$.

13 $41 + b < 1 + 41$.

14 $9 + 4 > 4 + b$.

15 $2 \cdot 3 > 3 \cdot a$.

16 $19 + b = b + 19$.

17 $a \cdot 24 > 24 \cdot 2$.

18 $45 + b > 45 + 13$.

KEEPING SKILFUL

For exercises 1 through 9, convert each numeral to a decimal numeral.

1 425_{six}

2 1011_{two}

3 683_{nine}

4 550_{seven}

5 3109_{eleven}

6 214_{five}

7 312_{six}

8 1239_{twelve}

9 528_{nine}

For exercises 10 through 15, convert each decimal numeral to a numeral in the numeration system whose base is expressed within parentheses.

10 205 (base three)

11 147 (base five)

12 631 (base four)

13 7523 (base seven)

14 95 (base two)

15 8761 (base nine)

For exercises 16 through 21, convert each numeral to a base-twelve numeral.

16 11010_{two}

17 708

18 734_{eight}

19 1120_{three}

20 396

21 34102_{five}

SPECIAL CHALLENGE

You know that both addition and multiplication of natural numbers have the commutative property. You also know that N is closed under the operations of addition and multiplication. Suppose we invent a new binary operation called "star." We will use an asterisk (*) to indicate the operation of "star."

The meaning of the operation of "star" is expressed by the following sentence: $a * b = a + 2b$.

When you perform the operation of "star," you find the sum of the first number and twice the second number. If N is closed under "star," a natural number is obtained from $a * b$ for each replacement of a and b. The universe for a and b is N.

What replacements were made for a and b in $a * b$ to obtain each number named below?

A $5 + 2(7)$ **B** $9 + 2(4)$ **C** $69 + 2(141)$

D Do you think that N is closed under the operation of "star"? Explain your answer.

If the operation of "star" is commutative, a true statement is obtained for each replacement of a and b in $a * b = b * a$.

E Replace a by 3 and b by 4 in $a * b = b * a$. You must decide if $3 * 4 = 4 * 3$. Is $3 * 4 = 3 + 2(4)$? Is $4 * 3 = 4 + 2(3)$?

F Is $3 + 2(4) = 4 + 2(3)$? Is $3 * 4 = 4 * 3$?

G Replace a by 7 and b by 15 in $a * b = b * a$. Is $7 * 15 = 15 * 7$? Explain your answer.

H Is the operation of "star" commutative? Explain your answer.

Each of the sentences below expresses the meaning of an invented binary operation. Tell whether or not each operation is commutative. Give three examples to illustrate each of your answers. The universe for a and b is N.

I $a \P b = 3a + b$. **K** $a \bowtie b = 2a + 2b$.

J $a \dagger b = 4a - b$. **L** $a \# b = 5a - 3b$.

The associative properties

From the commutative properties, you know that the order in which you add or multiply two natural numbers does not affect the sum or product. In this lesson you will study another property of both addition and multiplication of natural numbers.

Look at symbol A in D1. You know that addition is a binary operation. This means that, when you find the sum of more than two numbers, you must work with one pair of numbers at a time.

A In symbol B in D1, the sum of what pair of numbers is expressed within the parentheses? Onto what natural number can you map this pair of numbers by addition?

B Look at symbol C. How was the number 7 obtained? Onto what natural number can you map (7, 6) by addition?

C Is the number that you named for exercise B the sum of 4, 3, and 6?

Notice that to find the sum of 4, 3, and 6, you first found the sum of 4 and 3 and then found the sum of 7 and 6. Now you will find the sum by grouping 4, 3, and 6 in another way to see whether or not you obtain the same sum.

A $4 + 3 + 6$
B $(4 + 3) + 6$
C $7 + 6$

D1

D Look at symbol D in D2. The sum of what pair of numbers is expressed within the parentheses? Is this the same pair of numbers that you named for symbol B in D1?

E Onto what natural number can you map (3, 6) by addition?

F Look at symbol E. How was the number 9 obtained? Onto what natural number can you map (4, 9) by addition?

G Is $(4 + 3) + 6 = 4 + (3 + 6)$?

Notice how the parentheses help you to think about grouping numbers into pairs when you add. The symbol $(4 + 3) + 6$ indicates that 6 has been added to the sum of 4 and 3. The symbol $4 + (3 + 6)$ indicates that the sum of 3 and 6 has been added to 4.

H Does each sentence in D3 express a true statement?

From the statements expressed in D3, you can see that the way in which you group three natural numbers does not affect the sum. These statements are examples of a property concerning addition of natural numbers. This property is the *associative (ə sō′shēə′tiv) property of addition of natural numbers.* The property is expressed below.

The universe for a, b, *and* c *is N. For each replacement of* a, b, *and* c, *a true statement is obtained from* $(a + b) + c = a + (b + c)$.

I Three statements are expressed in D4. For each statement, tell what replacements were made for the variables in $(a + b) + c = a + (b + c)$. Is each statement true? Upon what property does your answer depend?

J Look again at symbol A in D1. Explain why you may accept the symbol $4 + 3 + 6$ as a name of a number.

Now you will learn how you may group numbers in different ways when you find the product of more than two natural numbers.

A $4 + 3 + 6$
B $(4 + 3) + 6$
C $7 + 6$

D1

D $4 + (3 + 6)$
E $4 + 9$

D2

F $(14 + 8) + 5 = 14 + (8 + 5)$.
G $(52 + 20) + 19 = 52 + (20 + 19)$.
H $(112 + 13) + 1 = 112 + (13 + 1)$.

D3

Universe for a, b, and c = N.
$(a + b) + c = a + (b + c)$.

I $(9 + 8) + 16 = 9 + (8 + 16)$.
J $(1 + 45) + 24 = 1 + (45 + 24)$.
K $(13 + 11) + 19 = 13 + (11 + 19)$.

D4

A $2 \cdot 5 \cdot 8$
B $(2 \cdot 5)8$
C $2(5 \cdot 8)$

D5

Look at symbol A in D5. You know that multiplication is a binary operation. This means that, when you find the product of more than two numbers, you must work with one pair of numbers at a time.

A In symbol B in D5, the product of what pair of numbers is expressed within the parentheses? Onto what natural number can you map this pair of numbers by multiplication?

Notice that in symbol B there is no raised dot between the numeral $(2 \cdot 5)$ and the numeral 8. The parentheses tell you that 8 has been multiplied by the product $2 \cdot 5$.

B Onto what natural number can you map (10, 8) by multiplication?

C In symbol C in D5, the product of what pair of numbers is expressed within the parentheses?

D Onto what natural number can you map (2, 40) by multiplication?

E Is the number you named for exercise D the same as the number you named for exercise B?

F Is $(2 \cdot 5)8 = 2(5 \cdot 8)$?

Notice how the parentheses help you to think about grouping numbers in pairs when you multiply. The symbol $(2 \cdot 5)8$ indicates that 8 has been multiplied by the product of 2 and 5. The symbol $2(5 \cdot 8)$ indicates that the product of 5 and 8 has been multiplied by 2.

G Does each sentence in D6 express a true statement?

From the statements expressed in D6, you can see that the way in which you group three natural numbers does not affect the product. These statements are examples of a property concerning multiplication of natural numbers. This property is the *associative property of multiplication of natural numbers*. The property is expressed below.

The universe for a, b, *and* c *is N. For each replacement of* a, b, *and* c, *a true statement is obtained from* (ab)c = a(bc).

H Three statements are expressed in D7. For each statement, tell what replacements were made for the variables in $(ab)c = a(bc)$. Is each statement true? Upon what property does your answer depend?

I Look again at symbol A in D5. Explain why you may accept the symbol $2 \cdot 5 \cdot 8$ as a name of a number.

D $(7 \cdot 6)14 = 7(6 \cdot 14)$.

E $(25 \cdot 18)39 = 25(18 \cdot 39)$.

F $(123 \cdot 1)14 = 123(1 \cdot 14)$.

D6

Universe for a, b, and c = N.

$(ab)c = a(bc)$.

G $(9 \cdot 12)17 = 9(12 \cdot 17)$.

H $(82 \cdot 13)5 = 82(13 \cdot 5)$.

I $(239 \cdot 4)11 = 239(4 \cdot 11)$.

D7

Universe for a, b, and c = N.

$(a - b) - c = a - (b - c)$.

$(4 - 3) - 1 = 4 - (3 - 1)$.

D8

The following exercises will help you decide whether or not subtraction and division of natural numbers have the associative property.

A Study the condition expressed in D8. What replacements were made for the variables in the condition to obtain the statement that is expressed?

B Is $(4 - 3) - 1 = 4 - (3 - 1)$? Notice that the parentheses are used in the same way as they were used in addition and multiplication. The parentheses indicate which subtraction has been done first.

C Does subtraction of natural numbers have the associative property? Explain your answer.

On the basis of just one example you know that subtraction of natural numbers does not have the associative property. If subtraction of natural numbers did have the associative property, then you would obtain a true statement

$(60 \div 6) \div 2 = 60 \div (6 \div 2)$.

D9

A $36 + (83 + 97) = (36 + 83) + 97$.
B $(36 + 83) + 97 = (83 + 36) + 97$.
C $36 + (83 + 97) = (83 + 36) + 97$.

D10

D $(8 \cdot 73)59 = 59(8 \cdot 73)$.
E $59(8 \cdot 73) = (59 \cdot 8)73$.
F $(8 \cdot 73)59 = (59 \cdot 8)73$.

D11

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from $(a - b) - c = a - (b - c)$ for each replacement of a, b, and c. You have used one set of replacements for a, b, and c for which you did not obtain a true statement from $(a - b) - c = a - (b - c)$. You know, therefore, that subtraction of natural numbers does not have the associative property.

D Read the sentence in D9. Is $(60 \div 6) \div 2 = 60 \div (6 \div 2)$?

E Does division of natural numbers have the associative property? Explain your answer.

The following exercises will help you see how the commutative and associative properties are used in mathematics.

A Read the sentences in D10. Does sentence A express a true statement? Upon what property does your answer depend?

B Does sentence B in D10 express a true statement? Upon what property does your answer depend?

C You know that $36 + (83 + 97)$ is the same as $(36 + 83) + 97$. You know that $(36 + 83) + 97$ is the same as $(83 + 36) + 97$. Does sentence C in D10 express a true statement?

The method used in exercises A, B, and C can be used to decide whether or not the sentences in D11 express true statements.

D Does sentence D in D11 express a true statement? Upon what property does your answer depend?

E Does sentence E in D11 express a true statement? Upon what property does your answer depend?

F Is the number $(8 \cdot 73)59$ the same as the number $59(8 \cdot 73)$? Is the number $59(8 \cdot 73)$ the same as the number $(59 \cdot 8)73$?

G If you know that a first number is the same as a second number, and that the second number is the same as a third number, do you also know that the first number is the same as the third number? Explain why sentence F in D11 expresses a true statement.

Now you know that both addition and multiplication of natural numbers have the associative property. You also know that subtraction and division of natural numbers do not have the associative property.

On your own

For each of exercises 1 through 10, use the commutative and associative properties to decide whether the sentence expresses a true statement or a false statement. For each true statement, name the property upon which your answer depends.

1 $(43 + 16) + 9 = 43 + (16 + 9)$.
2 $34 \div 5 = 5 \div 34$.
3 $(17 - 4) - 2 = 17 - (4 - 2)$.
4 $(67 \cdot 15)3 = 67(15 \cdot 3)$.
5 $44 \div 11 = 11 \div 44$.
6 $14 - 6 = 6 - 14$.
7 $(13 + 5) + 7 = (5 + 13) + 7$.
8 $(12 \div 2) \div 2 = 12 \div (2 \div 2)$.
9 $27 + (5 + 2) = (27 + 5) + 2$.
10 $26 - (4 - 0) \neq (26 - 4) - 0$.

For each condition expressed in exercises 11 through 24, tabulate the solution set. You can use what you have learned about the commutative and associative properties to help you find the solutions without computing. The universe for the variables is N.

11 $(81 + 14) + 10 = 81 + (14 + c)$.
12 $a \cdot 85 = 85 \cdot 12$.
13 $27 + b = 19 + 27$.
14 $(44 \cdot b)3 = (b \cdot 44)3$.
15 $(a \cdot 12)15 < 1(12 \cdot 15)$.
16 $(a + 71) + 12 > 3 + (71 + 12)$.
17 $(19 + 21) + c = 11 + (19 + 21)$.
18 $(37 + b) + 14 = 37 + (b + 14)$.
19 $(1 + 0)16 < (b + 1)16$.
20 $(17 \cdot 23)0 \neq 17(23 \cdot c)$.
21 $(47 \cdot 4)4 = a(4 \cdot 4)$.
22 $6 \cdot b < 5 \cdot 6$.
23 $45 + 32 > a + 45$.
24 $(17 + 9) + 5 = 5 + (17 + b)$.

For each of exercises 25 through 34, tell why the numeral in column B names the same number as the numeral in column A. Use the commutative and associative properties to explain your answers. For example, you could answer exercise 25 by writing the following:

$37 + (76 + 92) = 37 + (92 + 76)$. The commutative property of addition.

$37 + (92 + 76) = (37 + 92) + 76$. The associative property of addition.

	A	B
25	$37 + (76 + 92)$	$(37 + 92) + 76$
26	$14(7 \cdot 9)$	$(7 \cdot 9)14$
27	$19(23 \cdot 8)$	$(19 \cdot 8)23$
28	$(21 \cdot 19)6$	$19(21 \cdot 6)$
29	$(37 \cdot 28) + 5$	$5 + (37 \cdot 28)$
30	$(24 + 18) + 12$	$(12 + 24) + 18$
31	$3(7 + 41)$	$(41 + 7)3$
32	$(17 + 2)8$	$8(17 + 2)$
33	$(45 \cdot 6)3$	$(3 \cdot 45)6$
34	$13 + (9 \cdot 2)$	$(2 \cdot 9) + 13$

KEEPING SKILFUL

For each condition expressed in exercises 1 through 9, use {23, 5, 9, 13, 4} as the universe for the variable. Tabulate the solution set of each condition.

1 $x + 18 < 35$.
2 $36x = 468$.
3 $147 - x > 96$.
4 $x + 97 = 120$.
5 $64 + x > 87$.
6 $39x < 351$.
7 $x + 5 > 14 \wedge 64 - x > 45$.
8 $x < 2016 \div 96 \wedge 75 + x < 88$.
9 $x > 5 \wedge x + 49 > 62$.
10 Sets K and L are tabulated below. Tabulate $K \times L$. $K = \{0, 1, 2, 3, 4\}$. $L = \{0, 1, 2\}$.

For each condition expressed in exercises 11 through 14, the universe for (m, n) is $K \times L$, which you tabulated for exercise 10. For each exercise, make graphs of the solution sets of both simple conditions. Then tabulate the solution set of the compound condition.

11 $m + n > 2 \wedge 3 - n = m$.
12 $m > n \wedge m + n > 3$.
13 $5 - m = n \wedge m + 2 < 8$.
14 $m + n < 5 \wedge m + n > 1$.

For each condition expressed in exercises 15 through 20, the universe for (r, s) is $C \times C$. Tabulate the solution set of each condition.

15 $5/7 \sim r/s \wedge r + s < 50$.
16 $r/s \sim 8/3 \wedge r - s = 15$.
17 $s < r + 6 \wedge r/s \sim 3/4$.
18 $18/12 \sim r/s \wedge r + s > 4$.
19 $98/161 \sim r/23 \wedge r + s = 111$.
20 $r/13 \sim s/104 \wedge 254 - s = 118$.

For each ordered pair named below, first name the natural number onto which you can map the ordered pair by addition. Then name the natural number onto which you can map the ordered pair by multiplication.

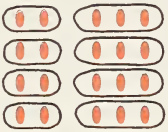
21 (11, 23)
22 (28, 49)
23 (113, 78)
24 (56, 210)
25 (457, 93)
26 (108, 630)
27 (152, 341)
28 (270, 610)
29 (901, 208)

The distributive property

You have learned to use the associative property in grouping numbers. In this lesson you will learn about another property used in grouping numbers. This property applies when addition is used with multiplication.

Think of the markers below as representing objects. Study the arrangement of objects opposite exercise A. The work below shows two ways to find the total number of objects.

A Are there 4 groups of 2 objects each? How many groups of 3 objects each are there? Is there a total of $(4 \cdot 2) + (4 \cdot 3)$ objects?



$(4 \cdot 2) + (4 \cdot 3)$

B What is the product of 4 and 2? Of 4 and 3? Does the numeral $8 + 12$ express the same number as the numeral $(4 \cdot 2) + (4 \cdot 3)$?

$8 + 12$

C Onto what natural number can you map (8, 12) by addition?

D The objects have been regrouped below. Now there are 4 groups of objects. Are there $2 + 3$ objects in each group? Are there $4(2 + 3)$ objects in all?

$4(2 + 3)$

E Onto what natural number can you map (2, 3) by addition? Does the numeral $4 \cdot 5$ express the same number as the numeral $4(2 + 3)$?

$4 \cdot 5$

Universe for a and b = N.

A $(6 \cdot 5) + (6 \cdot 9) = 6(5 + 9)$.

B $(10 \cdot 3) + (10 \cdot 7) = a(3 + 7)$.

C $9(b + 4) = (9 \cdot 18) + (9 \cdot 4)$.

D1

D $2(13 + 1) = (2 \cdot 13) + (2 \cdot 1)$.

E $8(10 + 7) = (8 \cdot 10) + (8 \cdot 7)$.

F $17(6 + 9) = (17 \cdot 6) + (17 \cdot 9)$.

D2

Universe for a, b, and c = N.

$a(b + c) = (ab) + (ac)$.

G $3(14 + 9) = (3 \cdot 14) + (3 \cdot 9)$.

H $16(12 + 8) = (16 \cdot 12) + (16 \cdot 8)$.

I $41(37 + 23) = (41 \cdot 37) + (41 \cdot 23)$.

J $3(3 + 3) = (3 \cdot 3) + (3 \cdot 3)$.

D3

F Onto what natural number can you map (4, 5) by multiplication?

G Is $(4 \cdot 2) + (4 \cdot 3) = 8 + 12$? Is $4(2 + 3) = 4 \cdot 5$? Is $(4 \cdot 2) + (4 \cdot 3) = 4(2 + 3)$?

Notice how the parentheses help you to think about grouping numbers in pairs. The numeral $(4 \cdot 2) + (4 \cdot 3)$ indicates that first the product of 4 and 2 and the product of 4 and 3 have been found. Then the sum of these products has been obtained. The numeral $4(2 + 3)$ indicates that the sum of 2 and 3 has been multiplied by 4.

The following exercises will help you develop a property that relates multiplication and addition of natural numbers.

H Read sentence A in D1. Since 6 is the multiplier of the number 5 and also of the number 9, we say that 6 is the *common multiplier* of 5 and 9. Does sentence A express a true statement?

I Read sentence B in D1. The number 10 is the common multiplier of what numbers? What replacement can you make for a so that you obtain a true statement?

J Read sentence C. What number is the common multiplier of the replacement for b and 4? If you replace b by 18, will you obtain a true statement? How do you know?

K Look at D2. Does each sentence express a true statement?

Study sentence D in D2. Notice that when you multiply the sum of 13 and 1 by 2, you obtain the same number as when you add the product of 2 and 13 and the product of 2 and 1.

L In each sentence in D2, is the product that is expressed at the left of the symbol for equality the same number as the sum of the products expressed at the right?

You have been studying an important property of natural numbers. This property is the *distributive* (dis trib′ū tiv) *property of multiplication over addition of natural numbers.* The property is expressed below.

The universe for a, b, *and* c *is N. For each replacement of* a, b, *and* c, *a true statement is obtained from* a(b + c) = (ab) + (ac).

M There are two ways to find the number obtained from $a(b + c)$ after you have replaced each variable. Describe these two ways.

N Look at D3. For each statement expressed, tell what replacements were made for the variables in $a(b + c) = (ab) + (ac)$. Is each statement true? How do you know?

O Look at D4. How does the condition expressed differ from $a(b + c) = (ab) + (ac)$?

You will decide if $(b + c)a = (ba) + (ca)$ is satisfied by all possible replacements for the variables. As you continue your study of mathematics, you will often develop *proofs* like the proof developed in exercises P through U that follow. You use the properties you have already accepted to prove that, for each replacement of a, b, and c in $(b + c)a = (ba) + (ca)$, you obtain a true statement.

Universe for a, b, and c = N.
$(b + c)a = (ba) + (ca)$.

D4

Universe for a, b, and c = N.
K $(b + c)a = a(b + c)$.
L $a(b + c) = (ab) + (ac)$.
M $(ab) + (ac) = (ba) + (ca)$.

D5

P Look at D5. What is the universe for the variables?

Q Read sentence K in D5. For each replacement of a, b, and c in the condition expressed by sentence K, will you obtain a true statement? Upon which property does your answer depend, the associative property, the commutative property, or the distributive property?

R Read sentence L in D5. For each replacement of a, b, and c in the condition expressed, will you obtain a true statement? Upon what property does your answer depend?

S Read sentence M. For each replacement of the variables in the condition expressed, will you obtain a true statement? Upon what property does your answer depend?

T Now think about the conditions expressed in D5. After you make replacements, you know that $(b + c)a$ is equal to $a(b + c)$; that $a(b + c)$ is equal to $(ab) + (ac)$; and that $(ab) + (ac)$ is equal to $(ba) + (ca)$. Is $(b + c)a$ equal to $(ba) + (ca)$?

U For each replacement of a, b, and c in $(b + c)a = (ba) + (ca)$, will you obtain a true statement? Explain your answer. Remember that the universe for a, b, and c is N.

Now you know that multiplication distributes over addition of natural numbers. You know that the distributive property can be expressed by the sentence $a(b + c) = (ab) + (ac)$ or by the sentence $(b + c)a = (ba) + (ca)$. The distributive property can also be expressed by the sentence $(ab) + (ac) = a(b + c)$ or by the sentence $(ba) + (ca) = (b + c)a$.

Many times throughout your study of arithmetic, you have found the product of two numbers like 53 and 6. When you did this, you really were using the distributive property. When you multiplied 53 by 6, you found the product by thinking of $6(50 + 3)$.

A Read sentence A in D6. Does sentence A express a true statement?

B Does sentence B express a true statement? Upon what property does your answer depend?

C Do sentences C and D express true statements?

D Is the product of 53 and 6 equal to 318?

E Think of the product of 46 and 4. Is $4(46) = 4(40 + 6)$?

F Is $4(40 + 6) = (4 \cdot 40) + (4 \cdot 6)$? Upon what property does your answer depend?

G Is $(4 \cdot 40) + (4 \cdot 6) = 160 + 24$, or 184?

In exercises E, F, and G, sentences show how the distributive property is used to help you obtain the product of 46 and 4. Write similar sentences to show how the distributive property is used to help find each of the products expressed below. Notice the three different symbols used to express multiplication.

H 8×47 **I** $7 \cdot 92$ **J** $3(79)$

Now you will decide if addition distributes over multiplication of natural numbers.

A Read the sentence in D7. Explain the condition expressed.

A $6(53) = 6(50 + 3)$.
B $6(50 + 3) = (6 \cdot 50) + (6 \cdot 3)$.
C $(6 \cdot 50) + (6 \cdot 3) = 300 + 18$.
D $300 + 18 = 318$.

D6

Universe for a, b, and c = N.
$a + (bc) = (a + b)\,(a + c)$.
$3 + (4 \cdot 7) = (3 + 4)\,(3 + 7)$.

D7

E $3 + (4 \cdot 7) = 3 + 28$.
F $(3 + 4)(3 + 7) = 7 \cdot 10$.

D8

Universe for a, b, and c = N.
$a(b - c) = (ab) - (ac)$.
$5(6 - 2) = (5 \cdot 6) - (5 \cdot 2)$.

D9

B What replacements were made for the variables in $a + (bc) = (a + b)(a + c)$ to obtain the statement expressed in D7?

If the statement expressed in D7 is true, then the number you obtain when you add the product $4 \cdot 7$ to 3 should be equal to the number you obtain when you add each of the numbers, 4 and 7, to 3 and then multiply these sums.

C Look at D8. Does each sentence express a true statement?

D Look again at D7. Does the sentence $3 + (4 \cdot 7) = (3 + 4)(3 + 7)$ express a true statement? How do you know?

E Does addition distribute over multiplication of natural numbers?

Next you will decide if multiplication distributes over subtraction of natural numbers.

F Look at D9. What replacements were made for the variables in $a(b - c) = (ab) - (ac)$ to obtain the statement expressed?

G Does the sentence $5(6 - 2) = (5 \cdot 6) - (5 \cdot 2)$ express a true statement?

H Now use 4, 3, and 7 as replacements for a, b, and c in $a(b - c) = (ab) - (ac)$. In an earlier lesson you learned that $3 - 7$ is not a natural number. Is $4(3 - 7)$ a natural number? Is $(4 \cdot 3) - (4 \cdot 7)$ a natural number?

Using only natural numbers, you cannot decide whether or not $4(3 - 7) = (4 \cdot 3) - (4 \cdot 7)$ is a true statement.

I Write two sentences that express true statements obtained from $a(b - c) = (ab) - (ac)$. Use replacements different from the ones used to obtain the statement expressed in D9. The universe for a, b, and c is N.

J From exercises G and I, you know that true statements can be obtained from $a(b - c) = (ab) - (ac)$. If a natural number is obtained from $b - c$, how do you know that the replacement for b is greater than or equal to the replacement for c?

K If the replacement for b is greater than or equal to the replacement for c, is a true statement always obtained from $a(b - c) = (ab) - (ac)$? If the replacement for b is greater than or equal to the replacement for c, does multiplication distribute over subtraction?

Now you know what it means for an operation to distribute over another operation. You have learned that multiplication distributes over addition of natural numbers.

On your own

For each of exercises 1 through 14, decide whether the sentence expresses a true statement or a false statement. For each true statement, name the property that you used to make your decision.

1 $(2 \cdot 8) + (7 \cdot 8) = (2 + 7)8$.
2 $19 + 73 = 73 + 19$.
3 $(16 \cdot 5)3 = 16(5 \cdot 3)$.
4 $7 + (8 \cdot 2) = (7 + 8)(7 + 2)$.
5 $4(21 + 9) = (4 \cdot 21) + (4 \cdot 9)$.
6 $(16 + 11)3 = (16 \cdot 3) + (11 \cdot 3)$.
7 $(13 + 8) + 4 = 13 + (8 + 4)$.
8 $13 \cdot 21 = 21 \cdot 13$.
9 $14 - 3 = 3 - 14$.
10 $(18 + 2) + 7 = (2 + 18) + 7$.
11 $12(2 - 6) = (12 \cdot 2) - (12 \cdot 6)$.
12 $(15 - 2) - 1 = 15 - (2 - 1)$.
13 $6(3 \cdot 11) = 6(11 \cdot 3)$.
14 $(24 \div 12) \div 2 = 24 \div (12 \div 2)$.

For each condition expressed in exercises 15 through 27, tabulate the solution set. U = N.

15 $n(5 + 2) = (n \cdot 5) + (n \cdot 2)$.
16 $a(5 - 2) = (a \cdot 5) - (a \cdot 2)$.
17 $12x = (8x) + (4x)$.
18 $(2x) + (3x) = 5x$.
19 $a(14 + 33) = (2 \cdot 14) + (2 \cdot 33)$.
20 $8(49 - 25) = (8 \cdot 49) - (x \cdot 25)$.
21 $10 = x(3 + 2)$.
22 $a(5 - 2) = 12$.
23 $n(92 + 87) < (5 \cdot 92) + (5 \cdot 87)$.
24 $(9 \cdot 13) - (a \cdot 7) > 9(13 - 7)$.
25 $(24 + 7)5 = (5 \cdot 24) + (5 \cdot x)$.
26 $(16 \cdot 21) - (10 \cdot 21) = (16 \cdot 10)y$.
27 $n(14 + 4) = 54$.

A sum is expressed in each exercise below. Use the distributive property to find a product that is equal to the sum. For exercise 28, first find a common factor (other than 1) of 15 and 35. Since 5 is a common factor of 15 and 35, you could write the following sentences:

$15 + 35 = (5 \cdot 3) + (5 \cdot 7)$.
$(5 \cdot 3) + (5 \cdot 7) = 5(3 + 7)$.
Therefore, $15 + 35 = 5(10)$.

28 $15 + 35$
29 $35 + 14$
30 $45 + 18$
31 $35 + 42$
32 $46 + 32$
33 $154 + 24$
34 $22 + 121$
35 $75 + 110$
36 $51 + 34$

APPLYING MATHEMATICS

For each problem in this lesson, write a sentence that expresses a condition. Then tabulate the solution set of the condition and give the answer to the problem. The universe is given for each problem.

1 The regular price of a certain piano was $937. During a sale, the net price of this piano was $750. What was the amount of the discount? U = N.

2 On a semester test, Clarence had 7 times as many correct answers as incorrect answers. He had 91 correct answers. How many incorrect answers did he have on this test? U = C.

3 Ralph had 5 sports magazines. He gave some, but not all, of these magazines away. How many magazines could he have given away, and how many magazines could he have kept? U = C × C.

4 Eighty-four persons entered the Mid-City tennis tournament. 25% of these persons were under 20 years of age. How many persons under 20 entered this tournament? U = C.

5 Three bars of a certain soap cost 51¢. At this rate, how much will 5 bars of this soap cost? U = C.

6 Last week Mr. Carlson drove 235 mi. on Monday, 299 mi. on Wednesday, and 317 mi. on Friday. How many miles in all did he drive on these three days? U = N.

THINK You can use $235 + 299 + 317 = x$ as the condition for the problem. Why?

7 During a sale, Wilma received a 20% discount on a woolen skirt. The discount amounted to $4. What was the regular price of this skirt? U = C.

8 At the beginning of the football season, Bob weighed more than 184 lb. By the end of the season, he had lost 14 lb. and weighed less than 176 lb. How much could Bob have weighed at the beginning of the season? U = N.

9 Miss Gunderson's take-home pay is 82% of her salary. Her yearly salary is $6500. Her yearly take-home pay amounts to how many dollars? U = C.

10 Ruth has more blouses than sweaters. If she had 3 more blouses, she would have 8 blouses. How many blouses does she have? How many sweaters can she have? U = N × N.

11 Jack delivered 7 handbills for each 4 handbills that Ross delivered. If Jack had delivered 75 fewer, he would have delivered 86 handbills. How many handbills did each of the boys deliver? U = C × C.

12 For a school carnival, Ellen made 6 more posters than Marie. Together, they made 20 posters. How many posters did each of the girls make? U = N × N.

13 600 students are enrolled at a certain university. 47% of the students are women. How many men are enrolled at this university? U = C × C.

14 In 1959, 5,017,686 motor vehicles were registered in Canada. Of these vehicles, 1,040,366 were registered in Quebec, and 27,502 were registered in Prince Edward Island. How many vehicles in all were registered in provinces other than Quebec and Prince Edward Island? U = N × N.

15 What numbers greater than 15 can be subtracted from 92 so that each difference is greater than 65? U = N.

16 Certain rate pairs are equivalent to 16/8. The first component of each of these rate pairs is less than 11. What are the rate pairs? U = C × C.

17 The sum of two numbers is less than 6. The second number is greater than 2. What can the numbers be? U = N × N.

CHECKING UP

If you have trouble with these two tests, you can find help in lessons 60 through 64.

Test 67

Which of the following sentences express true ideas? Which express false ideas?

1 The set of odd numbers greater than 100 is closed under multiplication.

2 (5, 4) and (10, 2) can be mapped onto the same natural number by multiplication.

3 Addition distributes over multiplication of natural numbers.

4 For any two finite sets A and B, $n(A \times B)$ is not the same number as $n(B \times A)$.

5 For any two disjoint sets V and W, $n(V) + n(W) = n(V \cup W)$.

Test 68

Some of the properties of addition and multiplication of natural numbers are expressed below. For each exercise, give the name of the property. The universe for each variable is N.

6 For each replacement of a, b, and c, a true statement is obtained from $a(b + c) = (ab) + (ac)$.

7 For each replacement of a and b, a true statement is obtained from $ab = ba$.

8 For each replacement of a, b, and c, a true statement is obtained from $(a + b) + c = a + (b + c)$.

9 For each replacement of a, b, and c, a true statement is obtained from $(ab)c = a(bc)$.

Use the properties that you named for exercises 6 through 9 to decide whether each sentence below expresses a true or a false statement. If a statement is true, name the property or properties upon which your answer depends.

10 $(3 \cdot 5) + (3 \cdot 8) = (5 + 8)3$.

11 $(4 + 10) + 2 = 2 + (10 + 4)$.

12 $(31 + 4)9 = 31(9 \cdot 4)$.

13 $5(31 + 7) = (5 \cdot 7) + (5 \cdot 31)$.

End-of-block tests on natural numbers and on operations of the set of natural numbers

Simplifying expressions

In this lesson you will study different ways to name numbers. You will also learn what is meant by an expression and by the standard name of a number.

A Look at D1. Numeral A tells you that the number $9 + 7$ was obtained by adding 9 and 7. What does each of the other numerals tell you?

B You know that the numeral 16 names the same number as the numeral $9 + 7$. Does the numeral 16 contain a symbol for an operation?

The numeral 16 is a *standard name*. Any numeral that is in the list of symbols 0, 1, 2, . . . is a standard name of a natural number. The standard names in common use are those that are developed according to the properties of the decimal numeration system that you have studied.

Each numeral in D1 contains a symbol for an operation. A numeral that contains one or more symbols for operations is an *expression*. Each numeral in D1 is an expression. Notice that numerals that are expressions contain both standard names and symbols for operations.

C What two standard names are contained in the expression $8 \cdot 2$? What symbol for an operation is contained in the expression $8 \cdot 2$?

A $9 + 7$ **C** $19 - 3$

B $8 \cdot 2$ **D** $32 \div 2$

D1

Expressions; standard names; simplifying expressions; order of operations

D What standard names are contained in the expression $19 - 3$? What operation symbol is contained in the expression?

E Give the standard name of the number named by each expression in D1. Do the four expressions name the same number?

The sentence $9 + 7 = 16$ tells you that the same number is named by the expression $9 + 7$ and by the standard name 16. Write a similar sentence involving each of the following expressions:

F $15 - 9$ **G** $42 \div 7$ **H** $23 + 8$ **I** $6 \cdot 50$

In your earlier work in arithmetic, you used computation to find standard names of numbers. You should understand the difference between an operation and computation. The *operation* of multiplication maps (38, 75) onto the product 38×75. To find the standard name of this product, you use *computation*. Computation includes the familiar addition, subtraction, multiplication, and division processes. Using the multiplication process, you *compute* to obtain the numeral 2850 as the standard name of the number 38×75.

J Is the numeral $(4 \cdot 5) + (4 \cdot 3)$ an expression? How do you know? What operation symbols does it contain?

K Give the standard name of $(4 \cdot 5) + (4 \cdot 3)$.

L Is $(4 \cdot 5) + (4 \cdot 3) = 32$ a true statement?

When you compute to find the standard name of a number represented by an expression, you *simplify* the expression.

M What operation symbols does the expression $(6 \cdot 7) - (6 \cdot 2)$ contain? Simplify the expression.

N Now look at D2. What operation symbols does expression E contain?

Since expression E does not contain parentheses, the expression can be interpreted in two ways. Expressions F and G indicate these two interpretations.

A $9 + 7$ **C** $19 - 3$
B $8 \cdot 2$ **D** $32 \div 2$

D1

E $4 + 3 \cdot 6$
F $(4 + 3)6$
G $4 + (3 \cdot 6)$

D2

H $9 + 24 \div 4$ **K** $3 \cdot 2 + 3 \cdot 3$
I $4 \cdot 2 + 7$ **L** $42 \div 2 - 18 \div 3$
J $25 - 5 \cdot 2$ **M** $3 \cdot 4 + 12 \div 4$

D3

O The parentheses in expression F tell you that 6 has been multiplied by the sum $4 + 3$. Simplify expression F.

P The parentheses in expression G tell you that the product $3 \cdot 6$ has been added to 4. Simplify expression G.

Q Is the standard name that you obtained for expression F the same as the standard name that you obtained for expression G?

The product of $(4 + 3)$ and 6 is different from the sum of 4 and $(3 \cdot 6)$. To decide which of these numbers is expressed by the numeral $4 + 3 \cdot 6$, you need to know how to interpret the expression. In expressions like $4 + 3 \cdot 6$ or $4 + 10 \div 2$ or $4 \times 8 - 2$, there is a special procedure to follow. You always perform the operations of multiplication or division before you perform the operations of addition or subtraction. In an expression like $7 - 2 + 3$ that involves only addition and subtraction, you perform the operations in the order in which they occur. Thus, $7 - 2 + 3 = 8$. In an expression like $10 \div 2 \times 5$ that involves only multiplication and division, you also perform the

operations in the order in which they occur. Thus, $10 \div 2 \times 5 = 25$. Of course, if the expression includes parentheses, the parentheses will indicate the order of operations.

R Now you know how you should simplify expression E in D2. What is the standard name of $3 \cdot 6$?

S What is the standard name of $4 + 18$?

T Is $4 + 3 \cdot 6 = 22$ a true statement? How do you know?

U Look at expression H in D3. How do you know that this expression names the same number as the expression $9 + (24 \div 4)$? Simplify expression H.

V Simplify each of the other expressions in D3.

W Are all the expressions in D3 names of the same number?

Now you know what is meant by expressions and by standard names of numbers. You also know how to simplify an expression to find the standard name of the number.

On your own

For each number named in exercises 1 through 4, write five expressions that name the given number.

1 6 **2** 18 **3** 25 **4** 27

For each of exercises 5 through 16, simplify the expression.

5 $9 + 6 \cdot 2$

6 $8 + 7 - 13$

7 $18 \div 6 + 7$

8 $16 - 5 \cdot 3$

9 $13 - 24 \div 8$

10 $24 \div 2 \cdot 3$

11 $14 - 3 + 6$

12 $8 \cdot 9 + 8 \cdot 3$

13 $(2 + 3)(2 + 7)$

14 $9 \cdot 7 \div 3$

15 $7 + 8 \cdot 3 + 6$

16 $16 + (6 + 2)5$

For each of exercises 17 through 26, tell whether the sentence expresses a true statement or a false statement.

17 $10 + 3 - 11 = 2$.

18 $6(7) + 6(2) = 6(9)$.

19 $14 \div 7 \cdot 8 = 10$.

20 $4(8 + 2) = 34$.

21 $18 + 7 \cdot 2 = (25)2$.

22 $4(17 - 3) = 56$.

23 $48 \div 6 - 3 = 2 \cdot 8$.

24 $(5 + 8)(2 + 3) = 5(7 + 6)$.

25 $(14 + 9)4 + 2 = (23)6$.

26 $72 \div 8 + 4 = 13$.

Copy the sentences in exercises 27 through 35. Then place parentheses in each sentence to make it express a true statement. For exercise 27 you should write: $(5 + 4) \cdot (3 + 1) = 36$.

27 $5 + 4 \cdot 3 + 1 = 36$.

28 $13 - 8 - 3 = 8$.

29 $60 = 12 \cdot 7 - 2$.

30 $36 \div 3 \cdot 3 + 1 = 5$.

31 $3 \cdot 6 + 4 - 2 = 28$.

32 $13 = 14 \div 2 - 1 - 1$.

33 $6 + 3 \cdot 2 + 4 = 54$.

34 $8 = 18 + 2 \div 5 + 4$.

35 $42 \div 6 + 1 = 6$.

KEEPING SKILFUL

For each of exercises 1 through 8, write a sentence that expresses the standard set of the natural number named.

1 3 **3** 0 **5** 5 **7** 8

2 11 **4** 7 **6** 1 **8** 14

For each condition expressed in exercises 9 through 14, tabulate the solution set. Then make a graph of the solution set. U = N.

9 $x + 5 < 13$.

10 $101 - n = 96$.

11 $17 - x > 8$.

12 $162 \div 27 > s$.

13 $38x = 266$.

14 $m + 15 \neq 15$.

For each ordered pair named in exercises 15 through 23, tell what natural number can be added to the second component to obtain the first component.

15 (70, 50) **18** (27, 8) **21** (220, 136)

16 (40, 15) **19** (52, 19) **22** (372, 296)

17 (58, 20) **20** (117, 49) **23** (87, 87)

6 66 Exploring ideas

The natural-number system

In this lesson you will learn what we mean when we speak of the natural-number system. In this system we include the set of natural numbers, the operations of addition and multiplication, and some of the properties of these operations.

A Describe the set of numbers that is tabulated in D1.

B What two operations are named in D1? Is the set of natural numbers closed under both of these operations?

C You have learned that a property is a condition from which a true statement is obtained for each replacement of the variables. In D1, the sentence $a + b = b + a$ expresses the commutative property of addition. Which sentence expresses the commutative property of multiplication?

D To which operation does the property $(a + b) + c = a + (b + c)$ apply? To which operation does the property $(ab)c = a(bc)$ apply?

E Which property expressed in D1 relates multiplication and addition?

The set of natural numbers, together with the operations of addition and multiplication and some of their properties, forms the *natural-number system*. The properties of the operations are the closure properties, the commutative properties, the associative properties, and the distributive property.

Notice from the definition of the natural-number system that the system includes a set of objects, two operations, and certain properties of these operations. As you continue your study of mathematics, you will become familiar with other number systems. We will call a set of objects a number system if the set behaves like the natural-number system with respect to two operations and their properties. The objects in such a set will be called numbers. Much of your future work in mathematics will deal with other number systems.

Now you will learn how to decide if a set and two operations have the necessary properties to form a number system. For sets whose members are natural numbers, you will use the operations of addition and multiplication.

F Look at D2. Is each member of set E a natural number? Is the number 2 a factor of each member of E? Describe set E.

$N = \{0, 1, 2, \ldots\}$.

Operations →	Addition	Multiplication
Closure properties →	$a + b$ is a member of N.	ab is a member of N.
Commutative properties →	$a + b = b + a$.	$ab = ba$.
Associative properties →	$(a + b) + c = a + (b + c)$.	$(ab)c = a(bc)$.
Distributive property →	$a(b + c) = ab + ac$.	

D1

$E = \{0, 2, 4, \ldots\}$.

D2

Universe for a, b, and c = E.

$a + b = b + a$.

A $2 + 8 = 8 + 2$.

B $22 + 34 = 34 + 22$.

$ab = ba$.

C $2 \cdot 8 = 8 \cdot 2$.

D $22 \cdot 34 = 34 \cdot 22$.

$(a + b) + c = a + (b + c)$.

E $(6 + 12) + 30 = 6 + (12 + 30)$.

F $(14 + 2) + 28 = 14 + (2 + 28)$.

$(ab)c = a(bc)$.

G $(18 \times 4) \times 10 = 18 \times (4 \times 10)$.

H $(32 \times 20) \times 8 = 32 \times (20 \times 8)$.

$a(b + c) = ab + ac$.

I $4(12 + 8) = 4(12) + 4(8)$.

J $2(6 + 10) = 2(6) + 2(10)$.

D3

natural-number system (nach′ə rəl). The set of natural numbers together with the operations of addition and multiplication and the following properties: The set is closed under the operations of addition and multiplication; both operations are commutative and associative; multiplication distributes over addition.

First you will decide if E is closed under addition and multiplication. To do this, you must show that, for each replacement of a and b from E, $a + b$ and ab are even numbers.

G Use 6 as a replacement for a and 14 as a replacement for b in both $a + b$ and ab. Are 6 and 14 even numbers? How do you know?

H Is $6 + 14$ an even number? Is 6×14 an even number?

I Select another replacement from E for a and for b. Use these replacements in $a + b$. Is the sum an even number?

J Use the replacements that you selected for a and b in ab. Is the product an even number?

K Do you think that E is closed under addition and multiplication? Explain your answer.

Even though, for many different replacements, $a + b$ and ab are members of E, this does not prove that E is closed under addition and multiplication. In an earlier lesson you proved that for each replacement of a, b, and c, $(b + c)a = ba + ca$. Now you will use exercises L and M to prove that set E is closed under addition and multiplication.

L You know that 2 is a factor of every even number. Let $2m$ and $2n$ represent even numbers. Use the distributive property to show that 2 is also a factor of the sum of each pair of even numbers. The universe for m and n is N.

M Show that the product of any two even numbers is also an even number. Use $2m$ and $2n$. The universe for m and n is N.

Next you will see if addition and multiplication are commutative and associative in set E. You will also see if multiplication distributes over addition.

N Look at D3. Think about the statements expressed by sentences A through D. Is each statement true?

O Do you think that addition and multiplication are commutative in set E?

P Now think about the statements expressed by sentences E through H. Is each statement true?

Q Do you think that addition and multiplication are associative in set E?

R Think about the statements expressed by sentences I and J. Are the statements true?

S Do you think that multiplication distributes over addition in the set of even numbers?

On the basis of the examples we have used, we will accept the following ideas as true: The set of even numbers is closed under addition and multiplication; addition and multiplication of even numbers are commutative and associative; and multiplication distributes over addition in the set of even numbers.

T Is the set of even numbers a number system with respect to addition and multiplication? Explain your answer.

Now you know that the natural-number system includes a set, two operations, and some properties of these operations. You also know how to decide whether or not a given set forms a number system with respect to two given operations.

On your own

For each of exercises 1 through 16, tell whether the sentence expresses a true statement or a false statement. If a statement is true, name the property that you used to make your decision.

1 $4 + 6 = 6 + 4$.
2 $(7 \cdot 9)11 = 7(9 \cdot 11)$.
3 $4(3 + 1) = 4(3) + 4(1)$.
4 $14 + 37$ is a natural number.
5 $16 \cdot 8 = 8 \cdot 16$.
6 $(5 + 12) + 1 = 5 + (12 + 1)$.
7 $7 + (6 \cdot 2) = (7 + 6)(7 + 2)$.
8 $0 + 13$ is a natural number.
9 $196 + 1 = 196$.
10 $17 + 35 = 35 + 17$.
11 $(18 + 26) + 2 = 18 + (26 + 2)$.
12 $7 \cdot 12$ is a natural number.
13 $0 \cdot 16 = 16 \cdot 0$.
14 $(8 \cdot 11) + 4 = (8 \cdot 11) + (8 \cdot 4)$.
15 $(14 \cdot 13)1 = 14(13 \cdot 1)$.
16 $(27 + 0) + 3 = 27 + (0 + 3)$.

Tabulate the solution set of each condition expressed in exercises 17 through 24. U = N.

17 $7(4) + 8(4) = a(15)$.
18 $2(5c) < (2 \cdot 5)c$.
19 $14x = 11x + 3x$.
20 $(19 + 14) + x = 19 + (x + 14)$.
21 $y + 15 = 9(6 - 4)$.
22 $5 \cdot 7 - 4 > m(3 + 5)$.
23 $3k = 6 \cdot 2 + 4 \cdot 3$.
24 $2 \cdot 8 - 3 \cdot 5 > x$.

For each set tabulated in exercises 25, 26, and 27, tell whether or not the set, together with addition and multiplication, forms a number system. If the set is not a number system, tell what property is not satisfied.

25 {1, 3, 5, . . .}
26 {4, 8, 12, . . .}
27 {3, 5, 6, 9, 10, 12, 15, 18, 20, . . .}

KEEPING SKILFUL

Find the sum.

1 4184, 9845, 4746, 8505
2 2326, 34, 677, 5830, 74, 864, 10292
3 9106, 4345, 19325, 8154
4 46352, 33049, 69248, 93460
5 45305, 75021, 61318, 31855, 7601
6 144137, 905111, 879629, 92684

Find the difference.

7 4140, 1331
8 7052, 6974
9 5033, 4513
10 39663, 17544
11 52701, 38375
12 63302, 8715
13 165206, 96761
14 499893, 218168

Multiply

15 81 by 16.
16 157 by 27.
17 536 by 89.
18 681 by 703.
19 928 by 260.
20 606 by 880.
21 358 by 423.
22 813 by 448.

Divide

23 1736 by 31.
24 6570 by 85.
25 6818 by 24.
26 24174 by 79.
27 7737 by 375.
28 32548 by 412.
29 36931 by 102.
30 71232 by 224.

A finite number system

In previous lessons you worked with the set of natural numbers and the natural-number system. As you know, the set of natural numbers is an infinite set. In this lesson you will work with a finite set of natural numbers and decide whether or not a given finite set can form a number system with respect to two operations.

A Sentence A in D1 tells you that the quotient of 13 and 5 is 2, remainder 3. Does sentence A express a true statement?

B Does each of the other sentences in D1 express a true statement?

C Look at D2. Is the set tabulated in D2 the set whose members are the remainders expressed in D1?

D Choose any three natural numbers. Divide each of them by 5. For each number you chose, tell what quotient and what remainder you obtained.

E Is each of the remainders that you obtained for exercise D a member of set T?

F When you divide a natural number by 5, the remainder can be one of what numbers?

G Does set T contain all the numbers you named for exercise F? Is T a finite set?

Before you can decide whether or not set T, together with two operations, forms a number system, you must know what the two operations are. Then you must decide whether or not these operations have the properties you have studied.

A $13 \div 5 = 2$, remainder 3.
B $146 \div 5 = 29$, remainder 1.
C $89 \div 5 = 17$, remainder 4.
D $2 \div 5 = 0$, remainder 2.
E $65 \div 5 = 13$, remainder 0.

D1

$T = \{0, 1, 2, 3, 4\}$.

D2

$T = \{0, 1, 2, 3, 4\}$.

A $4 \oplus 3 = 2$.
B $3 \oplus 2 = 0$.
C $2 \oplus 4 = 1$.
D $4 \oplus 4 = 3$.

D3

The first operation you are going to use involves both addition and division. We will call this invented operation *addivision*. We will call the result of using this operation on two numbers the *addiv* of the two numbers.

A Look at sentence A in D3. Notice the symbol between the numerals 4 and 3. Read this symbol "addivided by." To find the "addiv" of 4 and 3, using the operation of addivision, find the sum of 4 and 3 and divide this sum by 5. The remainder is the addiv of 4 and 3. Will the number you obtain be a member of T?

B Is the number 7 the sum of 4 and 3? When you divide 7 by 5, is there a remainder of 2? Does sentence A express a true statement?

C Look at sentence B. When you divide the sum of 3 and 2 by 5, is there a remainder of 0? What is the addiv of 3 and 2? Does sentence B express a true statement?

D How do you know that sentences C and D in D3 express true statements?

E D4 contains a table that expresses the addiv of any two members of T. To find a name for the addiv of 4 and 1, first find the red numeral 4 in the top row. Then look down the column below the red numeral 4 until you find the numeral opposite the red numeral 1. What member of T is named here?

F Name the other pairs of members of T whose addiv is 0.

G Is the addiv of any two members of T also a member of T? Is set T closed under addivision?

The second operation you are going to use involves both multiplication and division. We will call this invented operation *muldivision*. We will call the result of using this operation on two numbers the *muldiv* of the two numbers.

H Look at sentence E in D5. Notice the symbol between the numerals 2 and 3. Read this symbol "muldivided by." To find the "muldiv" of 2 and 3, first find the product of 2 and 3 in the usual way. Then divide the product of 2 and 3 by 5. The remainder is the muldiv of 2 and 3. When you do this, will the number you obtain be a member of T?

I When you divide 6 by 5, is there a remainder of 1? Does sentence E express a true statement?

J Look at sentence F in D5. When you divide the product of 4 and 2 by 5, is there a remainder of 3? Does sentence F express a true statement?

K How do you know that sentence G in D5 expresses a true statement?

L The table in D6 is an incomplete muldivision table. When completed, this table will express the muldiv of any two members of T. To find a name for the muldiv of 1 and 3, first find the red numeral 1 in the top row. Then look down the column below the red numeral 1 until you find the numeral opposite the red numeral 3. What member of T is named here?

M Copy and complete the table shown in D6.

N Is the muldiv of any two members of T also a member of T? Is set T closed under muldivision?

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

D4

$T = \{0, 1, 2, 3, 4\}$.

E $2 \otimes 3 = 1$.

F $4 \otimes 2 = 3$.

G $3 \otimes 4 = 2$.

D5

$\otimes$	0	1	2	3	4
0	0				
1	0	1			
2	0	2	4		
3	0	3	1	4	
4	0	4	3	2	1

D6

You have a set T that is closed under two operations. Now you will decide if addivision and muldivision are commutative and associative in T and if muldivision distributes over addivision.

A Look at sentence H in D7. Use the table in D4 to answer the following questions: What member of T is the addiv of 4 and 1? What member of T is the addiv of 1 and 4?

B Does sentence H express a true statement? How do you know?

C Look at sentence I in D7. Use the muldivision table you completed to answer the following questions: What member of T is the muldiv of 3 and 4? What member of T is the muldiv of 4 and 3?

D Does sentence I express a true statement? How do you know?

E When you replace the variables in $a \oplus b = b \oplus a$ by members of T, is it possible to obtain a false statement? When you replace the variables in $a \otimes b = b \otimes a$ by members of T, is it possible to obtain a false statement?

F Do you think that addivision is commutative in set T? Do you think that muldivision is commutative in set T? Explain your answers.

G Now look at sentence J in D7. To simplify the expression $(4 \oplus 2) \oplus 3$, first find the addiv of 4 and 2. Is 1 the addiv of 4 and 2? How do you know? Now find the addiv of 1 and 3. What number do you obtain?

H Next simplify the expression $4 \oplus (2 \oplus 3)$. Find the addiv of 2 and 3. What number do you obtain? Now find the addiv of 4 and 0. What number do you obtain?

I Does sentence J express a true statement?

J Look at sentence K in D7. To simplify the expression $(2 \otimes 4) \otimes 2$, first find the muldiv of 2 and 4. Is 3 the muldiv of 2 and 4? How do you know? Now find the muldiv of 3 and 2. What number do you obtain? Next simplify $2 \otimes (4 \otimes 2)$.

K Does sentence K express a true statement?

L When you use the members of T as replacements for the variables in $(a \oplus b) \oplus c = a \oplus (b \oplus c)$, is it possible to obtain a false statement? When you replace the variables in $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ by members of T, is it possible to obtain a false statement?

D7

Universe for a, b, and c = T.

$T = \{0, 1, 2, 3, 4\}$.

$a \oplus b = b \oplus a$.

H $4 \oplus 1 = 1 \oplus 4$.

$a \otimes b = b \otimes a$.

I $3 \otimes 4 = 4 \otimes 3$.

$(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

J $(4 \oplus 2) \oplus 3 = 4 \oplus (2 \oplus 3)$.

$(a \otimes b) \otimes c = a \otimes (b \otimes c)$.

K $(2 \otimes 4) \otimes 2 = 2 \otimes (4 \otimes 2)$.

$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$.

L $2 \otimes (4 \oplus 3) = (2 \otimes 4) \oplus (2 \otimes 3)$.

M Do you think that addivision is associative in set T? Do you think that muldivision is associative in set T? Explain your answers.

N Now look at sentence L in D7. Simplify $2 \otimes (4 \oplus 3)$. Simplify $(2 \otimes 4) \oplus (2 \otimes 3)$. Does sentence L express a true statement?

O When you use the members of T as replacements for the variables in $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$, is it possible to obtain a false statement?

P Do you think that muldivision distributes over addivision in T? Explain your answer.

Now you know that addivision and muldivision are closed in set T. You also know that you cannot obtain any false statements from the conditions expressed in D7. Thus, addivision and muldivision are both commutative and associative in T, and muldivision distributes over addivision.

Q Does set T, together with the operations of addivision and muldivision, form a number system? Explain your answer.

In this lesson and in lesson 66 you have worked with different examples of number sys-

$F = \{0, 1, 2\}$.

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\otimes$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

D8

tems. For each example you noticed that you had a set, two operations, and some properties of the operations. These different systems have a common *structure*. You will learn more about the structures of mathematical systems as you continue your study of mathematics.

Now you know how to decide whether or not a finite set, together with two operations, forms a number system.

On your own

D8 shows an addivision and a muldivision table for a set of numbers. Use D8 as you answer the questions in exercises 1 through 9. For these exercises, addivision and muldivision involve division by 3.

1 Look at the tabulation of set F in D8. When you divide a natural number by 3, the remainder can be one of what three numbers? Does set F contain all the numbers you named?

2 Is set F closed under addivision? Explain your answer.

3 Is set F closed under muldivision? Explain your answer.

4 Is addivision commutative in set F? Give examples to explain your answer.

5 Is muldivision commutative in set F? Give examples to explain your answer.

6 Is addivision associative in set F? Give examples to explain your answer.

7 Is muldivision associative in set F? Give examples to explain your answer.

8 Does muldivision distribute over addivision in set F? Give examples to explain your answer.

9 Does set F, together with the operations of addivision and muldivision, form a number system? Explain your answer.

For each of exercises 10 through 19, set $G = \{0, 1, 2, 3, 4, 5, 6\}$. Notice that the members of G are all the numbers that the remainder can be when you divide natural numbers by 7.

10 To find the addiv of any two members of G, first find the sum of the two members. Then divide this sum by 7. The remainder is the addiv of the two members. Make an addivision table that expresses the addiv of each pair of members of G.

11 To find the muldiv of any two members of G, first find the product of the two members. Then divide this product by 7. The remainder is the muldiv of the two members. Make a muldivision table that expresses the muldiv of each pair of members of G.

12 Is set G closed under addivision? How do you know?

13 Is set G closed under muldivision? How do you know?

14 Is addivision commutative in set G? Give examples to explain your answer.

15 Is muldivision commutative in set G? Give examples to explain your answer.

16 Is addivision associative in set G? Give examples to explain your answer.

17 Is muldivision associative in set G? Give examples to explain your answer.

18 Does muldivision distribute over addivision in set G? Give examples to explain your answer.

19 Does set G, together with the operations of addivision and muldivision, form a number system? Explain your answer.

Zero and one

You have studied some of the properties of addition and multiplication of natural numbers. In this lesson you will study some properties of addition and multiplication that concern the numbers zero and one.

A Look at the tabulations in D1. What natural number is associated with set E? With set F? With set G? Remember that the number associated with a set is the number of members in the set. Tabulate E ∪ F.

B Is the sum of 0 and 4 associated with the set you tabulated for exercise A?

> REMINDER
> The sum of two numbers is the number associated with the union of two disjoint sets.
> *See lesson 60, page 257.*

C Is set F the same as the set E ∪ F? Is E ∪ F = F? Is 0 + 4 = 4?

D Tabulate E ∪ G. Is the sum of 0 and 5 associated with the set you just tabulated?

E Is set G the same as the set E ∪ G? Is E ∪ G = G? Is 0 + 5 = 5?

F Suppose that A is any set associated with a natural number. Is { } ∪ A = A?

G You know that 0 = n{ }. Suppose also that a = n(A). Remember that the symbol a = n(A) is read "a equals the number of members in set A." Is $0 + a$ associated with the union of the empty set and set A? Are $0 + a$ and a associated with the same set? Are $0 + a$ and a the same number? The universe for a is N.

$E = \{\ \}$.
$F = \{5, 6, 7, 8\}$.
$G = \{30, 40, 50, 60, 70\}$.

D1

$U = N$.
$0 + a = a$.
A $0 + 8 = 8$.
B $0 + 99 = 99$.
C $0 + 0 = 0$.

D2

You have learned that the sum of 0 and any natural number is the same as, or identical with, the natural number. We say that the number 0 is the *identity element for addition of natural numbers*. This property is the *identity-element property of addition of natural numbers*. The property is expressed below.

The universe for a *is N. For each replacement of* a, *a true statement is obtained from* 0 + a = a.

H For each statement expressed in D2, tell what replacement was made for the variable in $0 + a = a$. Does each sentence express a true statement?

I Is $a + 0 = a$ also a property? Explain your answer. Remember that if a condition is a property, you can obtain a true statement for each replacement of the variable by a member of the universe. The universe for a is N.

J You know that subtraction is related to addition. The condition $11 - 7 = x$ has the same solution set as $11 = 7 + x$. What is the solution set of $11 - 7 = x$? What is the solution set of $11 = 7 + x$? The universe for x is N.

K Look at D3. Since subtraction is related to addition, think of what each condition means in terms of addition. The condition expressed by sentence D also means $a = a + 0$. In other words, you must find what natural numbers can replace a so that $a = a + 0$. Tabulate the solution set of $a - a = 0$.

L Is the condition expressed by sentence D a ? property? Explain your answer.

M Now look at sentence E in D3. The condi- ? tion $a - 0 = a$ also means $a = 0 + a$. Is $a - 0 = a$ a property? Explain your answer.

N Look at sentence F. The condition $0 - a = a$ also means $0 = a + a$. For each replacement of a, do you obtain a true statement from $0 = a + a$? Is $0 - a = a$ a property?

Next you will decide if there is a number that is an identity element for multiplication of natural numbers.

A Look at D4. What natural number is associated with set M? With set K? With set L?

B Tabulate $M \times K$. Is the product $1 \cdot 3$ associated with $M \times K$?

REMINDER

The product of two numbers is the number associated with the Cartesian set of two sets.

See lesson 61, page 262.

C Look at D5. Is $M \times K$ equivalent to K? Explain your answer.

REMINDER

Two sets that can be put in one-to-one correspondence are equivalent sets.

See lesson 57, page 247.

U = N.

D $a - a = 0$.

E $a - 0 = a$.

F $0 - a = a$.

D3

$M = \{6\}$.

$K = \{7, 14, 21\}$.

$L = \{1, 3, 5, 7, 9, 11\}$.

D4

$K = \{7, 14, 21\}$.

$M \times K = \{(6, 7), (6, 14), (6, 21)\}$.

D5

D Is $1 \cdot 3 = 3$?

E Look again at D4. Tabulate $M \times L$. Is $1 \cdot 6$ associated with this set?

F Is $M \times L$ equivalent to L? Is $1 \cdot 6 = 6$?

G Suppose that $1 = n(B)$ and that A is any set associated with a natural number. Is $B \times A$ equivalent to set A?

H Suppose also that $a = n(A)$. With what Cartesian set is $1 \cdot a$ associated? Are $1 \cdot a$ and a associated with equivalent sets? Are $1 \cdot a$ and a the same number? The universe for a is N.

You have learned that the product of 1 and any natural number is identical with the natural number. The set of natural numbers also contains an *identity element for multiplication of natural numbers*. The identity element is the number 1. This property is the *identity-element property of multiplication of natural numbers*. The property is expressed below.

The universe for a *is N. For each replacement of the variable, a true statement is obtained from* $1 \cdot a = a$.

I For each statement expressed in D6, tell what replacement was made for a in $1 \cdot a = a$. Does each sentence express a true statement?

J Is the condition $a \cdot 1 = a$ also a property? Explain your answer.

K You know that division is related to multiplication. The condition $12 \div 4 = x$ has the same solution set as $12 = x(4)$. What is the solution set of $12 \div 4 = x$? Of $12 = x(4)$? The universe for x is N.

Various symbols may be used to express division. You know that the symbol $12 \div 4$ expresses "12 divided by 4." The symbol at the right shows another way to express "12 divided by 4." Since symbols like 12/4 are often ratios that express rate pairs, you must decide from the situation whether the symbol 12/4 expresses "12 divided by 4" or the rate pair 12 to 4.

12/4

Look at D7. The symbol a/a has exactly the same meaning as the symbol $a \div a$. Since division is related to multiplication, think of what each condition means in terms of multiplication. Notice that the universe for a in all these conditions does not contain zero. Later in this lesson you will learn why zero is not included.

L The condition expressed by sentence J in D7 also means that $a = 1 \cdot a$. Is the condition $a/a = 1$ a property? Explain your answer.

M Now look at sentence K in D7. The condition $a/1 = a$ also means that $a = a \cdot 1$. How do you know that $a/1 = a$ is a property?

N Look at sentence L. The condition $1/a = a$ also means that $1 = a \cdot a$. Do you obtain a true statement from $1 = a \cdot a$ for each replacement of a? Is $1/a = a$ a property?

Now you will study the rôle of the number zero in multiplication and division of natural numbers.

U = N.

$1 \cdot a = a$.

G $1 \cdot 16 = 16$.

H $1 \cdot 43 = 43$.

I $1 \cdot 1 = 1$.

D6

U = C.

J $a/a = 1$.

K $a/1 = a$.

L $1/a = a$.

D7

Q = { }.

R = {6, 12}.

D8

A Look at D8. What natural number is associated with set Q? With set R?

B You know that to find Q × R you pair each member of set Q with each member of set R. Since there are no members in Q, there are no ordered pairs in Q × R. Is Q × R the empty set? Is $0 \cdot 2$ associated with Q × R?

C Are $0 \cdot 2$ and 0 associated with the same set? Is $0 \cdot 2 = 0$?

D Suppose that A is any set associated with a natural number. Is { } × A = { }?

E Suppose also that $a = n(A)$. You know that $0 = n\{\ \}$. With what Cartesian set is $0 \cdot a$ associated? Is $0 \cdot a$ the number of members in the empty set? The universe for a is N.

You know that the product of zero and any natural number is the number 0. This property is the *zero property of multiplication of natural numbers*. The property is expressed below.

The universe for a *is N. For each replacement of* a, *a true statement is obtained from* $0 \cdot a = 0$.

F Is $0 \cdot 17 = 0$? Is $0 \cdot 98 = 0$? Is $0 \cdot 0 = 0$?

G How do you know that $a \cdot 0 = 0$ is a property?

The properties $0 \cdot a = 0$ and $a \cdot 0 = 0$ involve a very important mathematical idea. Notice that, if at least one of two factors is zero, then the product of the two factors is zero.

Now you will decide if a true statement is obtained from $0/a = 0$ for each replacement of a by a member of $\{1, 2, 3, \ldots\}$. Remember that $0/a = 0$ also means that $0 = 0 \cdot a$.

H Do you obtain a true statement from $0/a = 0$ for each replacement of a from $\{1, 2, 3, \ldots\}$? Explain your answer.

Now you will think about $a/0$ when a is replaced by a member of $\{1, 2, 3, \ldots\}$. That is, you will decide whether or not symbols like $1/0$, $2/0$, $3/0$, and so on have any meaning.

I Suppose that $a/0$ is equal to a natural number b. You know that $a/0 = b$ also means $a = b \cdot 0$. Is there any number that, when multiplied by 0, gives a number other than 0?

J Suppose that a in $a = b \cdot 0$ is replaced by any counting number. Is there a natural number b such that a true statement will be obtained from $a = b \cdot 0$?

There are no replacements for a and b that satisfy $a/0 = b$. Therefore, such symbols as $1/0$, $2/0$, $3/0$, and so on are meaningless.

K Now suppose that you replace a in $a/0$ by 0. You obtain $0/0$. Suppose that $0/0$ is equal to a natural number b. You know that $0/0 = b$ also means $0 = b \cdot 0$. Is $0 = 4 \cdot 0$? Is $0 = 56 \cdot 0$?

L If you replace b in $0 = b \cdot 0$ by any natural number, do you obtain a true statement? Does any natural number satisfy $0/0 = b$?

Since any replacement of b satisfies $0/0 = b$, $0/0$ can be any number. We say that $0/0$ is *indeterminate*.

To emphasize that the symbols $1/0$, $2/0$, $3/0$, and so on are meaningless, and also that $0/0$ is indeterminate, we say that *division by zero is impossible*.

In this lesson you have studied some properties of addition and multiplication that involve zero and one.

On your own

For each of exercises 1 through 8, simplify the expression.

1 $16 - 0$
2 $3(42 + 0)$
3 $4 \cdot 1 + 8 \cdot 1$
4 $0 \div 9 + 3$
5 $(2 \cdot 0)5$
6 $1(49 + 108)$
7 $14 + 0 + 19$
8 $(35 + 1)2$

For each of the conditions expressed in exercises 9 through 12, tabulate the solution set. U = N.

9 $14(2x) = 28$.
10 $0 \div 13 + 5 = m$.
11 $x(3 \cdot 12) = 36$.
12 $(42 + 5)x = 0$.

13 K = {0, 2, 4, . . .}. Does K contain the identity element for addition? For multiplication?

14 L = {1, 3, 5, . . .}. Does L contain the identity element for addition? For multiplication?

KEEPING SKILFUL

For each ordered pair named in exercises 1 through 9, tell by what number you can multiply the second component to obtain the first component.

1 (75, 25)
2 (20, 1)
3 (80, 20)
4 (45, 3)
5 (121, 11)
6 (81, 27)
7 (1512, 42)
8 (4437, 87)
9 (1820, 65)

For each condition expressed in exercises 10 through 19, tabulate the solution set. U = C.

10 $8/9 \sim 64/n$.
11 38 is $s\%$ of 19.
12 $4/3 \sim x/51$.
13 4% of 250 is n.
14 48 is $c\%$ of 60.
15 210 is 70% of b.
16 $17/23 \sim x/138$.
17 $x/42 \sim 185/210$.
18 288 is 120% of t.
19 85% of 320 is r.

SPECIAL CHALLENGE

In lesson 67 you learned that set T, together with the operations of addivision and muldivision, forms a number system. $T = \{0, 1, 2, 3, 4\}$.

A Make an addivision table that expresses the addiv of each pair of members of T.

B Make a muldivision table that expresses the muldiv of each pair of members of T.

Tabulate the solution set of each condition expressed in exercises C through J. U = T.

C $3 \oplus 4 = x$.

D $2 \otimes 1 = x$.

E $0 \oplus x = x$.

F $x \otimes 2 = 2$.

G $x \oplus 3 = 3$.

H $3 \otimes 4 = x$.

I $1 \otimes x = x$.

J $4 \oplus 1 = x$.

K Is 0 the identity element for addivision in set T? Explain your answer.

L Is 1 the identity element for muldivision in set T? Explain your answer.

Suppose that there are two other operations, "subdivision" ($\ominus$) and "didivision" ($\oslash$), for set T. The sentences below express the meanings of these two operations:

$3 \ominus 1 = x$ also means $3 = 1 \oplus x$.

$4 \oslash 2 = x$ also means $4 = x \otimes 2$.

What number is named in each of exercises M through U? Use the tables you made for exercises A and B to help you.

M $3 \ominus 1$

N $4 \ominus 2$

O $2 \ominus 3$

P $1 \ominus 4$

Q $4 \oslash 2$

R $3 \oslash 2$

S $2 \oslash 1$

T $1 \oslash 4$

U $3 \oslash 4$

V Is set T closed under subdivision? Under didivision? Explain your answers.

Tabulate the solution set of each condition expressed in exercises A through J. U = T.

A $1 \ominus 3 = x$.

B $2 \oslash 4 = x$.

C $0 = x \oplus 2$.

D $3 = 1 \otimes x$.

E $1 = x \otimes 4$.

F $2 = 4 \oplus x$.

G $4 \oplus 1 = x \ominus 3$.

H $x = (2 \ominus 3) \oslash 4$.

I $x = (1 \oslash 4) \ominus 2$.

J $2 \otimes 3 = x \ominus 4$.

APPLYING MATHEMATICS

For each problem in this lesson, write a sentence that expresses a condition. Then tabulate the solution set of the condition and give the answer to the problem. The universe is given for each problem.

1 Mrs. Rowe's monthly salary is $482. After deductions are made for insurance and taxes, she receives $395. How many dollars are deducted from her salary each month for insurance and taxes? U = N.

2 Last year, Mr. Ferguson had 4 times as many acres under cultivation as he had in pasture. He had 268 acres under cultivation. How many acres did he have in pasture last year? U = C.

3 Mr. Davies receives a 7% commission on his total monthly sales. One month his total sales amounted to $4200. How much was his commission for that month? U = C.

4 A stamp collector sold 35% of his stamps for more than $978. He sold 420 stamps. How many stamps did he have before he sold the 420 stamps? U = C.

5 Seven years ago, Mark was less than 18 years old. Nine years from now, he will be over 28. How old can Mark be now? U = N.

6 Two hundred votes were cast during an election for student council president. Bill received 136 of these votes. Bill received what per cent of all the votes cast? U = C.

7 Mike painted fewer than 12 storm windows on Friday and Saturday. He painted the same number of windows each day. How many storm windows could he have painted on each of the two days? $U = C \times C$.

8 A certain machine can fold 10,800 envelopes in 12 minutes. At this rate, how many envelopes can this machine fold in 8 minutes? U = C.

9 Together, Chuck and Steve scored 19 points during a basketball game. If Steve had scored 7 fewer points, he would have scored the same number of points as Chuck. How many points did each of the boys score? $U = N \times N$.

10 In all, there are 12 colleges in the provinces of Alberta and Newfoundland. There are 5 times as many colleges in Alberta as in Newfoundland. How many colleges are there in each of these provinces? $U = C \times C$.

11 Mr. Larson borrowed \$1650 at 6% interest. At the end of one year, he repaid the loan with interest. How much did he pay at the end of one year? $U = C \times C$.

12 Lois had \$41. She used some of this money to buy a skirt that cost \$13, a sweater that cost \$9, and a blouse that cost \$4. How much money did she have left after she had bought the skirt, sweater, and blouse? $U = N \times N$.

13 51 is 6% of what number? $U = C$.

14 What numbers are greater than the product of 63 and 19? $U = N$.

15 When a first number is subtracted from a second number, the difference is 917. When the first number is added to 796, the sum is 2243. What are the numbers? $U = N \times N$.

16 What number is 147% of 4000? $U = C$.

17 The sum of two numbers is 8158. The first number is equal to the product of 214 and 38. What are the numbers? $U = N \times N$.

18 The sum of two numbers is less than 12. The first number is 3 more than the second number. What can the numbers be? $U = N \times N$.

19 235 is what per cent of 188? $U = C$.

20 Certain rate pairs are equivalent to 28/32. The sum of the components of each of these rate pairs is less than 15. What are the rate pairs? $U = C \times C$.

21 The sum of two numbers is 349. The product of 7 and the second number is 791. What are the numbers? $U = N \times N$.

CHECKING UP

If you have trouble with these two tests, you can find help in lessons 65 through 68.

Test 69

Simplify each of the following expressions.

1 $(15 \div 3) \times 4$
2 $6(7 + 2) + 6(7 - 2)$
3 $9 \div 9 + 10 \div 10$
4 $7 \times 3 + 3 \times 2$
5 $13 + 7 \times 2$
6 $48 \div 8 + 3(5 + 2)$

For each of exercises 7 through 12, decide which property of addition or multiplication of natural numbers tells you that the statement expressed is a true statement.

7 $3 + 20 = 20 + 3$.
8 $7 = 1 \cdot 7$.
9 $2(9 \cdot 3) = (2 \cdot 9)3$.
10 $9 + 26$ is a natural number.
11 $23 + 0 = 23$.
12 $5(9 + 2) = (9 + 2)5$.

Tabulate the solution set of each condition expressed in exercises 13 through 16. $U = N$.

13 $x + 13 = 8(0 + 5)$.
14 $0 \cdot 1 + 8 = x(7 - 3)$.
15 $3(4 + 2) + x = 4 \cdot 3 + 6$.
16 $(14 + 9) + x = 14 + (x + 9)$.

Test 70

$Q = \{0, 1, 2, 3\}$. The members of Q are all the numbers the remainder can be when you divide a natural number by 4. To find the addiv of any two members of Q, first find the sum of the two numbers. Then divide the sum by 4. The remainder is the addiv.

17 Explain how to find the muldiv of any two members of Q.

18 Make an addivision table that expresses the addiv of each pair of members of Q.

19 Make a muldivision table that expresses the muldiv of each pair of members of Q.

20 Does set Q, together with the operations of addivision and muldivision, form a number system? Explain your answer.

End-of-block tests on simplifying expressions and on the natural-number system

Composite numbers and prime numbers

In this lesson you will use what you know about the factors of a number to learn more about the counting numbers. Remember that the set of counting numbers is {1, 2, 3, . . .} and that the letter C is a name for this set.

A Look at D1. For each statement expressed in the display, what replacements were made for the variables in $a \times b = c$?

B How do you know that 2 is a factor of 6? How do you know that 6 is a factor of 6?

> REMINDER
> If a true statement is obtained from $a \times b = c$ when a, b, and c are replaced by members of C, then the replacement for a is a factor of the replacement for c. The replacement for b is also a factor of the replacement for c.
> *See lesson 53, page 226.*

C Does {1, 2, 3, 6} contain all the factors of 6? Explain your answer.

D Tabulate the set of factors of 8. Of 12. Of 13.

E Is 1 a factor of every member of C? Is every member of C a factor of itself?

Now you will learn how to find all the factors of a counting number.

F Is 2 a factor of 2? How do you know?

G Is 2 a factor of 2×2, or 4? Of 2×3, or 6? How do you know?

Universe for a, b, and c = C.
$a \times b = c$.
A $3 \times 2 = 6$.
B $6 \times 1 = 6$.

D1

1	2	3	~~4~~$_2$	5
~~6~~$_{2,3}$	7	~~8~~$_2$	~~9~~$_3$	~~10~~$_2$
11	~~12~~$_{2,3}$	13	~~14~~$_2$	~~15~~$_3$
~~16~~$_2$	17	~~18~~$_{2,3}$	19	~~20~~$_2$
~~21~~$_3$	~~22~~$_2$	23	~~24~~$_{2,3}$	25
~~26~~$_2$	~~27~~$_3$	~~28~~$_2$	29	~~30~~$_{2,3}$

D2

H Is 2 a factor of 8? Of 10? Of 12?

I Is each number that has 2 as a factor an even number?

J Do odd numbers have 2 as a factor?

K Look at the chart in D2. Find the names of the numbers that are greater than 2 and that have 2 as a factor. Is the name of each of these numbers crossed out? Is the numeral 2 crossed out?

Notice the small numeral 2 that is shown below the name of each even number greater than 2.

L Now think about the number 3. Is 3 a factor of 3? Of 3×2, or 6? Of 3×3, or 9? Of 3×4, or 12?

M Look again at D2. Find the names of the numbers that are greater than 3 and that have 3 as a factor. Is the name of each of these numbers crossed out? Is the numeral 3 crossed out?

N Notice the small numerals 2 and 3 below the numeral 6. Why do both the numerals 2 and 3 appear below the numeral 6? Which other numerals in the chart in D2 are names of numbers that have both 2 and 3 às factors?

The type of chart shown in D2 was invented by a Greek scientist and mathematician named Eratosthenes (er′ə tos′thə nēz′), who lived more than two thousand years ago. The chart is called the "Sieve of Eratosthenes." It is called a "sieve" because factors are used to "sift out" certain numbers.

O D3 shows the Sieve of Eratosthenes for the numbers from 1 through 12. The numeral 4 is crossed out because the number 4 has 2 as a factor. Are 1 and 4 also factors of 4? Tabulate the set of factors of 4.

P Tabulate the set of factors of each of the other numbers whose names are crossed out in D3.

Q How do you know that each number whose ? name is crossed out must have 1 and itself as factors?

Now you will make a Sieve of Eratosthenes for the numbers from 1 through 50.

A Make a chart of the names of the counting numbers from 1 through 50. Write the numerals in vertical columns and leave enough space between the numerals for your work.

B First cross out the name of each number greater than 2 that has 2 as a factor. Write the numeral 2 next to each of these names.

C Then write the numeral 3 next to the name of each number greater than 3 that has 3 as a factor. Be sure to cross out each of these numerals if it has not already been crossed out.

D Now think about the number 4. Is the nu- ? meral 4 crossed out in your chart? Why? Is the name of each of the other numbers that have 4 as a factor crossed out? Why? Write the numeral 4 after the name of each number greater than 4 that has 4 as a factor.

E Continue to work in this manner with 5, 6, 7, and so on. Be sure that you have crossed out all numerals except the numerals for numbers that have no factors other than 1 and the numbers themselves. Opposite each crossed-out numeral you should have the names of *all* the

1	2	3	~~4~~$_2$	5
~~6~~$_{2,3}$	7	~~8~~$_2$	~~9~~$_3$	~~10~~$_2$
11	~~12~~$_{2,3}$	13	~~14~~$_2$	~~15~~$_3$
~~16~~$_2$	17	~~18~~$_{2,3}$	19	~~20~~$_2$
~~21~~$_3$	~~22~~$_2$	23	~~24~~$_{2,3}$	25
~~26~~$_2$	~~27~~$_3$	~~28~~$_2$	29	~~30~~$_{2,3}$

D2

1	2	3	~~4~~$_2$
5	~~6~~$_{2,3}$	7	~~8~~$_{2,4}$
~~9~~$_3$	~~10~~$_{2,5}$	11	~~12~~$_{2,3,4,6}$

D3

com pos ite number (kom′pə zit). A member of {1, 2, 3, . . .} that has more than two factors. The factors of 8 are 1, 2, 4, and 8; therefore, 8 is a composite number. The only factors of 5 are 1 and 5; thus 5 is not a composite number.

prime number (prīm). A member of {1, 2, 3, . . .} that has exactly two factors. The only factors of 7 are 1 and 7; therefore, 7 is a prime number.

factors of the number named except 1 and the number itself. Part of your work might look like the work in D4.

F How do you know that each number whose ? name is crossed out in your chart must have more than two factors? What is the greatest factor that you named?

Counting numbers that have more than two factors are *composite* numbers.

G Tabulate the set of numbers from 1 through ? 10 that have exactly two factors. Is 1 a member of this set? Explain your answer.

Numbers that have exactly two factors are *prime* numbers. Remember that we are referring only to the counting numbers.

H Why is 1 not a prime number? Is 2 a prime number? Is 3 a prime number?

I Is 8 a prime number? Is 14 a prime number? Is any even number greater than 2 a prime number? Explain your answer.

J Is every prime number a factor of itself? What is the other factor of a prime number?

K You know that 5^2 means "5 squared," or 5×5. We also call 5^2 "the *square* of 5." Is $5^2 = 25$? Tabulate the set of factors of 25.

L What number is the square of 7? Tabulate ? the set of factors of the square of 7. A number that is a square of a prime number has how many factors?

You have learned what composite numbers and prime numbers are. In the exercises that follow, you will learn more about prime numbers.

A Make a list of the names of the prime numbers that are less than 100. The Sieve of Eratosthenes that you made will help you with the numbers through 50.

B Study the list that you made for exercise A. Do the prime numbers occur regularly in {1, 2, 3, . . ., 99}? Explain your answer.

29
30 2, 3, 5, 6, 10, 15
31
32 2, 4, 8, 16
33 3, 11
34 2, 17

D4

A $4 = 2 + 2$.
B $6 = 3 + 3$.
C $8 = 3 + 5$.
D $10 = 5 + 5$.
E $12 = 5 + 7$.
F $14 = 7 + 7$.

D5

C There are some pairs of prime numbers that are separated by only one other number. What is the first pair like this that is named in your list?

Two prime numbers that are separated by only one other number are *twin primes*. The difference of twin primes is two.

D Are 29 and 31 twin primes? Are 37 and 41 twin primes? Name five twin primes that are counting numbers less than 100.

Mathematicians have wondered if there is a greatest pair of numbers that are twin primes. No one yet knows the answer, but there is good evidence to support the belief that the set of twin primes is an infinite set.

E Look at D5. Sentence B tells you that 6 is the sum of the two numbers 3 and 3. Is 3 a prime number? The number 8 is the sum of what two primes? Are 6 and 8 even numbers? Is 4 the sum of two primes? Is 10? Is 12? Is 14? Are 4, 10, 12, and 14 even numbers?

F For each of the even numbers from 16 through 36, write a sentence that expresses that number as the sum of two prime num-

$U = \{2, 3, 4, \ldots\}$.

x	$2x$	Prime numbers between x and $2x$
2	4	3
3	6	5
4	8	5, 7
5	10	7
14	28	17, 19, 23
15	30	17, 19, 23, 29
16	32	17, 19, 23, 29, 31

D6

bers. For 16 you could write either the sentence $16 = 3 + 13$ or the sentence $16 = 5 + 11$.

It appears that all even numbers greater than 2 can be thought of as the sum of two prime numbers, but mathematicians have not found a way to prove that this is true.

G Now look at D6. The universe is the set of counting numbers greater than 1. From the first row of the chart, you can see that the only prime number between 2 and 2×2, or 4, is 3. What prime number is between 3 and 2×3, or 6? What prime numbers are between 4 and 2×4, or 8? Between 5 and 2×5, or 10?

H What prime numbers are between 14 and 2×14, or 28? Between 15 and 2×15, or 30? Between 16 and 2×16, or 32?

I Make a chart like the one shown in D6. Use the numbers 6 through 13 as replacements for x.

Mathematicians have proved that, no matter how great the number selected, there is always at least one prime number between the selected number and two times that number.

Now you know what composite numbers and prime numbers are.

On your own

Tabulate the set of factors of each number named in exercises 1 through 8.

1 27	**3** 19	**5** 57	**7** 167
2 3	**4** 32	**6** 120	**8** 227

9 Which of the numbers named in exercises 1 through 8 are prime numbers? Which are composite numbers?

10 Set A is the set of factors of a prime number. How many members does A have?

11 Set K is the set of factors of 16; set L is the set of factors of 4. Is L a subset of K? Explain your answer.

Which numbers named in exercises 12 through 20 are even numbers? Which numbers are odd?

12 2×5	**15** $2 + 16$	**18** $128 - 37$
13 $3 + 7$	**16** $7 + 8$	**19** $5(9 + 13)$
14 $6 \times 5 \times 3$	**17** $3(4 + 7)$	**20** $4(3 + 11)$

For each ordered pair named in exercises 21 through 29, name two factors of the first component that are also factors of the second component.

21 (7, 14)	**24** (10, 16)	**27** (14, 56)
22 (6, 8)	**25** (45, 21)	**28** (54, 108)
23 (100, 200)	**26** (26, 130)	**29** (39, 117)

Tell what prime numbers are factors of each number named in exercises 30 through 38.

30 51	**33** 286	**36** 178
31 84	**34** 660	**37** 132
32 89	**35** 1001	**38** 195

For each ordered pair named in exercises 39 through 50, find the greatest number that is a factor of both components.

39 (30, 60)	**43** (45, 50)	**47** (36, 51)
40 (7, 11)	**44** (112, 96)	**48** (77, 84)
41 (70, 130)	**45** (18, 90)	**49** (600, 120)
42 (98, 1)	**46** (25, 75)	**50** (104, 136)

The unique factorization property

In this lesson you will learn to express counting numbers greater than 1 as the product of prime numbers.

A Look at sentence A in D1. The number 24 is expressed as the product of 4 and 6. Are 4 and 6 prime numbers? You can think of 4 as the product of what primes? You can think of 6 as the product of what primes?

B If you think of 4 as 2×2, and of 6 as 2×3, then you can think of 24 as $2 \times 2 \times 2 \times 3$. Are 2 and 3 primes? Does sentence D express 24 as the product of primes?

C Is $2^3 \times 3^1$ the same number as $2 \times 2 \times 2 \times 3$? Does sentence E express a true statement? Explain your answers.

> REMINDER
>
> Numbers like 2^3 are powers. The base of 2^3 is 2, and the exponent of 2^3 is 3. The exponent tells how many times the base is used as a factor. In 2^3, 2 is used as a factor three times. $2^3 = 2 \times 2 \times 2$ is a true statement.
>
> *See lesson 53, page 227.*

D Look at sentences F and G in D2. Which factor of 20×5 is not a prime? How can you obtain $2 \times 10 \times 5$ from 20×5?

E Which factor of $2 \times 10 \times 5$ is not a prime? How can you obtain $2 \times 2 \times 5 \times 5$ from $2 \times 10 \times 5$?

D1

A $24 = 4 \times 6$.
B $4 = 2 \times 2$.
C $6 = 2 \times 3$.
D $24 = 2 \times 2 \times 2 \times 3$.
E $24 = 2^3 \times 3^1$.

D2

F $100 = 20 \times 5$.
G $100 = 2 \times 10 \times 5$.
H $100 = 2 \times 2 \times 5 \times 5$.
I $100 = 2^2 \times 5^2$.

D3

J $350 = 10 \times 35$.
K $350 = 2 \times 5 \times 35$.
L $350 = 2 \times 5 \times 5 \times 7$.

F When you think of 100 as $2 \times 2 \times 5 \times 5$, are you thinking of 100 as a product of primes? Does sentence H express a true statement?

G Is $2^2 \times 5^2$ the same number as $2 \times 2 \times 5 \times 5$? How do you know? Does sentence I also express a true statement?

H Look at sentence J in D3. Is either factor of 10×35 a prime?

I Look at sentence K. How can you obtain $2 \times 5 \times 35$ from 10×35? Which factor of $2 \times 5 \times 35$ is not a prime? This factor can be expressed as the product of what primes?

J Does sentence L express 350 as a product of primes? Use exponential notation to express 350 as the product of primes.

You have expressed the number 24 as $2^3 \times 3^1$. You have expressed the number 100 as $2^2 \times 5^2$, and you have expressed the number 350 as $2^1 \times 5^2 \times 7^1$. In each case you have thought of a number as a product of primes.

Mathematicians have shown that any composite number can be thought of as a product of primes. When you express a number as a product of prime numbers, this product is a *complete factorization* of the given number.

Mathematicians think of a prime number as a complete factorization of itself. 2 is the complete factorization of 2, and 3 is the complete factorization of 3. This rule applies to every prime.

K Look at D4. Which sentences express numbers as products of primes? Is 4×3 a complete factorization of 12?

Now you will learn how to find a complete factorization of 12. The least prime is 2. If you divide 12 by 2, you obtain 6. You know that 2 and 6 are factors of 12. But, since 6 is not a prime, you divide again by 2. If you divide 6 by 2, you obtain 3. Thus, there are two factors of 2 and one factor of 3 in the product 12.

L Is $2 \times 2 \times 3$ a complete factorization of 12?

M Now think about the number 168. Is 2 a factor of 168? $168 = 2 \times 84$. Is 2 a factor of 84? $84 = 2 \times 42$. Is 2 a factor of 42? $42 = 2 \times 21$. Is 2 a factor of 21?

N How many times have you used 2 as a factor in exercise M?

O How many factors of 3 are there in the product 21? $21 = 3 \times 7$. Is 7 a prime number?

P Is $2 \times 2 \times 2 \times 3 \times 7$ a complete factorization of 168? Express $2 \times 2 \times 2 \times 3 \times 7$ by using exponential notation.

Q Look again at D4. Find a complete factorization of 16. Of 144. Use exponential notation to express your answers.

Notice that there is a remainder of 0 when a number is divided by one of its factors. For example, 4 is a factor of 12; when 12 is divided by 4, the remainder is 0. We say that, if the remainder is 0 when a first number is divided by a second number, then the first number is *divisible* by the second number. A number is divisible by each of its factors. 12 is divisible by each of its factors, 1, 2, 3, 4, 6, and 12.

You can use division to test whether or not a number is a factor of another number. 17 is not divisible by 5 because when 17 is divided by 5, there is a remainder of 2. Therefore, 5 is not a factor of 17.

R Is 144 divisible by 36? Is 36 a factor of 144? Is 14 a factor of 52?

Mathematicians have shown that each counting number greater than 1 can be expressed as a product of primes in just one way. Changing the order of the factors does not form a different product. For example, the product $2 \times 5 \times 3$ is the only complete factorization of 30. However, you can think of the complete factorization of 30 as $5 \times 3 \times 2$.

You have been studying a property of the counting numbers greater than 1. The property is expressed in this way.

M $9 = 3 \times 3$.
N $12 = 4 \times 3$.
O $168 = 4 \times 42$.
P $16 = 4 \times 4$.
Q $70 = 5 \times 7 \times 2$.
R $144 = 2 \times 72$.

D4

$M = \{1, 2, 4, 8\}$.
$N = \{1, 2, 4, 5, 10, 20\}$.

D5

S $100 = 2 \times 5 \times 2 \times 5$.
T $210 = 2 \times 5 \times 3 \times 7$.

D6

complete factorization (fak′tər ə zā′shən). A number expressed as a product of primes. 3×5 is a complete factorization of 15. $3^2 \times 2$ is a complete factorization of 18. 9×2 is not a complete factorization of 18.

Except for the order in which the factors occur, every counting number greater than 1 can be factored into primes in only one way.

This property is the *unique factorization property of the counting numbers.* The word "unique," as used here, means "able to be done in just one way."

S If you considered 1 a prime, would there be a unique factorization property of the counting numbers greater than 1? Explain your answer.

Next you will see how the factors of one number are related to the factors of another number.

A Look at D5. Which set is the set of factors of 8? Which set is the set of factors of 20?

B Is 2 a factor of 8? Is 2 also a factor of 20? Is {2} a subset of M ∩ N? When a number is a factor of each of two numbers, it is a *common factor* of the two numbers.

C How do you know that 1, 2, and 4 are common factors of 8 and 20?

D Tabulate the set of common factors of 8 and 20. Is this set a subset of the set of factors of 8? Is this set a subset of the set of factors of 20?

E Tabulate the set of factors of 12. Tabulate the set of factors of 30. Tabulate the set of common factors of 12 and 30.

F Tabulate the set of common factors of 5 and 3. Of 5 and 7. Of 7 and 3.

G How do you know that {1} is the set of common factors of any pair of prime numbers?

H Is {1, 2, 4} the set of common factors of 12 and 16? What is the greatest member of {1, 2, 4}? This number is the *greatest common factor* of 12 and 16. The greatest common factor of two numbers is the greatest number that is a factor of both numbers.

I Is {1, 2, 7, 14} the set of common factors of 14 and 28? What is the greatest common factor of 14 and 28? Explain why a number greater than 14 cannot be a common factor of 14 and 28.

greatest common factor. The greatest number that is a factor of two or more counting numbers. 15 is the greatest common factor of 30 and 45. 7 is the greatest common factor of 14, 35, and 49.

When numbers have many factors, it is not convenient to tabulate the set of factors of each number in order to determine their greatest common factor. You will now learn another way to determine the greatest common factor of two numbers.

J Look at D6. Is 100 expressed as a product of prime numbers? Is 210 expressed as a product of prime numbers? Are these complete factorizations?

K What prime numbers are factors of both 100 and 210? What is the product of these prime factors? Is this product the greatest common factor of 100 and 210?

L Find the complete factorization of 300. Of 550.

M What prime numbers are factors of both 300 and 550?

You have discovered that 2 and 5 are the common prime factors of 300 and 550. Notice that 5 is used twice as a factor of both 300 and 550. This means that 5^2 is a factor of both 300 and 550. Notice also that 2 is used twice as a factor of 300 but only once as a factor of 550. Therefore, 2^1, but not 2^2, is a common factor of 300 and 550. You can use 5^2 and 2^1 to find the greatest common factor of 300 and 550. The power of a common prime factor depends on the greatest number of times the prime is used as a factor in *both* numbers. 5^2 and 2^1 are the greatest powers of 5 and 2 that are common factors of both 300 and 550.

N What is the product of 5^2 and 2^1? Is this product the greatest common factor of 300 and 550?

O Now think about the numbers 24, 36, and 120. Are 2 and 3 the only common prime factors of these numbers?

P Is 2^2 the greatest power of 2 that is a common factor of 24, 36, and 120?

Q Is 3, or 3^1, the greatest power of 3 that is a common factor of 24, 36, and 120?

R What is the product of 2^2 and 3? What is the greatest common factor of 24, 36, and 120?

S Find the complete factorization of 14, 21, and 28.

T What is the greatest common factor of 14, 21, and 28?

U Find the greatest common factor of 100, 125, and 75.

You have learned how to find the complete factorization of a number. You have also learned how to find the greatest common factor of two or more numbers.

On your own

For each pair of numbers named in exercises 1 through 6, first tabulate the set of factors of each number. Then tabulate the set of common factors of each pair of numbers.

1 6, 8 **3** 27, 15 **5** 30, 72

2 15, 30 **4** 36, 27 **6** 16, 12

7 For exercises 1 through 6, name the greatest common factor of each pair of numbers.

8 What is the greatest common factor of 1 and 6? Of 1 and 8? Of 1 and 300? Of 1 and any member of {1, 2, 3, . . .}?

9 Do any two members of {1, 2, 3, . . .} have a common factor? If so, what is it?

10 Tabulate a set of three composite numbers whose only common factor is 1.

For each pair of numbers named in exercises 11 through 19, first find the complete factorization of each number. Then name the greatest common factor of the two numbers.

11 24, 60 **14** 25, 125 **17** 252, 672

12 36, 90 **15** 60, 84 **18** 306, 1173

13 72, 108 **16** 165, 234 **19** 2040, 2184

For each of exercises 20 through 25, determine the least prime factor of the number named. Remember that 1 is not a prime.

20 115 **22** 321 **24** 539

21 135 **23** 484 **25** 121

26 What number is the least prime factor of any even number?

Each of exercises 27 through 32 names two numbers. What is the least number that is divisible by both numbers?

27 1, 2 **29** 3, 2 **31** 4, 12

28 7, 2 **30** 4, 11 **32** 3, 9

APPLYING MATHEMATICS

For each problem, write a sentence that expresses a condition. Then tabulate the solution set of the condition and give the answer to the problem. The universe is given for each problem.

1 In the year 1951, 128,808 tons of copper were produced in the province of Ontario. There were 75,313 more tons of copper produced in Ontario in 1960 than were produced in 1951. How many tons of copper were produced in Ontario in 1960? $U = N$.

2 Mr. Peters drove 104 mi. in 2 hr. At this rate, how many miles can he drive in 3 hr.? $U = C$.

3 Altogether, Helen has fewer than 5 coins in her purse. She has only dimes and quarters. How many coins of each kind can she have? $U = C \times C$.

4 A large can of orange juice contains 46 oz. How many ounces will 6 of these cans contain? $U = C$.

5 The Spanish club has fewer than 15 members. Nine of the members are girls. How many boys can be members of the Spanish club? U = C.

6 Mr. Curtis, a stamp dealer, had fewer than 150 rare stamps to sell. He sold 39 of them and still had more than 62 left to sell. How many rare stamps could he have had before he sold the 39? U = N.

7 Larry had 82¢. After he had bought 12 pieces of candy priced at 2 pieces for 3¢, how much money did he have left? U = C × C.

8 Mrs. Roberts has a collection of antique glass. If she buys 18 more pieces of glass, she will have more than 100 pieces. If she sells 27 pieces, she will have fewer than 66 pieces. How many pieces of antique glass can Mrs. Roberts have in her collection? U = N.

9 Mr. Andrews borrowed $2250 from a bank. At the end of one year, he paid $135 interest on his loan. What rate of interest did the bank charge him? U = C.

10 Together, Sue and Carol bought fewer than 12 yd. of material. Sue bought 3 yd. of denim and 4 yd. of print. How many yards of material could Carol have bought? U = C × C.

11 Mr. Mitchell's monthly salary is $550. Each month 18% of his salary is deducted for insurance and taxes. What is his monthly take-home pay? U = C × C.

12 In the 1960 American census, the population of the United States was 179,323,000. Of these persons, 16,782,000 lived in New York, 10,081,000 lived in Illinois, and 15,717,000 lived in California. How many persons in the United States lived outside these three states at the time of the 1960 census? U = N × N.

13 Ellen answered all the questions on a test and had correct answers for 75% of them. She had 72 correct answers. How many incorrect answers did she have on this test? U = C × C.

14 Mr. Cass had 10 fir trees to sell. He sold some, but not all, of these trees. The number of trees he had left to sell was less than the number of trees he sold. How many trees could he have sold, and how many trees could he have left to sell? U = C × C.

15 Last year the Danville hockey team played 35 games. They won 3 games for each 2 games that they lost. There were no ties. How many games did they win and how many games did they lose last year? U = C × C.

16 Last year Mrs. Nichols knitted 4 more sweaters than pairs of socks. She knitted fewer than 16 sweaters and pairs of socks in all. How many sweaters and how many pairs of socks could Mrs. Nichols have knitted last year? U = C × C.

17 What number is the sum of 2467, 38961, and 15890? U = N.

18 17 is what per cent of 425? U = C.

19 The sum of two numbers is less than 24. Twice the second number is equal to 32. What can the numbers be? U = N × N.

20 What numbers less than the product of 32 and 17 are greater than the difference of 912 and 362? U = N.

21 Certain rate pairs are equivalent to 25/20. The sum of the components of each of these rate pairs is less than 75. What are the rate pairs? U = C × C.

22 The sum of two numbers is 21. The first number is equal to the product of five and the second number. What are the numbers? U = N × N.

23 What numbers are less than the quotient of 528 and 48? U = N.

24 A first number is 27 more than a second number. The sum of the first number and 37 is 112. What are the numbers? U = N × N.

25 What numbers are greater than the product of 1001 and 10? U = N.

CHECKING UP

The small numeral within parentheses tells what page to turn to for help if you need it.

Test 71

The six mathematical sentences below express properties of the operations of the set of natural numbers.

$m(n+p) = mn + mp$.

$(a+b)+c = a+(b+c)$.

$d+e$ is a member of N.

$x+y = y+x$.

$(pq)r = p(qr)$.

$rs = sr$.

For each property named in exercises 1 through 5, choose the sentence above that expresses that property. The universe for the variables is N.

1 Commutative property of addition (265)

2 Commutative property of multiplication (267)

3 Associative property of addition (270)

4 Associative property of multiplication (271)

5 Distributive property of multiplication over addition (275)

Test 72

What words or symbols best complete exercises 6 through 16?

6 The standard name of $6 \cdot 4$ is ~~~. (279)

7 A number is divisible by each of its ~~~. (300)

8 The identity element for addition is the number ~~~. (289)

9 Since each natural number has a natural number that immediately follows it, the set of natural numbers has the ~~~ property. (254)

10 The number associated with the union of two disjoint finite sets is the ~~~ of the numbers associated with the two sets. (257)

11 You use division to find the ~~~ of two numbers. (264)

12 Addition is a ~~~ mapping. (258)

13 The number associated with a Cartesian set $R \times S$ is the ~~~ of the numbers associated with sets R and S. (262)

14 An operation that involves an ordered pair of numbers is a ~~~ operation. (259)

15 A numeral that contains one or more symbols for an ~~~ is an expression. (279)

16 A number expressed as a product of primes is the complete ~~~ of the number. (300)

Test 73

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

17 {A, B, C, D} and {17, 34, 51} can be put in one-to-one correspondence. (246)

18 The set of natural numbers that are divisible by 6, together with the operations of addition and multiplication, forms a number system. (283)

19 The number 1 is a prime number. (297)

20 The number 0 is the successor of the natural number associated with { }. (254)

21 The standard set of any natural number is a finite set. (251)

22 If $A = \{5, 7\}$ and $B = \{8, 11, 4\}$, then $5 = n(A \cup B)$. (257)

23 The set of odd numbers is closed under addition. (259)

24 Multiplication is a binary operation. (263)

25 $1/a = a$ is a property when $U = C$. (291)

Test 74

From the list below each exercise, choose the word, words, or symbols that correctly complete the exercise.

26 When you simplify the expression $7 \cdot 3 + 4$, you obtain the numeral ~~~. (280)

a 84 **b** 49 **c** 14 **d** 25

27 The greatest common factor of 8 and 112 is ~~~. (301)

a 2 **b** 2^2 **c** 2^3 **d** 2^4

28 The number that is associated with the standard set whose greatest member is 9 is the natural number ~~~. (251)

a 8 b 9 c 10 d 11

29 The number 1 is the identity element for ~~~. (290)

a addition
b multiplication
c subtraction
d the union of sets

30 ~~~ is not a number system with respect to addition and multiplication. (283)

a {7, 9, 11, . . .}
b {12, 24, 36, . . .}
c {4, 8, 12, . . .}
d {60, 62, 64, . . .}

31 Counting numbers that have more than two factors are ~~~ numbers. (296)

a prime
b twin prime
c square
d composite

32 The number that is associated with {0, 1, 2, 3, 4, 5, 6, 7, 8} is the successor of ~~~. (254)

a 7 b 8 c 9 d 10

33 When $U = N \times N$, you always obtain a natural number from ~~~. (259)

a $x - y$
b $x + y$
c $x \div y$
d $2x - y$

34 $2 \cdot 3 \cdot 5 \cdot 7$ is the complete factorization of ~~~. (300)

a 180 b 140 c 210 d 240

35 The solution set of ~~~ is equivalent to {J, K, L, M, N}. (247)

a $x + 5 < 9$. $U = N$.
b $x + 18 < 25$. $U = C$.
c $12 - x > 6$. $U = N$.
d $x/y \sim 2/3 \wedge x + y < 30$. $U = C \times C$.

Test 75

36 Tabulate three proper subsets of {red, black}. (253)

37 What inequality involving the idea of "less than" gives the same information as $32 > 5$? (253)

38 How many natural numbers are between 6 and 26? (254)

39 Name the successor of 699. (254)

40 If F = {0, 1, 2, 3} and G = {0, 1, 2, 3, 4, 5, 6}, what natural number is associated with $F \times G$? (262)

41 How many even prime numbers are there? (296)

42 What is the complete factorization of 17? (300)

43 Tabulate the set of factors of 24. (301)

CHECKING UP

The small numeral within parentheses tells what page to turn to for help if you need it.

Test 76

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

1 Any set that is not a finite set is an infinite set. (23)

2 The set of even numbers is closed under subtraction. (260)

3 If set A has the same number of members as set B, then A and B are the same set. (247)

4 If point K separates $\overleftrightarrow{FG}$ into two half-lines, and $\overline{FG} \cap \{K\} = \{\ \}$, then points F and G are in the same half-line. (42, 102)

5 A plane figure is a geometric figure in which all points are coplanar. (48)

6 If two lines are subsets of the same plane, and their intersection is the empty set, the lines are skew lines. (101)

7 If F = {11, 12, 13} and G = {13, 14, 15}, then $9 = n(F \times G)$. (262)

8 For any two sets R and S, $R \cap S$ is a subset of set R. (86)

9 Two parallel planes are disjoint sets. (99)

10 The intersection of set Q with itself is set Q. (86)

11 If a line and a plane are parallel, then their intersection is the line. (101)

Test 77

What words or symbols best complete exercises 12 through 28?

12 The number 12 is the ~~~ of 12^3. (227)

13 A numeration system is a system for naming ~~~. (209)

14 The first period to the left of the billions period is the ~~~ period. (238)

15 Scientific notation is a method of expressing a number as the product of a number from 1 to 10 and a ~~~ of 10. (239)

16 The ~~~ of the numeral 750 is the numeral $7(10^2) + 5(10^1) + 0(1)$. (228)

17 The factors of 32 are ~~~. (226)

18 64 expressed as a power of 2 is ~~~. (227)

19 The expanded form of the numeral 56073 is the numeral ~~~. (228)

20 The number 17 is the ~~~ of 3^{17}. (227)

21 30,000 expressed in scientific notation is ~~~. (239)

22 The numbers 10^1, 10^2, and 10^3 are powers of ~~~. (227)

23 Six is the ~~~ of the numeral 302_{six}. (230)

24 {1, 2, 3, 4, 6, 9, 12, 18, 36} is the set of ~~~ of 36. (226)

25 The decimal numeral that names the number one hundred twenty-six trillion fifty-eight thousand three hundred sixty-nine is ~~~. (238)

26 Without using exponential notation, 6^3 is expressed by the numeral ~~~. (227)

27 The decimal numeral for forty-four quadrillion is ~~~. (238)

28 In the numeral 8120694, the digit ~~~ is associated with the fourth power of ten. (230)

Test 78

For exercises 29 through 38, use the coplanar geometric figures represented in D1. D is the center of the circle. Decide which of the Venn diagrams X, Y, or Z in D2 represents the set described in each exercise. The universe is plane ADF.

29 $\overline{BC} \cap$ circle D (102)

30 $\ell_1 \cap \triangle ABC$ (83)

31 $\ell_1 \cap \{F\}$ (102)

32 $\{D\} \cap$ circle D (83)

33 $\{C\} \cap \{X \mid \overline{DX} < \overline{DE}\}$ (83)

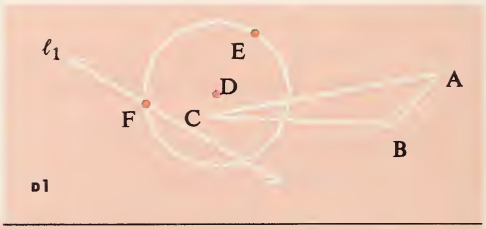
34 $\{X \mid \overline{DX} \cong \overline{DF}\} \cap \overline{AD}$ (83)

35 Exterior of $\triangle ABC \cap \overline{AB}$ (114)

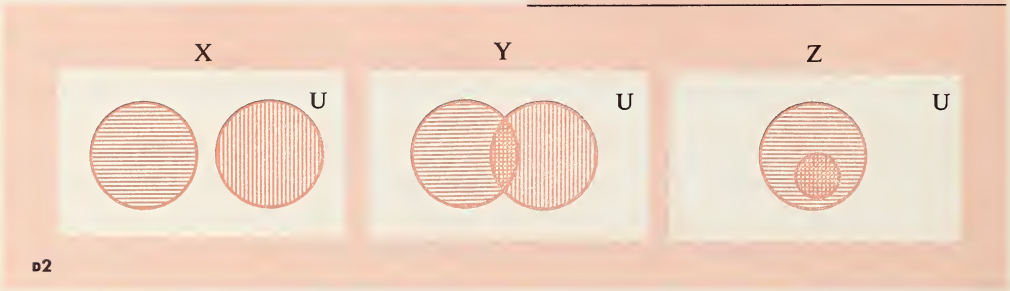
36 Interior of $\triangle ABC \cap \{X \mid \overline{DX} > \overline{DE}\}$ (114)

37 Circle D $\cap \overline{AC}$ (102)

38 $\ell_1 \cap \overline{BC}$ (102)



D1



D2

Test 79

For each problem, write a sentence that expresses a condition, tabulate the solution set, and give the answer to the problem. The universe is given for each problem.

39 There are 230 boys in Bell School. Of these boys, 161 take part in sports after school. How many boys in Bell School do not take part in after-school sports? U = N. (68)

40 Frank's bowling average is 136. In a game one day, Frank bowled fewer than 12 pins under his average. His score was less than 128. What could have been his score for that game? U = N. (94)

41 Miss Lee's salary was $4500 last year. She saved 3% of her salary and spent the rest. The part of her salary that she spent amounted to how many dollars? U = C × C. (195)

42 Sue spent $24 for clothes. Lucy spent 125% of that amount for clothes. How much money did Lucy spend? U = C. (190)

43 Mr. Jackson bought a set of drums at a 15% discount. The regular price of the drums was $60. How much did Mr. Jackson pay for the drums? U = C × C. (195)

44 The sum of two numbers is 12. The first number is equal to the product of 3 and the second number. What are the numbers? U = N × N. (160)

45 352 is what per cent of 550? U = C. (191)

46 80 is 16% of what number? U = C. (192)

47 24% of 950 is what number? U = C. (192)

48 10 is 1% of what number? U = C. (192)

49 Certain rate pairs are equivalent to 5/10. The product of the components of each of these rate pairs is less than 35. What are the rate pairs? U = C × C. (199)

50 Certain rate pairs are equivalent to 3/1. The sum of the components of each of these rate pairs is less than 20. What are the rate pairs? U = C × C. (199)

Unit 7

The rational numbers of arithmetic

lesson page

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7 71 Exploring ideas

Fractions and equivalent fractions

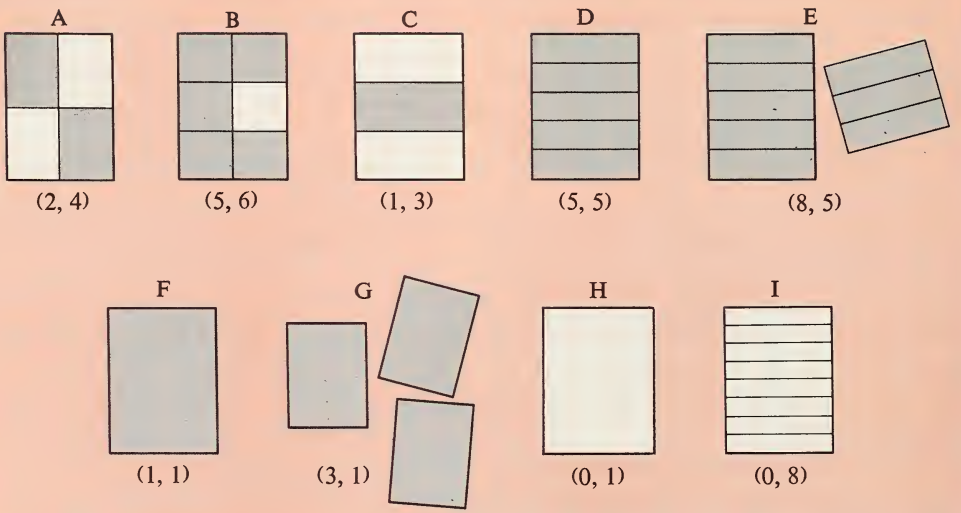
In unit 3, you used ordered pairs of natural numbers to solve problems that involved conditions in two variables. Then, in unit 4, you used a special kind of ordered pair to solve rate and comparison problems. In this lesson you will study another special kind of ordered pair of natural numbers.

The pictures in D1 show pieces of paper. Notice how the pieces are marked off into parts. Also notice the parts that are gray.

A Look at picture A. The piece of paper is marked off into four parts. Each of these four parts is the same size as each of the other parts. The parts you are going to think about are gray. How many of the parts are you to think about?

You can use (2, 4) to represent the number of parts in the whole piece and also the number of these parts that you are to think about. The second component of (2, 4) refers to the number of parts into which the whole piece is marked off. The first component refers to the number of these parts that you are to think about.

B Look at picture B. The whole piece of paper is marked off into how many parts that are the same size? How many of these parts are you to think about? You can use (5, 6) to represent the number of parts into which the piece is marked off and also the number of these parts that you are thinking about. What does the second component of (5, 6) refer to? What does the first component refer to?



D1

C Look at picture C. What ordered pair represents the number of parts into which the whole piece of paper is marked off and also the number of these parts that you are thinking about?

D Look at picture D. The whole piece of paper is marked off into how many parts? Notice that you are to think about all of these parts. Why can you use (5, 5) to represent the number of parts marked off and also the number of these parts that you are thinking of?

E Look at picture E. The whole piece of paper is marked off into how many parts that are the same size? How many of these parts are you to think about? Besides these five parts, the picture shows three more gray parts just like the five parts. You are to think about how many parts altogether?

F You can use (8, 5) to represent the number of parts into which the whole piece is marked off and also the total number of such parts that you are thinking about. The second component refers to the number of parts into which the whole piece is marked off. What does the first component refer to?

G Look at picture F. No parts are marked off in the whole piece of paper. Think of the whole piece as consisting of one part. How many parts are you to think about? Does (1, 1) represent the total number of parts and also the number of these parts that you are thinking about?

H Look at picture G. There is a whole piece of paper consisting of one part. Besides this part, the picture shows two more gray parts just like the first part. You are to think about how many parts altogether? What does each component of (3, 1) refer to?

I Look at picture H. Think of the whole piece of paper as consisting of one part. Are you to think about this part? Does (0, 1) represent the total number of parts and also the number of parts that you are thinking about? What does each component of (0, 1) refer to?

J Look at picture I. Why can you use (0, 8) to represent the total number of parts and also the number of parts that you are thinking about?

For each picture in D1, you decided into how many parts the whole piece of paper was marked off. You also decided how many of these parts you were to think about. Sometimes the number of parts you were thinking about was less than the number of parts in the whole. Sometimes the number of parts you were thinking about was the same as the number of parts in the whole. Sometimes the number of parts you were thinking about was greater than the number of parts in the whole. In each case, you used an ordered pair to represent the number of parts into which the whole was marked off and also the number of these parts you were thinking about. Each of these ordered pairs is a *fraction*. A fraction is used to represent parts of a whole.

A fraction is an ordered pair of natural numbers whose second component is not zero. The first component of the ordered pair is the *numerator* of the fraction. The second component is the *denominator* of the fraction.

Any natural number can be the numerator of a fraction. Any natural number *except zero* can be the denominator of a fraction.

K For each fraction expressed in D1, what is the numerator? What is the denominator?

Various symbols are used to express a fraction. D2 shows three symbols for the same fraction. You read each of these symbols "two

(2, 4) 2/4 $\frac{2}{4}$

D2

fourths" when it expresses a fraction. You can also use any one of the symbols in D2 to express a rate pair. Then you read each of these symbols "two to four." Whenever you see a symbol like one of those shown in D2, you must decide from what you are reading whether or not the symbol expresses a fraction. These decisions will be easier to make after you learn more about fractions.

Although different kinds of symbols can be used to express fractions, it is convenient to use the same symbol all the time. In this book we will use a symbol like the one below to express a fraction. This symbol expresses the same fraction as the symbol (3, 5) and is read "three fifths."

$$\frac{3}{5}$$

We give symbols for fractions a special name. We call them *fraction numerals*. The symbols $\frac{2}{4}$, $\frac{4}{5}$, $\frac{9}{7}$, and $\frac{3}{1}$ are fraction numerals.

L In D1 fractions were expressed by the symbols (2, 4), (5, 6), (1, 3), (5, 5), (8, 5), (1, 1), (3, 1), (0, 1), and (0, 8). Express each of these fractions by a fraction numeral of the kind we will use in this book. For example, use the symbol $\frac{2}{4}$ instead of the symbol (2, 4).

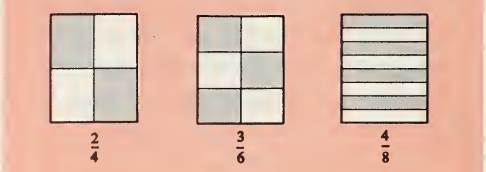
Read each of the fraction numerals in exercises M through R.

M $\frac{1}{2}$ **N** $\frac{2}{3}$ **O** $\frac{7}{5}$ **P** $\frac{9}{15}$ **Q** $\frac{3}{4}$ **R** $\frac{8}{6}$

So far in this lesson you have learned that a fraction is an ordered pair of natural numbers that represents parts of a whole, when the parts are the same size. Now look at D3. The same amount of paper is shaded in each picture. Notice, however, that, because each piece of paper is marked off into a different number of parts, a different fraction represents the number of parts in the whole and the number of these parts that you are to think about. Now you will learn how these different fractions are related.

(2, 4) 2/4 $\frac{2}{4}$

D2



D3

$$\frac{2}{4} \quad \frac{3}{6}$$
$$2 \times 6 = 3 \times 4.$$

D4

$$\frac{3}{6} \quad \frac{4}{8}$$
$$3 \times 8 = 4 \times 6.$$

D5

A First you will learn how $\frac{2}{4}$ and $\frac{3}{6}$ are related. Look at D4. What is the product of the numerator of $\frac{2}{4}$ and the denominator of $\frac{3}{6}$? What is the product of the numerator of $\frac{3}{6}$ and the denominator of $\frac{2}{4}$?

B Is the product of 2 and 6 the same as the product of 3 and 4? Is $2 \times 6 = 3 \times 4$?

C Next you will learn how $\frac{3}{6}$ and $\frac{4}{8}$ are related. Look at D5. What is the product of the numerator of $\frac{3}{6}$ and the denominator of $\frac{4}{8}$? What is the product of the numerator of $\frac{4}{8}$ and the denominator of $\frac{3}{6}$? Is the product of 3 and 8 the same as the product of 4 and 6? Is $3 \times 8 = 4 \times 6$?

D In exercises A, B, and C you worked with $\frac{2}{4}$ and $\frac{3}{6}$ and with $\frac{3}{6}$ and $\frac{4}{8}$. In each case you multiplied the numerator of one fraction and

the denominator of the other fraction to obtain two products. Are these products equal in each case?

For any two fractions, if these two products are equal, we say that the fractions are *equivalent fractions.*

E Look at D6. The display shows how to write and read a symbol for a fraction when variables are used for the numerator and the denominator.

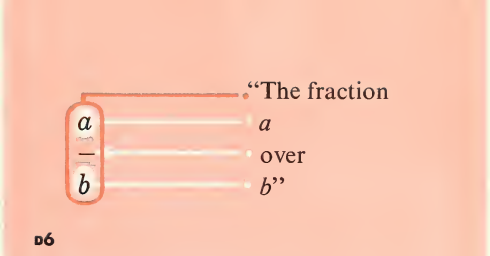
Now you will study the definition of equivalent fractions. Suppose that $\frac{a}{b}$ and $\frac{c}{d}$ are any two fractions. The universe for a and c is N, and the universe for b and d is C. Remember that zero is not a member of C. For each replacement of the variables, if $a \times d = c \times b$, then $\frac{a}{b}$ is equivalent to $\frac{c}{d}$. Also, for each replacement of the variables, if $\frac{a}{b}$ is equivalent to $\frac{c}{d}$, then $a \times d = c \times b$.

Notice that we have expressed the definition by saying "For each replacement of the variables, if $a \times d = c \times b$, then $\frac{a}{b}$ is equivalent to $\frac{c}{d}$." This is a shorter way of saying "For each replacement of the variables, if a true statement is obtained from $a \times d = c \times b$, then $\frac{a}{b}$ is equivalent to $\frac{c}{d}$." From now on in this book we will use the shorter form when we discuss true statements.

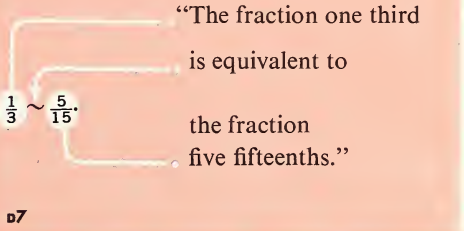
Use the definition of equivalent fractions to decide whether or not $\frac{1}{3}$ is equivalent to each of the fractions expressed in exercises F through K.

F $\frac{2}{6}$ **G** $\frac{4}{12}$ **H** $\frac{10}{30}$ **I** $\frac{15}{45}$ **J** $\frac{30}{90}$ **K** $\frac{5}{15}$

Notice that the definition of equivalent fractions is very much like the definition of equivalent ordered pairs that you learned in unit 4 (page 168). Remember that you used a symbol for equivalence in sentences about equivalent ordered pairs. You can also use the symbol for equivalence in sentences about equivalent fractions.



D6



D7

e quiv a lent fractions (i kwiv′ə lənt). The universe for a and c is N. The universe for b and d is C. For each replacement of the variables, if $a \times d = c \times b$, then $\frac{a}{b}$ is equivalent to $\frac{c}{d}$. Also, for each replacement of the variables, if $\frac{a}{b}$ is equivalent to $\frac{c}{d}$, then $a \times d = c \times b$.

L Look at D7. The display shows how to write and read a sentence about equivalent fractions.

Decide whether each of the following sentences expresses a true statement or a false statement.

M $\frac{4}{3} \sim \frac{12}{9}$. **P** $\frac{1}{2} \sim \frac{2}{1}$. **S** $\frac{4}{7} \sim \frac{12}{14}$.

N $\frac{20}{16} \sim \frac{5}{4}$. **Q** $\frac{0}{5} \sim \frac{0}{8}$. **T** $\frac{22}{2} \sim \frac{11}{2}$.

O $\frac{6}{8} \sim \frac{4}{6}$. **R** $\frac{9}{10} \sim \frac{36}{30}$. **U** $\frac{3}{3} \sim \frac{20}{20}$.

V Think about the fractions $\frac{0}{b}$ and $\frac{0}{d}$. If you replace b and d by any members of C, will you always obtain two fractions that are equivalent to each other? Explain your answer.

In this lesson you studied a special kind of ordered pair of natural numbers that represents parts of a whole. You learned that this kind of ordered pair is a fraction. You also learned what is meant by equivalent fractions.

On your own

Write the words that you should use to read each of the fraction numerals in exercises 1 through 8.

1 $\frac{5}{8}$ 3 $\frac{3}{3}$ 5 $\frac{4}{4}$ 7 $\frac{14}{28}$

2 $\frac{0}{2}$ 4 $\frac{17}{11}$ 6 $\frac{51}{30}$ 8 $\frac{0}{100}$

For each of exercises 9 through 14, decide whether or not the fraction expressed in the exercise is equivalent to $\frac{1}{4}$. If the fraction is not equivalent to $\frac{1}{4}$, explain how you know.

9 $\frac{2}{8}$ 10 $\frac{5}{1}$ 11 $\frac{10}{40}$ 12 $\frac{4}{16}$ 13 $\frac{25}{100}$ 14 $\frac{20}{100}$

For each of exercises 15 through 26, tell whether the sentence expresses a true statement or a false statement.

15 $\frac{10}{6} \sim \frac{5}{2}$. 19 $\frac{32}{16} \sim \frac{1}{2}$. 23 $\frac{0}{8} \sim \frac{0}{30}$.

16 $\frac{8}{5} \sim \frac{16}{10}$. 20 $\frac{3}{4} \sim \frac{3}{4}$. 24 $\frac{6}{10} \sim \frac{15}{25}$.

17 $\frac{72}{45} \sim \frac{24}{15}$. 21 $\frac{3}{5} \sim \frac{18}{20}$. 25 $\frac{7}{3} \sim \frac{3}{7}$.

18 $\frac{7}{8} \sim \frac{30}{40}$. 22 $\frac{12}{16} \sim \frac{9}{12}$. 26 $\frac{7}{7} \sim \frac{29}{29}$.

KEEPING SKILFUL

For exercises 1 through 8, tabulate the set described. $U = N \times N$.

1 $\{(m, n) \mid m + n = 14 \wedge m > n\}$

2 $\{(m, n) \mid 3964 + n = 7261 \wedge n + 709 = m\}$

3 $\{(m, n) \mid m + 8 = n \wedge 127 + m < 132\}$

4 $\{(m, n) \mid 15 - m = n \wedge n + 8 < m\}$

5 $\{(m, n) \mid m + n < 32 \wedge 6m = 162\}$

6 $\{(m, n) \mid m - 18 = n \wedge n < 6\}$

7 $\{(m, n) \mid 3m = n \wedge m + n = 12\}$

8 $\{(m, n) \mid 336 - m = 217 \wedge 119 - m > n\}$

For each of exercises 9 through 16, simplify the expression.

9 $7(3) + 6(19)$ 13 $5(27) \div 3$

10 $4(351 \div 3) + 13$ 14 $19 + (7 \cdot 9) - 23$

11 $476 - 29 + 56$ 15 $(19 + 7)(28 - 13)$

12 $476 - (29 + 56)$ 16 $68 \div 4 \cdot 9$

Tell what prime numbers are factors of each number named in exercises 17 through 24.

17 84 19 37 21 678 23 534

18 65 20 420 22 912 24 111

Rational numbers

In this lesson you will learn about sets of equivalent fractions. First, however, you will learn how to obtain fractions that are equivalent to a given fraction.

A Study D1. By what number can you multiply the numerator of $\frac{2}{3}$ to obtain the numerator of $\frac{6}{9}$? By what number can you multiply the denominator of $\frac{2}{3}$ to obtain the denominator of $\frac{6}{9}$? How can you obtain $\frac{6}{9}$ from $\frac{2}{3}$?

B How can you obtain $\frac{10}{15}$ from $\frac{2}{3}$?

C Is $\frac{6}{9} \sim \frac{2}{3}$? Is $\frac{10}{15} \sim \frac{2}{3}$? Explain your answers.

D What fraction do you obtain when you multiply both the numerator and the denominator of $\frac{2}{3}$ by 6? By 7? By 10? By 100? Is each fraction that you obtained equivalent to $\frac{2}{3}$? How do you know?

E If you multiply both the numerator and the denominator of $\frac{2}{3}$ by the same member of C, do you think that the fraction you obtain will be equivalent to $\frac{2}{3}$? Remember that $C = \{1, 2, 3, \ldots\}$.

D1

$\frac{2}{3}$ $3 \times 2 = 6.$ $\frac{6}{9}$

$3 \times 3 = 9.$

$\frac{6}{9} \sim \frac{2}{3}$.

$\frac{2}{3}$ $5 \times 2 = 10.$ $\frac{10}{15}$

$5 \times 3 = 15.$

$\frac{10}{15} \sim \frac{2}{3}$.

F Look at D2. By what number can you divide the numerator of $\frac{6}{12}$ to obtain the numerator of $\frac{1}{2}$? By what number can you divide the denominator of $\frac{6}{12}$ to obtain the denominator of $\frac{1}{2}$? How can you obtain $\frac{1}{2}$ from $\frac{6}{12}$?

G Is 6 a common factor of the numerator and the denominator of $\frac{6}{12}$? Is 2 also a common factor of the numerator and the denominator of $\frac{6}{12}$? How can you obtain $\frac{3}{6}$ from $\frac{6}{12}$?

REMINDER
A number that is a factor of two other numbers is a common factor of the two numbers.
See lesson 70, page 301.

H Is $\frac{1}{2} \sim \frac{6}{12}$? Is $\frac{3}{6} \sim \frac{6}{12}$? Explain your answers.

I If you divide both the numerator and the denominator of $\frac{6}{12}$ by a common factor, will you always obtain a fraction that is equivalent to $\frac{6}{12}$?

J If you divide both the numerator and the denominator of $\frac{15}{45}$ by a common factor, will you always obtain a fraction that is equivalent to $\frac{15}{45}$?

You can use variables to show that, if you multiply both the numerator and the denominator of any fraction by the same counting number, you obtain an equivalent fraction. You can also show that, if you divide both the numerator and the denominator of a fraction by a common factor, you obtain an equivalent fraction.

A By what member of C do you multiply the numerator and the denominator of $\frac{5}{7}$ to obtain $\frac{4 \cdot 5}{4 \cdot 7}$? You know that $\frac{5}{7} \sim \frac{4 \cdot 5}{4 \cdot 7}$ if $5(4 \cdot 7) = (4 \cdot 5)7$. Is $5(4 \cdot 7) = (4 \cdot 5)7$? Is $\frac{5}{7} \sim \frac{4 \cdot 5}{4 \cdot 7}$?

Now think of $\frac{a}{b}$ as any fraction. The universe for a is N. The universe for b is C. Next think of k as any member of C.

$\frac{6}{12}$ $\quad 6 \div 6 = 1.$ $\quad \frac{1}{2}$
$\quad 12 \div 6 = 2.$
$\frac{1}{2} \sim \frac{6}{12}.$

$\frac{6}{12}$ $\quad 6 \div 2 = 3.$ $\quad \frac{3}{6}$
$\quad 12 \div 2 = 6.$
$\frac{3}{6} \sim \frac{6}{12}.$

D2

Universe for a = N.
Universe for b and k = C.
A $a(kb) = (ak)b$.
B $(ak)b = (ka)b$.
C $a(kb) = (ka)b$.

D3

B When you multiply the numerator of $\frac{a}{b}$ by k, you obtain ka. What do you obtain when you multiply the denominator of $\frac{a}{b}$ by k? What fraction do you obtain if you multiply both the numerator and the denominator of $\frac{a}{b}$ by k?

If you can show that, for each replacement of the variables, $\frac{a}{b}$ is equivalent to $\frac{ka}{kb}$, then you know that when you multiply the numerator and the denominator of any fraction by the same counting number, you obtain an equivalent fraction.

C How do you obtain $a(kb) = (ka)b$ from ? $\frac{a}{b} \sim \frac{ka}{kb}$? Do you think that, for each replacement of the variables, $a(kb) = (ka)b$?

D You can use certain properties of multipli- ? cation of natural numbers to show that, for each replacement of the variables, $a(kb) = (ka)b$. Look at sentence A in D3. Which property tells you that, for each replacement of the variables, $a(kb) = (ak)b$?

E Read sentence B. Which property of multiplication of natural numbers tells you that, for each replacement of the variables, $(ak)b = (ka)b$?

F How do you obtain the condition expressed by sentence C from the conditions expressed by sentences A and B? For each replacement of the variables, is $a(kb) = (ka)b$?

G For each replacement of the variables, is $\frac{a}{b} \sim \frac{ka}{kb}$? If you multiply the numerator and the denominator of any fraction by the same member of C, do you obtain an equivalent fraction?

H Is the fraction $\frac{8}{6}$ the same fraction as $\frac{2 \cdot 4}{2 \cdot 3}$? Is 2 a common factor of 8 and 6?

I Divide both the numerator and the denominator of $\frac{2 \cdot 4}{2 \cdot 3}$ by 2. What fraction do you obtain? Is this fraction equivalent to $\frac{2 \cdot 4}{2 \cdot 3}$? To $\frac{8}{6}$?

J Is the fraction $\frac{7 \cdot 3}{7 \cdot 5}$ the same fraction as $\frac{21}{35}$? Is 7 a common factor of 21 and 35?

K Divide both the numerator and the denominator of $\frac{21}{35}$ by 7. What fraction do you obtain? Is this fraction equivalent to $\frac{21}{35}$? To $\frac{7 \cdot 3}{7 \cdot 5}$?

L Now think again of the fraction $\frac{ka}{kb}$. What is a common factor of ka and kb? For each replacement of the variables, $ka \div k = a$. For each replacement of the variables, is $kb \div k = b$?

M What fraction do you obtain when you divide the numerator and the denominator of $\frac{ka}{kb}$ by k? You already know that, for each replacement of the variables, $\frac{ka}{kb} \sim \frac{a}{b}$. If you divide both the numerator and the denominator of any fraction by a common factor, do you obtain an equivalent fraction?

You have been working with the *reduction property of fractions*. This property is expressed below.

If you multiply the numerator and the denominator of a given fraction by the same counting number, the fraction that you obtain is equivalent to the given fraction. If you divide the numerator and the denominator of a given fraction by a common factor, the fraction that you obtain is equivalent to the given fraction.

Universe for a = N.
Universe for b and k = C.

A $a(kb) = (ak)b$.
B $(ak)b = (ka)b$.
C $a(kb) = (ka)b$.

D3

"The set whose members are $\{\frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \ldots\}$ and all fractions equivalent to $\frac{1}{5}$"

D4

$D = \{\frac{8}{9}, \frac{16}{18}, \frac{24}{27}, \ldots\}$.
$E = \{\frac{6}{5}, \frac{12}{10}, \frac{18}{15}, \ldots\}$.
$F = \{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \ldots\}$.
$G = \{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \ldots\}$.

D5

The same property also may be expressed as follows:

The universe for a *is N. The universe for* b *and* k *is C. For each replacement of the variables,* $\frac{a}{b} \sim \frac{ka}{kb}$.

Decide whether each of the sentences below expresses a true statement or a false statement.

N $\frac{1}{4} \sim \frac{10}{40}$.
O $\frac{9}{6} \sim \frac{3}{2}$.
P $\frac{0}{6} \sim \frac{2}{12}$.
Q $\frac{11}{11} \sim \frac{11}{11}$.
R $\frac{35}{5} \sim \frac{5}{1}$.
S $\frac{3}{5} \sim \frac{15}{20}$.
T $\frac{12}{16} \sim \frac{3}{4}$.
U $\frac{5}{8} \sim \frac{15}{24}$.
V $\frac{21}{18} \sim \frac{7}{3}$.
W $\frac{8}{9} \sim \frac{32}{36}$.
X $\frac{6}{18} \sim \frac{2}{9}$.
Y $\frac{7}{49} \sim \frac{1}{7}$.

From the reduction property you know that you can use multiplication or division to obtain fractions equivalent to a given fraction.

A Name three fractions that are equivalent to $\frac{1}{5}$. Is the set that contains $\frac{1}{5}$ and all fractions equivalent to $\frac{1}{5}$ a finite set? An infinite set?

B Is the set that contains a given fraction and all fractions equivalent to the given fraction a finite set? An infinite set?

C D4 shows how to write and read a tabulation of the set whose members are $\frac{1}{5}$ and all fractions equivalent to $\frac{1}{5}$. Is $\frac{7}{35}$ a member of this set? Is $\frac{20}{100}$ a member of this set?

D Study the tabulations in D5. The names of three fractions are used in each tabulation. For each set, name two other fractions that are also members of that set.

E Look again at D5. Think about the fraction $\frac{8}{9}$. What are the factors of 8? Of 9? What is the common factor of 8 and 9?

F Think about the fraction $\frac{6}{5}$. What are the factors of 6? Of 5? What is the common factor of 6 and 5?

G Think about the fraction $\frac{2}{3}$. What are the factors of 2? Of 3? What is the common factor of 2 and 3?

H Think about the fraction $\frac{3}{4}$. What is the common factor of 3 and 4?

In the fractions $\frac{8}{9}$, $\frac{6}{5}$, $\frac{2}{3}$, and $\frac{3}{4}$, the numerator and the denominator have only the number 1 as a common factor. Numbers whose only common factor is the number 1 are *relatively prime numbers.*

REMINDER

A prime number is a counting number that has exactly two factors, itself and 1.

See lesson 69, page 296.

I You know that 8 and 9 are relatively prime numbers. Is 8 a prime number? Is 9 a prime number?

J Which of the numbers, 6 or 5, is a prime? Are both 2 and 3 prime numbers? Are both 3 and 4 prime numbers?

Notice that, if two numbers are relatively prime, it does not mean that either of the numbers is itself necessarily prime.

The fractions $\frac{8}{9}$, $\frac{6}{5}$, $\frac{2}{3}$, and $\frac{3}{4}$ are *basic fractions.* A basic fraction is a fraction whose numerator and denominator are relatively prime.

K You discovered that there is one basic fraction in each set tabulated in D5. Is there more than one basic fraction in any of these sets?

L Think about the fraction $\frac{4}{15}$. What are the factors of 4? Of 15? How do you know that $\frac{4}{15}$ is a basic fraction?

M Name three fractions that are equivalent to the fraction $\frac{4}{15}$.

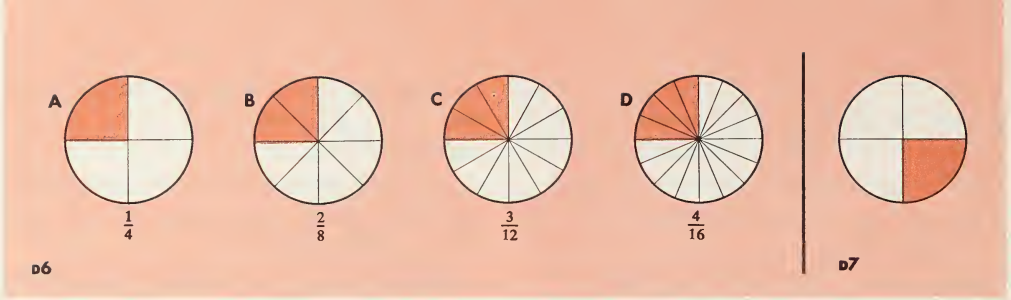
N Think about the fraction $\frac{12}{3}$. What are the factors of 12? Of 3? How do you know that $\frac{12}{3}$ is not a basic fraction?

O Name the basic fraction that is equivalent to $\frac{12}{3}$.

Now that you know how to obtain sets of equivalent fractions, you will learn more about these sets.

rel a tive ly prime numbers (rel′ə tiv lē prīm). Numbers whose only common factor is the number 1. The numbers 9 and 16 are relatively prime. 16 and 10 are not relatively prime.

ba sic fraction (bā′sik). A fraction whose numerator and denominator are relatively prime. $\frac{3}{4}$, $\frac{16}{9}$, and $\frac{7}{5}$ are basic fractions. $\frac{9}{12}$, $\frac{3}{4}$, and $\frac{4}{6}$ are not basic fractions.



A Look at D6. The disks shown in pictures A, B, C, and D are all the same size. Each disk is marked off into a different number of parts of the same size. The parts you are to think about are red. For each picture, what fraction represents the number of red parts and also the number of parts marked off? Is each of these fractions equivalent to each of the other fractions?

B Look again at D6. Is the amount that is red in each disk the same? Are 3 of the 12 parts of disk C the same amount as 4 of the 16 parts of disk D? Are 4 of the 16 parts of disk D the same amount as 2 of the 8 parts of disk B?

C Imagine that you are going to draw four more pictures of disks like those in D6. For each picture you will use the denominator of a different member of $\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \ldots\}$ to decide into how many parts of the same size you will mark off the disk. Could you use the denominator 20? The denominator 24? Name two more denominators that you could use.

D If you use the denominator 20, will you mark off a disk into 20 parts of the same size? Into how many parts will you mark off a disk if you use the denominator 24? If you use the denominator 32?

E Now imagine that you are going to color each of your disks so that the colored amount is the same as the red amount in pictures A, B, C, and D. If your disk is marked off into 20 parts, should you color 5 of these parts? Is $\frac{5}{20}$ a member of $\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \ldots\}$?

F How many parts should you color if your disk is marked off into 24 parts? How many parts should you color if your disk is marked off into 32 parts? Is the number of parts you color the same as the numerator of a member of $\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \ldots\}$?

Each member of $\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \ldots\}$ represents a different way to think about the same amount of the disk.

The disk shown in D7 is the same size as the disks shown in D6. The red amount of the disk shown in D7 is the same as the red amount in each picture in D6. You can associate any member of $\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \ldots\}$ with the red amount. Therefore, you can associate the whole set with the red amount.

$\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \ldots\}$ is a *rational number of arithmetic*. It is the number that is associated with the red amount of the disk pictured in D7.

A rational number of arithmetic is a set of equivalent fractions. A rational number of arithmetic contains a basic fraction and all fractions equivalent to the basic fraction. In this book, when we say "rational number," we will mean "rational number of arithmetic."

ra tion al number of arithmetic (rash′ən əl). A set of equivalent fractions. A rational number of arithmetic contains a basic fraction and all fractions equivalent to the basic fraction. $\{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \ldots\}$ and $\{\frac{5}{2}, \frac{10}{4}, \frac{15}{8}, \ldots\}$ are rational numbers.

Which sets of fractions tabulated below are rational numbers? Explain your answers.

G $\{\frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \ldots\}$ I $\{\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \ldots\}$

H $\{\frac{2}{7}, \frac{6}{14}, \frac{9}{21}, \ldots\}$ J $\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \ldots\}$

K Is it possible for a rational number to contain more than one basic fraction? Explain your answer.

Because a rational number is a set of equivalent fractions, you can use any member of the set to determine, or *indicate*, the rational number. For example, the fraction $\frac{2}{3}$ indicates a set of equivalent fractions that is a rational number. The fraction $\frac{4}{6}$ indicates a set of equivalent fractions that is a rational number. Since the fractions $\frac{2}{3}$ and $\frac{4}{6}$ are members of the same set of equivalent fractions, they indicate the same rational number. Therefore, the statement $\frac{4}{6} = \frac{2}{3}$ is true.

L For each replacement of n, does the fraction $\frac{2n}{3n}$ indicate the rational number $\frac{2}{3}$? The universe for n is C. For each replacement of n, does $\frac{2n}{3n} = \frac{2}{3}$? Explain your answers.

You have already learned that $\frac{a}{b}$ and $\frac{ka}{kb}$ are members of the same set of equivalent fractions. Thus, $\frac{a}{b}$ and $\frac{ka}{kb}$ indicate the same rational number. The rational numbers $\frac{a}{b}$ and $\frac{ka}{kb}$ are therefore equal. You have been studying a property of rational numbers. This property is the *reduction property of rational numbers.* The property is expressed below.

The universe for a *is N. The universe for* b *and* k *is C. For each replacement of* a, b, *and* k, $\frac{a}{b} = \frac{ka}{kb}$.

Decide whether each sentence below expresses a true statement or a false statement.

M $\frac{1}{2} = \frac{5}{10}$. O $\frac{7}{5} = \frac{14}{10}$. Q $\frac{3}{4} = \frac{9}{16}$.

N $\frac{3}{4} = \frac{4}{3}$. P $\frac{39}{13} = \frac{3}{1}$. R $\frac{25}{5} = \frac{100}{25}$.

Now you know how to use multiplication or division to obtain fractions equivalent to a given fraction. You also know that a rational number of arithmetic is a set of equivalent fractions.

On your own

Five fractions that are equivalent to $\frac{30}{50}$ are named in exercises 1 through 5. Tell how you can obtain each fraction from $\frac{30}{50}$.

1 $\frac{15}{25}$ 2 $\frac{6}{10}$ 3 $\frac{60}{100}$ 4 $\frac{3}{5}$ 5 $\frac{90}{150}$

Each of the fractions named in exercises 6 through 10 is equivalent to $\frac{70}{35}$. Tell how you can obtain each fraction from $\frac{70}{35}$.

6 $\frac{2}{1}$ 7 $\frac{140}{70}$ 8 $\frac{10}{5}$ 9 $\frac{14}{7}$ 10 $\frac{210}{105}$

What is the basic fraction of each rational number named in exercises 11 through 18?

11 $\frac{3}{18}$ 13 $\frac{18}{4}$ 15 $\frac{60}{25}$ 17 $\frac{17}{70}$

12 $\frac{3}{4}$ 14 $\frac{56}{64}$ 16 $\frac{100}{10}$ 18 $\frac{120}{30}$

Each of the sentences in exercises 19 through 24 expresses a statement about two rational numbers. For each exercise, tell whether the statement is true or false. Explain your answer.

19 $\frac{1}{3} = \frac{5}{15}$. 21 $\frac{3}{5} = \frac{9}{25}$. 23 $\frac{5}{3} = \frac{3}{5}$.

20 $\frac{5}{2} = \frac{10}{4}$. 22 $\frac{7}{8} = \frac{70}{80}$. 24 $\frac{6}{5} = \frac{36}{25}$.

KEEPING SKILFUL

Find the difference.

1 5003, 3198
2 1692, 792
3 7813, 5689
4 14865, 7619
5 36417, 29392
6 220467, 129385

Divide

7 918 by 27.
8 4796 by 68.
9 8487 by 41.
10 10002 by 84.
11 25676 by 279.
12 37648 by 104.
13 14358 by 112.
14 34020 by 243.

For each ordered pair named below, first name the natural number onto which you can map the ordered pair by addition. Then name the natural number onto which you can map the ordered pair by multiplication.

15 (13, 47) 18 (78, 302) 21 (216, 113)
16 (20, 39) 19 (452, 19) 22 (308, 490)
17 (126, 12) 20 (106, 49) 23 (186, 296)

7 73 Exploring ideas

Ordering of the rational numbers

You have learned that a rational number of arithmetic is a set of equivalent fractions. In this lesson you will first learn about the set whose members are all the rational numbers of arithmetic. Then you will learn how to use the idea of "less than" to order the members of this set.

A The names of some rational numbers are given in D1. Is the rational number $\frac{2}{2}$ the same rational number as $\frac{1}{1}$? Is the rational number $\frac{2}{2}$ named in D1? How do you know?

B Is $\frac{2}{4}$ named in D1? Is $\frac{9}{3}$ named?

C Is $\frac{3}{8}$ named in D1? Is $\frac{16}{9}$ named?

D Name six other rational numbers that are not named in D1.

E Think about the set of all possible basic fractions. Is this set an infinite set? Does each basic fraction indicate a different rational number?

F Now think about the set of all possible rational numbers of arithmetic that are indicated by basic fractions. Is this set an infinite set? Explain your answer.

We will use the symbol below at the right as a name of the set of all possible rational numbers of arithmetic. This symbol is read "the set of rational numbers of arithmetic."

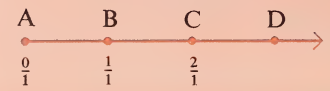
R_a

In earlier lessons you associated points in a line with natural numbers. You learned that a line that has points associated with natural numbers is called a natural-number line. In the following exercises, you will see that points in a line may also be associated with rational numbers.

$\frac{0}{1}$ $\frac{1}{1}$ $\frac{1}{2}$ $\frac{2}{1}$ $\frac{1}{3}$ $\frac{3}{1}$ $\frac{1}{4}$ $\frac{4}{1}$

$\frac{1}{5}$ $\frac{5}{1}$ $\frac{1}{6}$ $\frac{6}{1}$ $\frac{2}{5}$ $\frac{5}{2}$ $\frac{3}{4}$ $\frac{4}{3}$

D1



D2

G Look at the picture of a line in D2. What number has been assigned to point A? What number has been assigned to point B?

The points in the line associated with $\frac{0}{1}$ and $\frac{1}{1}$ determine a segment. In a line like the one represented in D2, you can locate the point associated with $\frac{1}{1}$ anywhere in the line to the right of the point associated with $\frac{0}{1}$. But, once you have located this point and the segment has been determined, it is convenient to use this segment to locate other points.

> REMINDER
> A segment is a set of points whose members are two given points and all points between these two.
> *See lesson 10, page 38.*

H Look again at D2. The segment determined by point $\frac{1}{1}$ and point C is congruent to the segment determined by point $\frac{0}{1}$ and point $\frac{1}{1}$. What number is associated with point C?

I The segment determined by points C and D is also congruent to the segment determined by points A and B. What rational number should you assign to point D?

J Explain how to locate points associated with $\frac{4}{1}$ and $\frac{5}{1}$ in the line represented in D2.

K Now look at the picture of a line in D3. $\overline{TW} \cong \overline{PT}$. What rational number should you assign to point W?

L What number is associated with point Q? With point R?

To locate the point associated with $\frac{1}{4}$, we separated the segment whose endpoints are $\frac{0}{1}$ and $\frac{1}{1}$ into four congruent segments, $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, and $\overline{ST}$. We assigned the rational number $\frac{1}{4}$ to point Q because Q is the endpoint at the right of the first of the four segments. We assigned $\frac{2}{4}$ to point R because R is the endpoint at the right of the second of the four segments.

M What rational number should you assign to point S?

N Why could you also assign $\frac{4}{4}$ to point T?

O Now imagine that you have separated $\overline{PT}$ into three congruent segments. You should assign the rational number $\frac{1}{3}$ to the right endpoint of the first of the three segments. To which point should you assign $\frac{2}{3}$?

P Look again at D3. $\frac{4}{3}$ is assigned to point U. Explain how point U was located.

Q How was the point associated with $\frac{3}{2}$ located? How was the point associated with $\frac{13}{5}$ located?

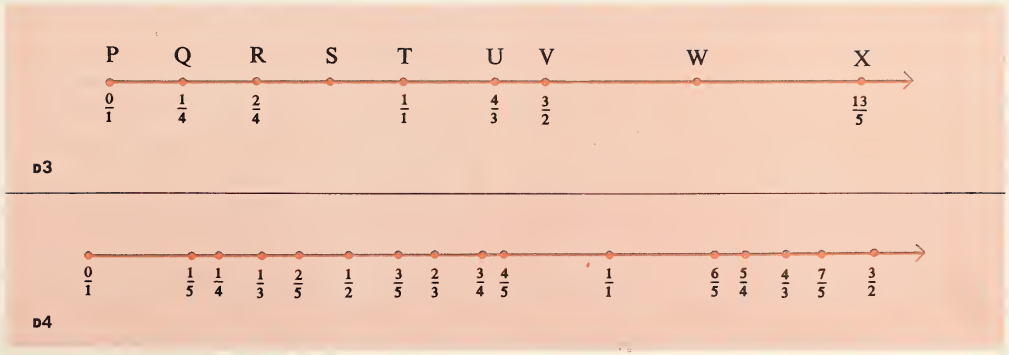
R Explain how to locate the point associated with the rational number $\frac{a}{b}$ in a line like the one represented in D3. The universe for a is N, and the universe for b is C.

A *rational-number line* is a set of points that are associated with rational numbers. Each member of the set is associated with a rational number, and each rational number is associated with a member of the set. The sets of points represented in D2 and D3 are examples of rational-number lines. Such pictures help you to think about sets of rational numbers.

S Another rational-number line is represented in D4. Tabulate the set of rational numbers named in D4 that are associated with points between $\frac{0}{1}$ and $\frac{1}{1}$. Are these the only rational numbers associated with points between $\frac{0}{1}$ and $\frac{1}{1}$? Do you think that there are infinitely many rational numbers associated with points between $\frac{0}{1}$ and $\frac{1}{1}$?

The arrow to the right of the picture of the line tells you that the points you can associate with rational numbers go on and on.

Suppose you want to locate the point associated with $\frac{4}{6}$ in the rational-number line represented in D4. First you locate points that separate the segment whose endpoints are $\frac{0}{1}$ and $\frac{1}{1}$ into six congruent segments. Then you assign $\frac{4}{6}$ to the right endpoint of the fourth segment.



Point $\frac{4}{6}$ is the same point as $\frac{2}{3}$. This is true because the right endpoint of the second of three congruent segments is the same point as the right endpoint of the fourth of six congruent segments. $\frac{2}{3}$ and $\frac{4}{6}$ are both associated with the same point. $\frac{2}{3}$ and $\frac{4}{6}$ are the same rational number.

T Suppose you locate the point associated with $\frac{6}{10}$. Is it the same point as the one associated with $\frac{3}{5}$? Are $\frac{3}{5}$ and $\frac{6}{10}$ the same rational number?

U Give three other names for the point associated with $\frac{2}{3}$. Give three other names for the point associated with $\frac{3}{5}$.

V How would you locate the point associated with $\frac{11}{10}$? Give three other names for this point.

W Are there infinitely many points in a ra- ? tional-number line? Are there infinitely many names for each point? Explain your answers.

So far in this lesson, you have learned about the set of rational numbers and about the rational-number line. In unit 6 you learned how to order the natural numbers using the idea of "less than." The rational numbers can also be ordered. A rational-number line helps you to think about this order.

A Look again at D4. Is point $\frac{1}{4}$ to the left of point $\frac{1}{2}$?

In a number line like the one represented in D4, if the point associated with one rational number is to the left of the point associated with a second rational number, then the first number is less than the second number. Since point $\frac{1}{4}$ is to the left of point $\frac{1}{2}$, the statement $\frac{1}{4} < \frac{1}{2}$ is true.

B Look again at D4. Is the point associated with $\frac{1}{2}$ to the left of the point associated with $\frac{4}{3}$? Is $\frac{1}{2} < \frac{4}{3}$? Explain your answer.

C Is $\frac{3}{4} < \frac{1}{1}$? Is $\frac{1}{3} < \frac{2}{5}$? Is $\frac{3}{2} < \frac{7}{5}$? Is $\frac{4}{3} < \frac{3}{4}$?

D Is any point located to the left of point $\frac{0}{1}$ in the rational-number line? Is there any rational number of arithmetic that is less than $\frac{0}{1}$?

E Look at D5. The sentence in A tells you that $\frac{5}{8}$ is less than what rational number?

In a rational-number line the point associated with $\frac{5}{8}$ is to the left of the point associated with $\frac{3}{4}$. From this you know that $\frac{5}{8}$ is less than $\frac{3}{4}$. Now you will learn how to decide whether or not $\frac{5}{8}$ is less than $\frac{3}{4}$ without using a rational-number line.

You know that fractions are used to indicate rational numbers. When we mention a numerator or a denominator, we will mean the numerator or denominator of a fraction. However, the fraction referred to will indicate a rational number.

F Look again at D5. The arrow in B indicates that the numerator of $\frac{5}{8}$ and the denominator of $\frac{3}{4}$ are to be multiplied. What is the product of 5 and 4?

G What does the arrow in C indicate? What is the product of 3 and 8?

H Does each sentence in D express a true statement?

From D you can see that the product of the numerator of the first fraction and the denominator of the second fraction is less than the product of the numerator of the second frac-

D4

A $\frac{5}{8} < \frac{3}{4}$.

B $\frac{5}{8}$ $\frac{3}{4}$

5×4

C $\frac{5}{8}$ $\frac{3}{4}$

3×8

D $\frac{5}{8}$ $\frac{3}{4}$

$5 \times 4 < 3 \times 8.$

$20 < 24.$

D5

$\frac{7}{6}$ $\frac{5}{5}$

$7 \times 5 < 5 \times 6.$

D6

Universe for a and c = N.

Universe for b and d = C.

$\frac{a}{b} < \frac{c}{d}$.

$a \times d < c \times b$.

E $\frac{2}{3} < \frac{4}{5}$.

F $2 \times 5 < 4 \times 3$.

D7

"less than" for rational numbers. The universe for a and c is N. The universe for b and d is C. For each replacement of the variables, if $ad < cb$, then $\frac{a}{b} < \frac{c}{d}$.

tion and the denominator of the first fraction. This means that the rational number indicated by the first fraction is less than the rational number indicated by the second fraction. In other words, since $5 \times 4 < 3 \times 8$ is a true statement, $\frac{5}{8} < \frac{3}{4}$ is also a true statement.

I What two rational numbers are named in D6? The arrows indicate that two pairs of numbers are to be multiplied. Name these pairs of numbers. Does the sentence in D6 express a true statement?

Since $7 \times 5 < 5 \times 6$ is a false statement, $\frac{7}{6} < \frac{5}{5}$ is also a false statement.

You can use variables to help you develop the definition of "less than" for rational numbers.

J Name the variables in the conditions expressed in D7. The universe for a and c is N. The universe for b and d is C. Can a and c be replaced by 0? Can b and d be replaced by 0? Explain your answers.

K Look at sentence E in D7. What replacements were made for the variables in $\frac{a}{b} < \frac{c}{d}$ to obtain the statement expressed by sentence E? Were the same replacements made in $a \times d < c \times b$ to obtain the statement expressed by sentence F?

L Does sentence F express a true statement? Since $2 \times 5 < 4 \times 3$ is a true statement, $\frac{2}{3} < \frac{4}{5}$ is also a true statement.

Now we can say exactly what we mean by the idea of "*less than*" *for rational numbers.* For each replacement of the variables, if $ad < cb$, then the rational number $\frac{a}{b}$ is less than the rational number $\frac{c}{d}$.

Which sentences below express true statements? Explain each answer by using the definition of "less than."

M $\frac{4}{3} < \frac{5}{4}$.

N $\frac{5}{3} < \frac{14}{8}$.

O $\frac{12}{2} < \frac{6}{1}$.

P $\frac{116}{59} < \frac{77}{40}$.

Q $\frac{3}{8} < \frac{7}{5}$.

R $\frac{3}{5} < \frac{3}{4}$.

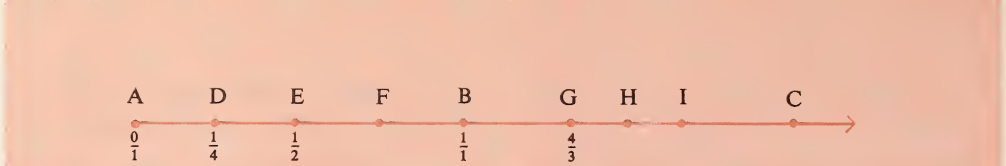
S $\frac{7}{26} < \frac{9}{20}$.

T $\frac{15}{60} < \frac{19}{41}$.

U Suppose that the same counting number is used to replace b and d in $\frac{a}{b} < \frac{c}{d}$. What can you say about the replacements that must be made for a and c to obtain a true statement from $\frac{a}{b} < \frac{c}{d}$? The universe for a and c is N.

Now you know what the idea of "less than" means for rational numbers. You also know how to use this idea to order these numbers. Next you will study the meaning of "greater than."

A You know that $\frac{2}{3} < \frac{4}{5}$. The inequality $\frac{4}{5} > \frac{2}{3}$ gives you the same information as $\frac{2}{3} < \frac{4}{5}$. Does $\frac{9}{10} > \frac{1}{2}$ give you the same information as $\frac{1}{2} < \frac{9}{10}$?



D8

B What inequality that involves the idea of "greater than" gives you the same information as $\frac{4}{3} < \frac{9}{2}$? As $\frac{2}{7} < \frac{3}{6}$? As $\frac{17}{5} < \frac{40}{3}$?

C If you obtain a true statement from $\frac{a}{b} > \frac{c}{d}$, ? how is the position of point $\frac{a}{b}$ related to the position of point $\frac{c}{d}$ in a number line like the ones you have studied in this lesson? The universe for a and c is N. The universe for b and d is C.

D If you obtain a true statement from $\frac{a}{b} > \frac{c}{d}$, ? how is the number obtained from ad related to the number obtained from cb? The universe for a and c is N. The universe for b and d is C.

You can see that the idea of "greater than" can be explained by using the idea of "less than" and that $\frac{c}{d} > \frac{a}{b}$ also means $\frac{a}{b} < \frac{c}{d}$.

In this lesson you have learned to associate the rational numbers of arithmetic with points in a line. You have also learned that you can use the idea of "less than" to order the rational numbers.

On your own

Give three other names for each of the rational numbers named in exercises 1 through 8.

1 $\frac{6}{12}$ 2 $\frac{700}{1}$ 3 $\frac{3}{7}$ 4 $\frac{20}{10}$ 5 $\frac{0}{900}$ 6 $\frac{15}{15}$ 7 $\frac{16}{48}$ 8 $\frac{1}{5}$

Use the number line represented in D8 as you answer the questions in exercises 9 through 14.

9 $\overline{BC} \cong \overline{AB}$. What rational number should you assign to point C?

10 $\overline{EF} \cong \overline{AD}$. What rational number should you assign to point F?

11 $\overline{BH} \cong \overline{AE}$. What rational number should you assign to point H?

12 $\overline{GI} \cong \overline{BG}$. What rational number should you assign to point I?

13 Give three other names for the point associated with $\frac{4}{3}$.

14 Give three other names for the point associated with $\frac{1}{4}$.

For each of exercises 15 through 18, arrange the rational numbers named in order according to the idea of "less than." Begin with the name of the least number.

15 $\frac{2}{3}$ $\frac{6}{3}$ $\frac{4}{3}$ $\frac{0}{3}$ $\frac{5}{3}$ $\frac{8}{3}$

16 $\frac{3}{12}$ $\frac{7}{12}$ $\frac{4}{12}$ $\frac{12}{12}$ $\frac{9}{12}$ $\frac{5}{12}$

17 $\frac{4}{10}$ $\frac{0}{10}$ $\frac{8}{10}$ $\frac{9}{10}$ $\frac{6}{10}$ $\frac{12}{10}$

18 $\frac{82}{100}$ $\frac{74}{100}$ $\frac{106}{100}$ $\frac{99}{100}$ $\frac{110}{100}$

Tell which sentences in exercises 19 through 27 express true statements and which express false statements.

19 $\frac{1}{2} < \frac{1}{1}$.
20 $\frac{7}{8} < \frac{5}{6}$.
21 $\frac{32}{35} < \frac{19}{20}$.
22 $\frac{0}{6} < \frac{1}{100}$.
23 $\frac{4}{3} > \frac{3}{2}$.
24 $\frac{3}{3} > \frac{11}{12}$.
25 $\frac{925}{4} < \frac{475}{2}$.
26 $\frac{4}{3} < \frac{24}{18}$.
27 $\frac{3}{50} > \frac{0}{80}$.

KEEPING SKILFUL

For each pair of numbers named in exercises 1 through 6, first find a complete factorization of each number. Then name the greatest common factor of the two numbers.

1 12, 40
2 24, 84
3 45, 72
4 35, 145
5 70, 102
6 132, 150

Tabulate each set described in exercises 7 through 11. $U = C \times C$.

7 $\{(c, d) | c/100 \sim d/5 \wedge d + 16 = 23\}$
8 $\{(c, d) | 7/9 \sim c/d \wedge c + d < 50\}$
9 $\{(c, d) | 3/2 \sim 21/d \wedge 104 - d = c\}$
10 $\{(c, d) | c/d \sim 3/5 \wedge c + d = 24\}$
11 $\{(c, d) | c/50 \sim 112/d \wedge c = 448 \div 8\}$

CHECKING UP

If you have trouble with these two tests, you should refer to lessons 71, 72, and 73.

Test 80

Each rational number named in exercises 1 through 10 is associated with the red amount of a disk pictured in the display below. Tell which picture each number refers to.

1 $\frac{2}{4}$ **3** $\frac{5}{4}$ **5** $\frac{10}{8}$ **7** $\frac{1}{2}$ **9** $\frac{4}{4}$

2 $\frac{6}{8}$ **4** $\frac{2}{2}$ **6** $\frac{3}{4}$ **8** $\frac{1}{1}$ **10** $\frac{9}{12}$

11 What number is the numerator of the fraction $\frac{3}{7}$?

12 What number is the denominator of the fraction $\frac{1000}{356}$?

Tell whether each of the sentences about fractions in exercises 13 through 21 expresses a true statement or a false statement.

13 $\frac{8}{6} \sim \frac{3}{4}$. **16** $\frac{12}{3} \sim \frac{9}{4}$. **19** $\frac{7}{9} \sim \frac{63}{81}$.

14 $\frac{18}{30} \sim \frac{12}{20}$. **17** $\frac{4}{5} \sim \frac{28}{35}$. **20** $\frac{15}{1} \sim \frac{60}{5}$.

15 $\frac{3}{12} \sim \frac{5}{20}$. **18** $\frac{3}{100} \sim \frac{100}{3}$. **21** $\frac{1000}{20} \sim \frac{100}{1}$.

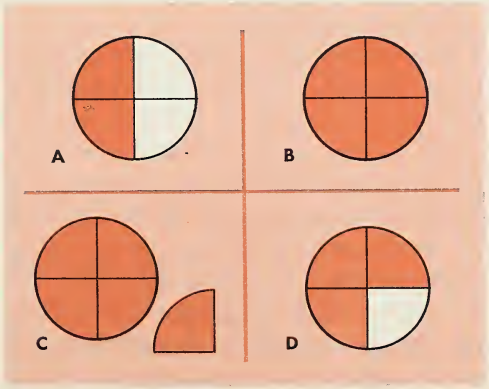
Each of the fractions named in exercises 22 through 31 is equivalent to $\frac{36}{72}$. Tell how you can obtain each fraction from $\frac{36}{72}$.

22 $\frac{3}{6}$ **24** $\frac{4}{8}$ **26** $\frac{12}{24}$ **28** $\frac{144}{288}$ **30** $\frac{9}{18}$

23 $\frac{72}{144}$ **25** $\frac{108}{216}$ **27** $\frac{1}{2}$ **29** $\frac{2}{4}$ **31** $\frac{180}{360}$

Test 81

32 Is $\frac{14}{15}$ a basic fraction? How do you know?



For each of exercises 33 through 37, tell what basic fraction indicates the rational number named.

33 $\frac{15}{18}$ **34** $\frac{12}{21}$ **35** $\frac{55}{100}$ **36** $\frac{27}{18}$ **37** $\frac{24}{32}$

For each of exercises 38 through 41, tell whether the sentence expresses a true statement or a false statement about rational numbers.

38 $\frac{7}{8} > \frac{6}{7}$. **39** $\frac{14}{28} = \frac{2}{7}$. **40** $\frac{16}{3} < \frac{21}{4}$. **41** $\frac{2}{5} < \frac{3}{9}$.

SPECIAL CHALLENGE

As you work with greater and greater natural numbers, you will find that there seem to be fewer primes in each succeeding thousand. There are:

168 primes between 1 and 1000,
135 primes between 1000 and 2000,
127 primes between 2000 and 3000, and
120 primes between 3000 and 4000.

Since there seem to be fewer primes in each succeeding thousand, you might think that there is a greatest prime. In other words, you might think that the set of primes is a finite set. If the set of primes is an infinite set, then there is no greatest prime number.

Either there is or there is not a greatest prime. First we will suppose that there is a greatest prime, and that the set of all prime numbers is a finite set. We will call this set "P." The members of P have been ordered by the idea of "less than" so that p_1 is the first prime, 2; p_2 is the second prime, 3; p_3 is the third prime, 5; and so on.

$P = \{p_1, p_2, p_3, \ldots, p_k\}$.

A What primes are represented by p_4, p_5, p_6, and p_7?

B The last member of P is represented by p_k. Since P is assumed to be a finite set, k must be a definite counting number. If we assume that P contains all the primes, does p_k represent the greatest prime number? Explain your answer.

C If P contains all the primes, every counting number greater than 1 must be divisible by at least one member of P. Why?

Now suppose there is a counting number that is obtained by using as factors all the primes contained in P and then adding 1. Use the letter c as a name for this number.

$c = (p_1 \times p_2 \times p_3 \times \ldots \times p_k) + 1.$

Next you will determine whether or not c has any prime factors greater than p_k.

D The number c is not divisible by p_1 because, when you divide c by p_1, you get $p_2 \times p_3 \times p_4 \times \ldots \times p_k$, remainder 1. If the number c were divisible by p_1, what would the remainder be?

E What quotient and what remainder do you get when you divide c by p_2? By p_3? By p_4? By p_k?

F How do you know that c is not divisible by any member of P? Is any member of P a factor of c? Explain your answer.

You know that it is possible to express every counting number greater than 1 as the product of primes. You also know that it is *not* possible to express c as the product of primes that are members of P.

G If c is a prime number, what is the only prime factor of c? Is this factor a member of P? Is this factor greater than p_k? Explain your answer.

H If c is a composite number, how do you know that you can express c as a product of primes?

I If c is a composite number, how do you know that each prime factor of c must be greater than p_k?

J We assumed that p_k was the greatest prime number. How do you know that the assumption is false?

K Is the set of prime numbers a finite set? An infinite set? Explain your answers.

7 74 Exploring ideas

Multiplication of rational numbers

In unit 6 you learned that multiplication of natural numbers is a mapping of the set of ordered pairs of natural numbers onto the set of natural numbers. You also learned that the product of two natural numbers is a natural number. In this lesson you will learn how to find the product of two rational numbers.

A Look at D1. What are the first and second components of each ordered pair of rational numbers named in the display?

B $(\frac{2}{3}, \frac{3}{4})$ is mapped onto the product $\frac{2}{3} \times \frac{3}{4}$. Describe the other mappings represented in D1.

C Read sentence A in D2. Sentence A expresses a true statement about the product of the rational numbers $\frac{2}{3}$ and $\frac{3}{4}$. The number $\frac{6}{12}$ is the product of $\frac{2}{3}$ and $\frac{3}{4}$. Is 6 the product of the numerators of $\frac{2}{3}$ and $\frac{3}{4}$? Is 12 the product of the denominators of $\frac{2}{3}$ and $\frac{3}{4}$? Remember that when we mention numerators and denominators, we are referring to the fractions that indicate the rational numbers.

D Sentence B also expresses a true statement about the product of two rational numbers. What are these numbers? Is $\frac{16}{10}$ their product?

The *product of two rational numbers* is a rational number indicated by a fraction. The numerator of the fraction is the product of the two numerators, and the denominator is the product of the two denominators. The work that follows will help you use variables to develop the definition of the product of two rational numbers.

Product of rational numbers; closure property of multiplication; well-defined property of multiplication

$(\frac{2}{3}, \frac{3}{4})$ $(\frac{8}{10}, \frac{2}{1})$ $(\frac{4}{5}, \frac{5}{4})$ $(\frac{32}{3}, \frac{0}{8})$

$\frac{2}{3} \times \frac{3}{4}$ $\frac{8}{10} \times \frac{2}{1}$ $\frac{4}{5} \times \frac{5}{4}$ $\frac{32}{3} \times \frac{0}{8}$

D1

A $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$.

B $\frac{8}{10} \times \frac{2}{1} = \frac{16}{10}$.

D2

C $\frac{2}{3} \times \frac{3}{4} = \frac{2 \cdot 3}{3 \cdot 4}$.

D $\frac{8}{10} \times \frac{2}{1} = \frac{8 \cdot 2}{10 \cdot 1}$.

E $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

D3

Universe for a and c = N.
Universe for b and d = C.

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

F $\frac{5}{8} \times \frac{1}{5} = \frac{5 \times 1}{8 \times 5}$.

G $\frac{7}{6} \times \frac{2}{3} = \frac{7 \times 2}{6 \times 3}$.

H $\frac{75}{1} \times \frac{1}{4} = \frac{75 \times 1}{1 \times 4}$.

I $\frac{3}{4} \times \frac{4}{3} = \frac{3 \times 4}{4 \times 3}$.

D4

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product of two rational numbers. The universe for a and c is N. The universe for b and d is C. For each replacement of a, b, c, and d, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

E Look at D3. Does sentence C express a true statement? Does sentence D express a true statement? Explain your answers.

F Now look at sentence E in D3. Think of $\frac{a}{b}$ and $\frac{c}{d}$ as any two rational numbers. The universe for a and c is N. The universe for b and d is C. Is the numerator of $\frac{ac}{bd}$ the product of a and c? Is the denominator of $\frac{ac}{bd}$ the product of b and d?

Now we can define the product, $\frac{a}{b} \times \frac{c}{d}$, of the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$. The universe for a and c is N. The universe for b and d is C. For each replacement of a, b, c, and d, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

G Read sentence F in D4. What replacements were made for the variables in $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ to obtain the statement expressed by sentence F? Is the statement true? How do you know?

H What replacements were made for the variables in $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ to obtain the statement expressed by sentence G? By sentence H? By sentence I? Is each of the statements true?

Decide whether each of the following sentences expresses a true statement or a false statement.

I $\frac{7}{8} \times \frac{1}{4} = \frac{7}{32}$.

J $\frac{2}{3}(\frac{1}{2}) = \frac{4}{3}$.

K $\frac{16}{5} \cdot \frac{3}{10} = \frac{48}{50}$.

L $\frac{0}{6} \times \frac{6}{1} = \frac{6}{6}$.

M $\frac{4}{3} \cdot \frac{1}{9} = \frac{5}{12}$.

N $\frac{11}{12}(\frac{12}{11}) = \frac{132}{132}$.

You know that the set of natural numbers is closed under multiplication. This means that the product of any two natural numbers is always a natural number. Now you will see how the definition of the product of two rational numbers assures you that the set of rational numbers also is closed under multiplication.

A Look again at D4. Is $\frac{ac}{bd}$ the product of $\frac{a}{b}$ and $\frac{c}{d}$?

B In this product, is ac a natural number? Is bd a counting number? Is $\frac{ac}{bd}$ a rational number? Explain your answers.

C Suppose that x and y are any two rational numbers, for example, $\frac{2}{5}$ and $\frac{4}{7}$. Is the product xy always a rational number?

You have learned that the product of any two members of R_a, the set of rational numbers of arithmetic, is always a member of R_a. Therefore, the set of rational numbers of arithmetic is closed under multiplication. This prop-

J $\frac{3}{8} \times \frac{2}{3} = \frac{6}{24}$.

K $\frac{6}{16} \times \frac{8}{12} = \frac{48}{192}$.

L $\frac{3}{8} \times \frac{6}{9} = \frac{18}{72}$.

M $\frac{6}{16} \times \frac{2}{3} = \frac{12}{48}$.

D5

N $\frac{9}{5} \cdot \frac{2}{7} = \frac{18}{35}$.

O $\frac{3}{5} \cdot \frac{6}{7} = \frac{18}{35}$.

P $\frac{3}{7} \cdot \frac{6}{5} = \frac{18}{35}$.

D6

erty is the *closure property of multiplication of rational numbers*. The property is expressed below.

The universe for x *and* y *is* R_a. *For each replacement of* x *and* y, *there is a number* xy *that is a member of* R_a.

D You know that multiplication of natural numbers is a binary operation. Do you think that multiplication of rational numbers is a binary operation?

Now you will study another important property of multiplication of rational numbers.

E Look at D5. Does each sentence express a true statement?

F In the statement expressed by sentence J, $\frac{6}{24}$ is the product of what two numbers?

G In the statement expressed by sentence K in D5, $\frac{3}{8}$ has been replaced by $\frac{6}{16}$, and $\frac{2}{3}$ has been replaced by $\frac{8}{12}$. Is $\frac{6}{16} = \frac{3}{8}$? Is $\frac{8}{12} = \frac{2}{3}$?

H In the statement expressed by sentence L, what replacement has been made for $\frac{2}{3}$? Is the replacement equal to $\frac{2}{3}$?

I In the statement expressed by sentence M, what replacement has been made for $\frac{3}{8}$?

J Is each of the products $\frac{48}{192}$, $\frac{18}{72}$, and $\frac{12}{48}$ equal to $\frac{6}{24}$?

Each of the following sentences expresses a true statement about the product of two numbers. Replace each number by an equal rational number. Then find their product. Is the product that you found the same number as the original product?

K $\frac{4}{5} \times \frac{2}{9} = \frac{8}{45}$.

L $\frac{3}{12} \times \frac{5}{2} = \frac{15}{24}$.

M $\frac{1}{2} \times \frac{4}{7} = \frac{4}{14}$.

N $\frac{1}{3} \times \frac{7}{6} = \frac{7}{18}$.

O $\frac{10}{11} \times \frac{2}{5} = \frac{20}{55}$.

P $\frac{3}{5} \times \frac{1}{4} = \frac{3}{20}$.

Q $\frac{1}{5} \times \frac{3}{7} = \frac{3}{35}$.

R $\frac{9}{8} \times \frac{11}{3} = \frac{99}{24}$.

S $\frac{15}{16} \times \frac{4}{4} = \frac{60}{24}$.

T When the two rational numbers that are used to obtain a product are replaced by equal rational numbers, does the product remain the same?

Now you know that, when two rational numbers that form a product are replaced by equal rational numbers, the product is still the same rational number. This property is the *well-defined property of multiplication of rational numbers*. The property is expressed below.

If one or both of the numbers used in forming the product of two rational numbers is replaced by an equal rational number, the product is still the same rational number.

U Study the sentences in D6. Does each sentence express a true statement? Is multiplication of rational numbers a many-to-one mapping? Explain your answer.

In this lesson you have learned how to find the product of two rational numbers. You have also studied the closure property and the well-defined property of multiplication of rational numbers.

On your own

Decide whether each of the following sentences expresses a true statement or a false statement.

1 $\frac{9}{10} \times \frac{1}{3} = \frac{9}{30}$.

2 $\frac{1}{5} \cdot \frac{5}{8} = \frac{6}{40}$.

3 $\frac{0}{72}(\frac{3}{3}) = \frac{3}{216}$.

4 $\frac{4}{7} \times \frac{1}{4} = \frac{4}{28}$.

5 $\frac{11}{12} \cdot \frac{6}{2} = \frac{66}{24}$.

6 $\frac{1}{8}(\frac{10}{10}) = \frac{80}{10}$.

7 $\frac{2}{3} \times \frac{65}{4} = \frac{130}{12}$.

8 $\frac{0}{20} \times \frac{13}{3} = \frac{0}{60}$.

For each of exercises 9 through 23, find the product of the numbers named in the exercise.

9 $\frac{7}{8}, \frac{1}{2}$
10 $\frac{4}{3}, \frac{2}{5}$
11 $\frac{9}{10}, \frac{0}{6}$
12 $\frac{18}{7}, \frac{2}{2}$
13 $\frac{100}{1}, \frac{3}{2}$
14 $\frac{15}{16}, \frac{12}{8}$
15 $\frac{9}{9}, \frac{3}{3}$
16 $\frac{4}{6}, \frac{6}{4}$
17 $\frac{37}{3}, \frac{1}{8}$
18 $\frac{5}{2}, \frac{5}{2}$
19 $\frac{0}{3}, \frac{1}{3}$
20 $\frac{3}{4}, \frac{7}{1}$
21 $\frac{25}{6}, \frac{8}{15}$
22 $\frac{3}{9}, \frac{9}{3}$
23 $\frac{50}{1}, \frac{9}{20}$

Each sentence in exercises 24 through 29 expresses a true statement about the product of two rational numbers. For each exercise, write three other sentences that express the same statement. Do this by first replacing one or both rational numbers by an equal rational number. Then find the product of the replacements. For exercise 24 you could write:

$\frac{9}{12} \times \frac{1}{8} = \frac{9}{96}$. $\frac{3}{4} \times \frac{2}{16} = \frac{6}{64}$. $\frac{6}{8} \times \frac{3}{24} = \frac{18}{192}$.

24 $\frac{3}{4} \times \frac{1}{8} = \frac{3}{32}$.
25 $\frac{0}{9} \cdot \frac{2}{4} = \frac{0}{36}$.
26 $\frac{3}{6}(\frac{9}{12}) = \frac{27}{72}$.
27 $\frac{1}{2}(\frac{2}{1}) = \frac{2}{2}$.
28 $\frac{4}{4} \cdot \frac{1}{5} = \frac{4}{20}$.
29 $\frac{0}{3} \times \frac{6}{1} = \frac{0}{3}$.

KEEPING SKILFUL

For each of exercises 1 through 8, suppose that J, K, and L are disjoint sets. Suppose also that $17 = n(J)$, $35 = n(K)$, and $129 = n(L)$. Tell what natural number is associated with the set named in each exercise.

1 $K \cup L$
2 $J \times K$
3 $L \times J$
4 $L \cup K$
5 $J \cup L$
6 $K \times L$
7 $J \times J$
8 $K \cup J$

For each condition expressed in exercises 9 through 14, first tabulate the solution set of the condition. Then make a graph of the solution set. $U = N \times N$.

9 $x + y = 10 \wedge x < y$.
10 $18 + y = 22 \wedge x + y < 7$.
11 $x + 6 = y \wedge x < 54 \div 18$.
12 $x < y + 4 \wedge y - 4 = 3$.
13 $127 + x < 136 \wedge x + 5 = y$.
14 $13 - x = y \wedge x + 9 < 15$.

Addition of rational numbers

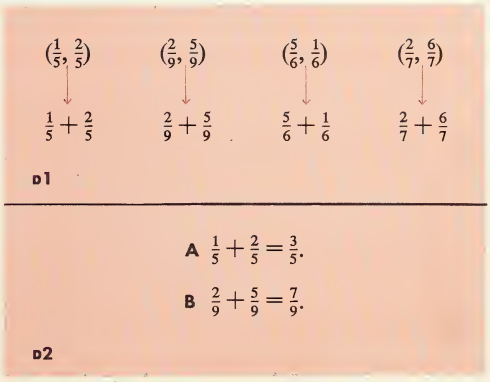
In unit 6 you learned that addition of natural numbers is a mapping of the set of ordered pairs of natural numbers onto the set of natural numbers. You also learned that the sum of two natural numbers is a natural number. In this lesson you will learn how to find the sum of two rational numbers.

A Look at D1. What are the first and second components of each ordered pair of rational numbers named in D1?

B $(\frac{1}{5}, \frac{2}{5})$ is mapped onto the sum $\frac{1}{5} + \frac{2}{5}$. Describe the other mappings represented in D1.

C Sentence A in D2 expresses a true statement about the sum of two rational numbers. The number $\frac{3}{5}$ is the sum of $\frac{1}{5}$ and $\frac{2}{5}$. Do $\frac{1}{5}$ and $\frac{2}{5}$ have the same denominator? What is the sum of the numerators of $\frac{1}{5}$ and $\frac{2}{5}$?

D Sentence B in D2 also expresses a true statement about the sum of two rational numbers.



What are the two numbers? What is the denominator of $\frac{2}{9}$? Of $\frac{5}{9}$? What is the sum of $\frac{2}{9}$ and $\frac{5}{9}$?

Suppose that any two rational numbers are indicated by two fractions that have the same denominator. The *sum of the two rational numbers* is a rational number indicated by a fraction. The denominator of this fraction is the same as the denominator of the two fractions. The numerator of the fraction is the sum of the two numerators. In the work that follows, you will use variables to develop the definition of the sum of two rational numbers.

E Look at D3. Does sentence C express the same statement as sentence A in D2? How do you know? Does sentence D express the same statement as sentence B in D2? How do you know?

F Look at sentence E in D3. Think of $\frac{a}{b}$ and $\frac{c}{b}$ as any two rational numbers. Do $\frac{a}{b}$ and $\frac{c}{b}$ have the same denominator? Remember that when we talk about numerators and denominators, we are referring to the fractions that indicate the rational numbers.

G $\frac{a+c}{b}$ is the sum of $\frac{a}{b}$ and $\frac{c}{b}$. Is the denominator of $\frac{a+c}{b}$ the same as the denominator of $\frac{a}{b}$ and $\frac{c}{b}$? Is the numerator of $\frac{a+c}{b}$ the sum of the numerators of $\frac{a}{b}$ and $\frac{c}{b}$?

Now we can define the *sum*, $\frac{a}{b}+\frac{c}{b}$, of the rational numbers $\frac{a}{b}$ and $\frac{c}{b}$. The universe for a and c is N. The universe for b is C. For each replacement of a, b, and c, $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$.

H Study sentence F in D4. To obtain the statement expressed by sentence F from $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$, a was replaced by 2; b was replaced by 3; and c was replaced by 5. Is the statement expressed by sentence F true? How do you know?

I What replacements were made for the variables in $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$ to obtain the statement expressed by sentence G? By sentence H? Is each of the statements true?

A $\frac{1}{5}+\frac{2}{5}=\frac{3}{5}$.

B $\frac{2}{9}+\frac{5}{9}=\frac{7}{9}$.

D2

Universe for a and c = N.
Universe for b = C.

C $\frac{1}{5}+\frac{2}{5}=\frac{1+2}{5}$.

D $\frac{2}{9}+\frac{5}{9}=\frac{2+5}{9}$.

E $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$.

D3

Universe for a and c = N.
Universe for b = C.

$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$.

F $\frac{2}{3}+\frac{5}{3}=\frac{2+5}{3}$.

G $\frac{7}{8}+\frac{1}{8}=\frac{7+1}{8}$.

H $\frac{5}{9}+\frac{2}{9}=\frac{5+2}{9}$.

D4

sum of two rational numbers. The universe for a and c is N. The universe for b is C. For each replacement of a, b, and c, $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$.

Decide whether each sentence below expresses a true statement or a false statement.

J $\frac{1}{2}+\frac{7}{2}=\frac{8}{2}$.
K $\frac{13}{17}+\frac{13}{17}=\frac{13}{34}$.
L $\frac{3}{4}+\frac{3}{4}=\frac{6}{4}$.
M $\frac{2}{7}+\frac{5}{7}=\frac{10}{7}$.
N $\frac{3}{5}+\frac{4}{5}=\frac{12}{25}$.
O $\frac{5}{6}+\frac{0}{6}=\frac{5}{6}$.
P $\frac{5}{6}+\frac{7}{6}=\frac{35}{36}$.
Q $\frac{1}{7}+\frac{1}{7}=\frac{2}{7}$.
R $\frac{5}{8}+\frac{1}{8}=\frac{5}{8}$.
S $\frac{0}{9}+\frac{11}{9}=\frac{11}{9}$.
T $\frac{8}{1}+\frac{5}{1}=\frac{13}{1}$.
U $\frac{3}{4}+\frac{7}{4}=\frac{10}{16}$.

You know that the set of natural numbers is closed under the operation of addition. This means that the sum of any two natural num-

bers is always a natural number. Now you will see how the definition of sum assures you that the set of rational numbers also is closed under addition.

A Look again at D4. Is $\frac{a+c}{b}$ the sum of $\frac{a}{b}$ and $\frac{c}{b}$?

B Why is $a + c$ a natural number? Why is b a counting number?

C Is $\frac{a+c}{b}$ a rational number?

D Is the sum of any two rational numbers always a rational number?

You have seen that the sum of any two members of R_a, the set of rational numbers of arithmetic, is always a member of R_a. Therefore, R_a is closed under addition. This property is the *closure property of addition of rational numbers*. The property is expressed below.

The universe for x *and* y *is* R_a. *For each replacement of* x *and* y, *there is a number* x + y *that is a member of* R_a.

E Do you think that addition of rational numbers is a binary operation?

In the last lesson you learned that the operation of multiplication of rational numbers is well defined. Now you will learn what is meant by the well-defined property of addition of rational numbers.

F Now read sentences I and J in D5. Does sentence I express a true statement about the sum of $\frac{1}{2}$ and $\frac{3}{2}$? Sentence J expresses a true statement about the sum of what two rational numbers?

G Are $\frac{1}{2}$ and $\frac{7}{14}$ equal rational numbers? How do you know? How do you know that $\frac{3}{2}$ and $\frac{21}{14}$ are equal rational numbers? Is $\frac{28}{14} = \frac{4}{2}$?

H Read sentences K and L in D5. Sentence K expresses a true statement about the sum of what two rational numbers? Is the sum of these two numbers also expressed by sentence L? Explain your answer.

I Is $\frac{18}{18} = \frac{3}{3}$?

I $\frac{1}{2} + \frac{3}{2} = \frac{4}{2}$.

J $\frac{7}{14} + \frac{21}{14} = \frac{28}{14}$.

K $\frac{2}{3} + \frac{1}{3} = \frac{3}{3}$.

L $\frac{12}{18} + \frac{6}{18} = \frac{18}{18}$.

D5

$\frac{1}{2} + \frac{2}{3}$

M $\frac{1 \times 3}{2 \times 3} + \frac{2 \times 2}{2 \times 3} = \frac{(1 \times 3) + (2 \times 2)}{2 \times 3}$.

N $\frac{1}{2} + \frac{2}{3} = \frac{(1 \times 3) + (2 \times 2)}{2 \times 3}$.

O $\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$.

D6

J When the two rational numbers that form a sum are replaced by equal rational numbers, does the sum remain the same?

Now you know that, when two rational numbers that form a sum are replaced by equal rational numbers, the sum is the same rational number. This property is the *well-defined property of addition of rational numbers*. The property is expressed below.

If the numbers used in forming the sum of two rational numbers are replaced by equal rational numbers, the sum is still the same rational number.

You have been working with the fractions that indicate the rational numbers $\frac{a}{b}$ and $\frac{c}{b}$. These fractions have the same denominator. Now you will learn to add two rational numbers when they are indicated by fractions with different denominators.

A The sum $\frac{1}{2} + \frac{2}{3}$ is expressed in D6. Do $\frac{1}{2}$ and $\frac{2}{3}$ have the same denominator? What is the denominator of $\frac{1}{2}$? Of $\frac{2}{3}$?

B Suppose that you multiply both the numerator and the denominator of $\frac{1}{2}$ by the denominator of $\frac{2}{3}$. Do you obtain $\frac{1 \times 3}{2 \times 3}$? Are $\frac{1}{2}$

and $\frac{1\times3}{2\times3}$ the same rational number? Explain your answer.

C Suppose that you multiply both the numerator and the denominator of $\frac{2}{3}$ by the denominator of $\frac{1}{2}$. Do you obtain $\frac{2\times2}{2\times3}$? Are $\frac{2}{3}$ and $\frac{2\times2}{2\times3}$ the same rational number? Explain your answer.

D In the sum $\frac{1}{2}+\frac{2}{3}$, can you replace $\frac{1}{2}$ by $\frac{1\times3}{2\times3}$? Can you replace $\frac{2}{3}$ by $\frac{2\times2}{2\times3}$? How do you know?

E Is $\frac{1\times3}{2\times3}\sim\frac{1}{2}$? Is $\frac{2\times2}{2\times3}\sim\frac{2}{3}$? Is the denominator of $\frac{1\times3}{2\times3}$ the same as the denominator of $\frac{2\times2}{2\times3}$? What is this denominator?

Notice that you have replaced each of the fractions, $\frac{1}{2}$ and $\frac{2}{3}$, by an equivalent fraction. Each fraction that you obtained has 6 as a denominator. The number 6 is a *common denominator* for $\frac{1}{2}$ and $\frac{2}{3}$. A number is a common denominator for two given fractions if it is the denominator of two fractions that are each equivalent to the given fractions.

F Read sentence M in D6. Think about the statement expressed and explain how the sum $\frac{(1\times3)+(2\times2)}{2\times3}$ was obtained.

G Look again at D6. To obtain the statement expressed by sentence N, $\frac{1\times3}{2\times3}$ was replaced by $\frac{1}{2}$, and $\frac{2\times2}{2\times3}$ was replaced by $\frac{2}{3}$. How do you know that these replacements do not affect the sum? Does sentence N express a true statement?

H In the statement expressed by sentence O, how was $\frac{7}{6}$ obtained? Does sentence O express a true statement?

I The sum $\frac{2}{7}+\frac{3}{5}$ is expressed in D7. If you multiply both the numerator and the denominator of $\frac{2}{7}$ by 5, do you obtain $\frac{2\times5}{7\times5}$? Are $\frac{2}{7}$ and $\frac{2\times5}{7\times5}$ the same rational number? How do you know?

J By what number do you multiply both the numerator and the denominator of $\frac{3}{5}$ to get $\frac{7\times3}{7\times5}$? How do you know that $\frac{3}{5}$ and $\frac{7\times3}{7\times5}$ are the same rational number?

$\frac{1}{2}+\frac{2}{3}$

M $\frac{1\times3}{2\times3}+\frac{2\times2}{2\times3}=\frac{(1\times3)+(2\times2)}{2\times3}$.

N $\frac{1}{2}+\frac{2}{3}=\frac{(1\times3)+(2\times2)}{2\times3}$.

O $\frac{1}{2}+\frac{2}{3}=\frac{7}{6}$.

D6

$\frac{2}{7}+\frac{3}{5}$

P $\frac{2\times5}{7\times5}+\frac{7\times3}{7\times5}=\frac{(2\times5)+(7\times3)}{7\times5}$.

Q $\frac{2}{7}+\frac{3}{5}=\frac{(2\times5)+(7\times3)}{7\times5}$.

R $\frac{2}{7}+\frac{3}{5}=\frac{31}{35}$.

D7

Universe for a and c = N.
Universe for b and d = C.

$\frac{a}{b}+\frac{c}{d}$

S $\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$.

T $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$.

D8

K In the sum $\frac{2}{7}+\frac{3}{5}$, how do you know that you can replace $\frac{2}{7}$ by $\frac{2\times5}{7\times5}$? How do you know that you can replace $\frac{3}{5}$ by $\frac{7\times3}{7\times5}$?

L What is the common denominator for $\frac{2\times5}{7\times5}$ and $\frac{7\times3}{7\times5}$? Are the denominators of $\frac{2}{7}$ and $\frac{3}{5}$ factors of this common denominator?

M In the statement expressed by sentence P, how was the sum $\frac{(2\times5)+(7\times3)}{7\times5}$ obtained?

N Does sentence Q express a true statement?

O Does sentence R express a true statement?

Use the method developed in D6 and D7 to find the sum of the numbers expressed in each exercise below.

P $\frac{3}{2},\frac{4}{5}$ **Q** $\frac{7}{11},\frac{2}{3}$ **R** $\frac{9}{5},\frac{5}{13}$ **S** $\frac{4}{3},\frac{8}{7}$

Now you are prepared to develop a property for adding two rational numbers when they are indicated by fractions that have different denominators. Think of $\frac{a}{b}$ and $\frac{c}{d}$ as any two rational numbers. The universe for a and c is N. The universe for b and d is C.

A Look at D8. For each replacement of a, b, and d, is $\frac{a}{b} = \frac{ad}{bd}$? For each replacement of b, c, and d, is $\frac{c}{d} = \frac{bc}{bd}$? Explain your answers.

B Study sentence S. In forming the sum, how do you know that you can replace $\frac{a}{b}$ by $\frac{ad}{bd}$? That you can replace $\frac{c}{d}$ by $\frac{bc}{bd}$?

C What is the common denominator for $\frac{ad}{bd}$ and $\frac{bc}{bd}$?

D How was the sum $\frac{ad+bc}{bd}$ obtained?

E Read sentence S again. For each replacement of a, b, c, and d, is $\frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}$?

F Study sentence T. For each replacement of a, b, c, and d, is $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$?

You have learned how to find the sum of two rational numbers when they are indicated by fractions that do not have a common denominator. This property is the *sum property of rational numbers.* The property is expressed below.

The universe for a *and* c *is N. The universe for* b *and* d *is C. For each replacement of* a, b, c, *and* d, $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

G For each statement expressed in D9, what replacements were made for the variables in $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$?

H Is each statement expressed in D9 true? How do you know?

Decide whether each sentence below expresses a true statement or a false statement.

I $\frac{3}{4} + \frac{1}{5} = \frac{4}{9}$.
J $\frac{2}{7} + \frac{4}{3} = \frac{8}{21}$.
K $\frac{2}{9} + \frac{1}{4} = \frac{17}{36}$.
L $\frac{1}{4} + \frac{5}{6} = \frac{26}{24}$.
M $\frac{9}{10} + \frac{0}{5} = \frac{45}{50}$.
N $\frac{2}{3} + \frac{11}{12} = \frac{13}{12}$.
O $\frac{5}{3} + \frac{4}{1} = \frac{17}{3}$.
P $\frac{7}{8} + \frac{4}{5} = \frac{28}{40}$.
Q $\frac{7}{1} + \frac{1}{7} = \frac{50}{7}$.

R You know that addition of rational numbers is a mapping of the set of ordered pairs of rational numbers onto the set of rational numbers. Study D10. Why is addition of rational numbers a many-to-one mapping?

Universe for a and c = N.
Universe for b and d = C.

$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

U $\frac{2}{3} + \frac{5}{7} = \frac{14+15}{21}$.

V $\frac{1}{8} + \frac{2}{9} = \frac{9+16}{72}$.

W $\frac{3}{5} + \frac{2}{11} = \frac{33+10}{55}$.

D9

X $\frac{2}{3} + \frac{1}{4} = \frac{11}{12}$.

Y $\frac{5}{12} + \frac{6}{12} = \frac{11}{12}$.

Z $\frac{10}{12} + \frac{1}{12} = \frac{11}{12}$.

D10

Now you know how to find the sum of two rational numbers. You also know that addition is well defined and that the set of rational numbers is closed under the operation of addition.

On your own

Tell which of the rational numbers named in exercises 1 through 8 are equal to the rational number $\frac{30}{84}$.

1 $\frac{6}{12}$
2 $\frac{15}{42}$
3 $\frac{60}{168}$
4 $\frac{3}{8}$
5 $\frac{10}{28}$
6 $\frac{120}{336}$
7 $\frac{5}{14}$
8 $\frac{90}{242}$.

For each of exercises 9 through 18, tell whether the sentence expresses a true statement or a false statement.

9 $\frac{3}{7} + \frac{2}{3} = \frac{23}{21}$.
10 $\frac{5}{6} + \frac{2}{6} = \frac{10}{6}$.
11 $\frac{3}{2} + \frac{5}{9} = \frac{34}{18}$.
12 $\frac{2}{10} + \frac{4}{7} = \frac{54}{70}$.
13 $\frac{5}{12} + \frac{3}{5} = \frac{8}{60}$.
14 $\frac{3}{8} + \frac{1}{8} = \frac{3}{64}$.
15 $\frac{2}{7} + \frac{0}{6} = \frac{0}{42}$.
16 $\frac{8}{1} + \frac{3}{1} = \frac{11}{1}$.
17 $\frac{13}{5} + \frac{12}{7} = \frac{151}{35}$.
18 $\frac{36}{29} + \frac{41}{17} = \frac{1801}{493}$.

For each of exercises 19 through 34, find the sum of the numbers named in the exercise. Use a fraction numeral to express each sum.

19 $\frac{5}{7}, \frac{3}{7}$ **20** $\frac{5}{6}, \frac{4}{7}$ **21** $\frac{2}{3}, \frac{4}{5}$ **22** $\frac{5}{12}, \frac{7}{12}$ **23** $\frac{3}{10}, \frac{4}{9}$ **24** $\frac{13}{4}, \frac{1}{3}$ **25** $\frac{5}{8}, \frac{9}{12}$ **26** $\frac{3}{12}, \frac{12}{11}$ **27** $\frac{3}{15}, \frac{14}{15}$ **28** $\frac{2}{1}, \frac{3}{12}$ **29** $\frac{6}{7}, \frac{8}{21}$ **30** $\frac{8}{11}, \frac{3}{10}$ **31** $\frac{3}{50}, \frac{8}{90}$ **32** $\frac{13}{1}, \frac{12}{1}$ **33** $\frac{17}{32}, \frac{15}{24}$ **34** $\frac{15}{18}, \frac{11}{20}$

35 Is the sum of $\frac{17}{263}$ and $\frac{927}{7267}$ a rational number? Explain your answer.

The sentences in exercises 36 through 41 express true statements about the sum of two rational numbers. For each exercise, write three other sentences that express the same statement. Do this by first replacing one or both of the rational numbers by an equal rational number. Then find the sum of the replacements. For exercise 36 you could write:

$\frac{4}{6} + \frac{1}{8} = \frac{38}{48}$. $\frac{4}{6} + \frac{2}{16} = \frac{76}{96}$. $\frac{2}{3} + \frac{3}{24} = \frac{57}{72}$.

36 $\frac{2}{3} + \frac{1}{8} = \frac{19}{24}$.

37 $\frac{6}{10} + \frac{2}{7} = \frac{62}{70}$.

38 $\frac{5}{6} + \frac{3}{21} = \frac{123}{126}$.

39 $\frac{2}{3} + \frac{3}{2} = \frac{13}{6}$.

40 $\frac{7}{12} + \frac{2}{5} = \frac{59}{60}$.

41 $\frac{16}{18} + \frac{3}{6} = \frac{25}{18}$.

KEEPING SKILFUL

For each condition expressed in exercises 1 through 8, tabulate the solution set. The universe for (x, y) is $C \times C$.

1 $x + y = 292 \wedge 7/19 \sim 91/x$.

2 $6/8 \sim x/y \wedge y + 9 < 30$.

3 $11/12 \sim x/y \wedge x + y = 69$.

4 $x/13 \sim 50/y \wedge 103 - x = 78$.

5 $x + 6 > y \wedge 7/8 \sim x/y$.

6 $x/4 \sim 27/y \wedge 11y = 132$.

7 $x/y \sim 6/22 \wedge 2x + y < 40$.

8 $x/15 \sim 52/y \wedge x = 494 \div 19$.

For each of exercises 9 through 17, find the product of the numbers named in the exercise.

9 $\frac{7}{2}, \frac{3}{4}$ **10** $\frac{0}{8}, \frac{11}{2}$ **11** $\frac{1}{3}, \frac{7}{9}$ **12** $\frac{17}{3}, \frac{3}{17}$ **13** $\frac{4}{11}, \frac{9}{1}$ **14** $\frac{6}{13}, \frac{2}{8}$ **15** $\frac{23}{1}, \frac{0}{5}$ **16** $\frac{5}{5}, \frac{7}{23}$ **17** $\frac{68}{1}, \frac{12}{13}$

Subtraction of rational numbers

You have learned how to find the sum of two rational numbers. Now you will learn how to find the difference of two rational numbers.

A Sentence A in D1 expresses a true statement about the difference of the numbers $\frac{4}{5}$ and $\frac{1}{5}$. Do $\frac{4}{5}$ and $\frac{1}{5}$ have the same denominator? Is $4 - 1$, or 3, the difference of the numerators of $\frac{4}{5}$ and $\frac{1}{5}$? Is $\frac{3}{5}$ the difference of $\frac{4}{5}$ and $\frac{1}{5}$?

B Sentence B in D1 also expresses a true statement about the difference of two rational numbers. What is the difference of $\frac{3}{7}$ and $\frac{3}{7}$?

You already know that subtraction is related to addition. You know, for example, that $8 - 3 = 5$ also means $8 = 3 + 5$. Subtraction of rational numbers is related to addition of rational numbers in the same way. $\frac{4}{5} - \frac{1}{5} = \frac{3}{5}$ also means $\frac{4}{5} = \frac{1}{5} + \frac{3}{5}$.

C Does $\frac{3}{7} - \frac{3}{7} = \frac{0}{7}$ also mean $\frac{3}{7} = \frac{3}{7} + \frac{0}{7}$?

D Look at sentence C in D2. You will decide if there is a rational number of arithmetic that is the difference of $\frac{5}{9}$ and $\frac{11}{9}$. What is the universe for x?

E Think about the condition expressed by sentence D in D2. Does $\frac{5}{9} - \frac{11}{9} = \frac{x}{9}$ also mean $\frac{5}{9} = \frac{11}{9} + \frac{x}{9}$? Explain your answer.

F Look at sentence E in D2. How do you know that, for each replacement of x, $\frac{11+x}{9}$ is the same as $\frac{11}{9} + \frac{x}{9}$? You must decide if there is a replacement for x that will make $\frac{11+x}{9}$ equal to $\frac{5}{9}$. Is there a natural number that satisfies $5 = 11 + x$? Is there a number that satisfies $\frac{5}{9} = \frac{11+x}{9}$?

A $\frac{4}{5} - \frac{1}{5} = \frac{3}{5}$.

B $\frac{3}{7} - \frac{3}{7} = \frac{0}{7}$.

D1

Universe for x = N.

C $\frac{5}{9} - \frac{11}{9} = \frac{x}{9}$.

D $\frac{5}{9} = \frac{11}{9} + \frac{x}{9}$.

E $\frac{5}{9} = \frac{11+x}{9}$.

D2

Universe for a and c = N.
Universe for b = C.

$$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}.$$

F $\frac{4}{5} - \frac{3}{5} = \frac{4-3}{5}$.

G $\frac{6}{7} - \frac{4}{7} = \frac{6-4}{7}$.

H $\frac{25}{21} - \frac{17}{21} = \frac{25-17}{21}$.

D3

difference of two rational numbers (dif′ər əns). The universe for a and c is N. The universe for b is C. For each replacement of a, b, and c, if a is greater than or equal to c, then $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$.

G Is the difference of $\frac{5}{9}$ and $\frac{11}{9}$ a member of R_a? Explain your answer.

H Is the difference of any two members of R_a always a member of R_a? Is R_a closed under subtraction? Explain your answer.

I Look again at the sentences in D1. The difference of $\frac{4}{5}$ and $\frac{1}{5}$ is $\frac{3}{5}$. Is the numerator 4 greater than the numerator 1? The difference of $\frac{3}{7}$ and $\frac{3}{7}$ is $\frac{0}{7}$. Is the numerator 3 equal to the numerator 3? Read sentence C in D2. Is the numerator 5 greater than or equal to the numerator 11? Is it less than the numerator 11?

Because the numerator 5 is less than the numerator 11, there is no member of R_a that is the difference of $\frac{5}{9}$ and $\frac{11}{9}$.

J Suppose that $\frac{a}{b}$ and $\frac{c}{b}$ are two rational numbers. Suppose also that a is greater than c. Is the difference of $\frac{a}{b}$ and $\frac{c}{b}$ a member of R_a? The universe for a and c is N. The universe for b is C.

K Next suppose that a is equal to c. Is the difference of $\frac{a}{b}$ and $\frac{c}{b}$ a member of R_a?

L Now suppose that a is less than c. Is the difference of $\frac{a}{b}$ and $\frac{c}{b}$ a member of R_a?

Now we can define the *difference*, $\frac{a}{b} - \frac{c}{b}$, *of two rational numbers*, $\frac{a}{b}$ and $\frac{c}{b}$. The universe for a and c is N. The universe for b is C. For each replacement of a, b, and c, if a is greater than or equal to c, then $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$.

M Look at D3. What is the universe for each variable in $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$?

N What replacements have been made for the variables in $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$ to obtain the statement that is expressed by sentence F? By sentence G? By sentence H?

O Does each sentence in D3 express a true statement? How do you know?

Decide whether each of the following sentences expresses a true statement or a false statement.

P $\frac{5}{7} - \frac{2}{7} = \frac{7}{7}$.

Q $\frac{11}{5} - \frac{4}{5} = \frac{7}{5}$.

R $\frac{14}{15} - \frac{14}{15} = \frac{0}{15}$.

S $\frac{7}{3} - \frac{2}{3} = \frac{5}{9}$.

T $\frac{11}{30} - \frac{1}{30} = \frac{10}{15}$.

U $\frac{17}{12} - \frac{16}{12} = \frac{1}{12}$.

V $\frac{7}{8} - \frac{11}{8} = \frac{4}{8}$.

W $\frac{1}{9} - \frac{0}{9} = \frac{1}{9}$.

X $\frac{5}{12} - \frac{1}{12} = \frac{6}{12}$.

You will often want to find the difference of two rational numbers when they are indicated by fractions that do not have the same denominator. To do this, you can use the same ideas you used to develop the sum property. From the well-defined property of addition of rational numbers you know that, if the rational numbers forming a sum are replaced by equal rational numbers, then the sum is still the same

rational number. It is also true that, if the rational numbers that form a difference are replaced by equal rational numbers, then the difference is still the same rational number.

A Look at D4. Is $\frac{2\times4}{3\times4}=\frac{2}{3}$? Is $\frac{3\times1}{3\times4}=\frac{1}{4}$? In the ? statement expressed by sentence I, how is the difference $\frac{(2\times4)-(3\times1)}{3\times4}$ obtained? Does sentence I express a true statement? Explain your answer.

B Does sentence J express a true statement? ? Does sentence K express a true statement? Explain your answers.

Now you will use variables to develop the difference property of rational numbers.

C Look at D5. For each replacement of a, b, and d, is $\frac{a}{b}=\frac{ad}{bd}$? For each replacement of b, c, and d, is $\frac{c}{d}=\frac{bc}{bd}$? Explain your answers.

D Does the difference $\frac{a}{b}-\frac{c}{d}$ remain the same when you replace $\frac{a}{b}$ by $\frac{ad}{bd}$ and $\frac{c}{d}$ by $\frac{bc}{bd}$?

E What is the denominator of $\frac{ad}{bd}$? Of $\frac{bc}{bd}$? Is bd a common denominator for $\frac{a}{b}$ and $\frac{c}{d}$? Is bd the product of the denominators of $\frac{a}{b}$ and $\frac{c}{d}$?

F How is the difference $\frac{ad-bc}{bd}$ obtained?

G Suppose that, after replacement of the variables, ad is greater than or equal to bc. Is $\frac{ad}{bd}-\frac{bc}{bd}$ equal to the rational number $\frac{ad-bc}{bd}$? Is $\frac{a}{b}-\frac{c}{d}$ equal to the rational number $\frac{ad-bc}{bd}$? Remember that the universe for a and c is N. The universe for b and d is C.

Now you know how to find the difference of two rational numbers when they are indicated by fractions that do not have the same denominator. This property is called the *difference property of rational numbers*. The property is expressed below.

The universe for a *and* c *is N. The universe for* b *and* d *is C. For each replacement of* a, b, c, *and* d, *if* ad *is greater than or equal to* bc, *then* $\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$.

H Look at D6. To obtain the statement expressed by sentence N, a was replaced by 3, b by 4, c by 2, and d by 3. What replacements were made for the variables in $\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$ to obtain the statement expressed by sentence O? By sentence P?

I Does each sentence in D6 express a true statement?

Decide whether each of the following sentences expresses a true statement or a false statement.

J $\frac{4}{5}-\frac{3}{4}=\frac{31}{20}$. **K** $\frac{5}{9}-\frac{1}{4}=\frac{4}{36}$. **L** $\frac{4}{7}-\frac{3}{11}=\frac{23}{77}$. **M** $\frac{7}{10}-\frac{2}{3}=\frac{1}{13}$. **N** $\frac{15}{16}-\frac{3}{4}=\frac{0}{64}$. **O** $\frac{27}{20}-\frac{5}{60}=\frac{1520}{1200}$.

$$\frac{2}{3}-\frac{1}{4}$$

I $\frac{2\times4}{3\times4}-\frac{3\times1}{3\times4}=\frac{(2\times4)-(3\times1)}{3\times4}$.

J $\frac{2}{3}-\frac{1}{4}=\frac{(2\times4)-(3\times1)}{3\times4}$.

K $\frac{2}{3}-\frac{1}{4}=\frac{5}{12}$.

D4

Universe for a and c = N.
Universe for b and d = C.

$$\frac{a}{b}-\frac{c}{d}$$

L $\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}$.

M $\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$.

D5

The universe for a and c = N.
The universe for b and d = C.

$$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}.$$

N $\frac{3}{4}-\frac{2}{3}=\frac{9-8}{12}$.

O $\frac{7}{9}-\frac{2}{5}=\frac{35-18}{45}$.

P $\frac{5}{11}-\frac{3}{8}=\frac{40-33}{88}$.

D6

Now you know that subtraction of rational numbers is related to addition of rational numbers. You also know how to find the difference of two rational numbers.

On your own

For each of exercises 1 through 12, tell whether the sentence expresses a true statement or a false statement.

1 $\frac{3}{8} - \frac{1}{3} = \frac{1}{24}$.

2 $\frac{4}{7} - \frac{1}{7} = \frac{5}{7}$.

3 $\frac{5}{8} - \frac{1}{3} = \frac{4}{24}$.

4 $\frac{3}{4} - \frac{5}{9} = \frac{7}{36}$.

5 $\frac{12}{5} - \frac{13}{6} = \frac{7}{30}$.

6 $\frac{4}{1} - \frac{5}{2} = \frac{1}{3}$.

7 $\frac{5}{6} - \frac{2}{7} = \frac{3}{42}$.

8 $\frac{4}{11} - \frac{4}{12} = \frac{4}{121}$.

9 $\frac{7}{9} - \frac{3}{4} = \frac{1}{36}$.

10 $\frac{12}{13} - \frac{7}{8} = \frac{5}{84}$.

11 $\frac{7}{6} - \frac{5}{6} = \frac{12}{36}$.

12 $\frac{5}{8} - \frac{0}{9} = \frac{53}{72}$.

For each ordered pair of rational numbers named in exercises 13 through 30, find the difference of the first and second components. Use a fraction numeral to express the difference.

13 $(\frac{5}{8}, \frac{3}{8})$

14 $(\frac{23}{12}, \frac{18}{12})$

15 $(\frac{5}{6}, \frac{5}{6})$

16 $(\frac{4}{3}, \frac{1}{2})$

17 $(\frac{7}{8}, \frac{1}{5})$

18 $(\frac{2}{5}, \frac{2}{6})$

19 $(\frac{4}{7}, \frac{3}{6})$

20 $(\frac{4}{9}, \frac{1}{3})$

21 $(\frac{7}{12}, \frac{2}{5})$

22 $(\frac{6}{7}, \frac{2}{3})$

23 $(\frac{11}{13}, \frac{5}{8})$

24 $(\frac{8}{11}, \frac{7}{10})$

25 $(\frac{9}{15}, \frac{1}{2})$

26 $(\frac{3}{1}, \frac{24}{23})$

27 $(\frac{3}{4}, \frac{1}{6})$

28 $(\frac{5}{6}, \frac{0}{7})$

29 $(\frac{20}{8}, \frac{5}{2})$

30 $(\frac{9}{11}, \frac{3}{8})$

For each ordered pair of rational numbers named in exercises 31 through 48, tell whether or not the difference of the components is a member of R_a. If the difference is a member of R_a, use a fraction numeral to express this difference.

31 $(\frac{1}{3}, \frac{1}{2})$

32 $(\frac{2}{5}, \frac{3}{5})$

33 $(\frac{1}{2}, \frac{1}{3})$

34 $(\frac{6}{7}, \frac{5}{6})$

35 $(\frac{8}{9}, \frac{9}{8})$

36 $(\frac{4}{9}, \frac{4}{9})$

37 $(\frac{4}{6}, \frac{5}{7})$

38 $(\frac{7}{3}, \frac{3}{7})$

39 $(\frac{13}{2}, \frac{21}{5})$

40 $(\frac{7}{11}, \frac{2}{3})$

41 $(\frac{4}{9}, \frac{1}{5})$

42 $(\frac{6}{1}, \frac{3}{5})$

43 $(\frac{5}{6}, \frac{0}{3})$

44 $(\frac{5}{6}, \frac{5}{11})$

45 $(\frac{1}{3}, \frac{3}{8})$

46 $(\frac{3}{10}, \frac{2}{15})$

47 $(\frac{25}{3}, \frac{24}{6})$

48 $(\frac{15}{8}, \frac{7}{4})$

APPLYING MATHEMATICS

For each of problems 1 through 13, write a sentence that expresses a condition. Tabulate the solution set of the condition and give the answer to the problem. The universe is given for each problem.

1 Mr. Howe's monthly salary is \$750. After deductions are made for taxes and insurance, his monthly take-home pay is \$607. How many dollars are deducted from his monthly salary for taxes and insurance? $U = N$.

2 Jerry answered fewer than 50 questions on a science test. He had more than 37 correct answers. How many questions could he have answered correctly? $U = N$.

3 Joyce and Donna went on a 7-mile bicycle ride. They rode the first 2 miles in 24 minutes. If they continued to ride at this same rate, how many minutes did it take them to ride the 7 miles? $U = C$.

4 One day Peter saw 6 tankers and freighters in the harbor. How many ships of each kind could he have seen that day? $U = C \times C$.

5 One hundred seventy-seven persons went to a bazaar on Thursday. Three times as many persons went to the bazaar on Friday as on Thursday. How many persons went to the bazaar on Friday? $U = C$.

6 On Saturday Bob and Terry washed fewer than 27 windows. Bob washed 5 windows for each 3 windows that Terry washed. How many windows could each of the boys have washed on Saturday? $U = C \times C$.

7 Mrs. Gordon bought 8 fewer chocolate doughnuts than plain doughnuts. She bought 24 plain and chocolate doughnuts in all. How many doughnuts of each kind did she buy? $U = N \times N$.

8 A manufacturer borrowed \$7200 from the Hillside Savings Bank. At the end of one year,

he paid the bank $7560. What rate of interest was he charged? $U = C \times C$.

9 Three hundred eighty-four students graduated from West High School this year. In the past, approximately 25% of the West graduating class has enrolled in university. If the per cent of students who enroll in university remains the same, about how many of this year's graduates will enroll in university? $U = C$.

10 Use your answer to problem 9 to find the answer to the following problem: Of the West students who can be expected to enroll in university this year, only about 72 will graduate. About what per cent of West students who might enroll in university this year will graduate? $U = C$.

11 Five main classes of animals belong to the vertebrates. Two of these classes are reptiles and amphibians. There are about 4×10^3 kinds of reptiles and 2×10^3 kinds of amphibians. About how many kinds of animals are either reptiles or amphibians? $U = N$.

THINK Is $4 \times 10^3 = 4000$?

12 Use the information given in problem 11 to find the answer to the following problem: There are approximately 4×10^4 kinds of vertebrates in all. About how many kinds of vertebrates are there other than reptiles? $U = N$.

13 A clothing store owner bought 250 neckties for a special sale. He sold 104 of these ties the first week, 97 the second week, and 49 the third week. How many of the 250 ties were left at the end of the third week? $U = N \times N$.

For problem 14, write a sentence that expresses a compound condition. Then give the answer to the problem. The universe for each of the variables is C.

14 Marilyn bought 4 cards of buttons for 15¢ a card and a spool of thread for 7¢. She gave the clerk 75¢. How many cents should she have received in change?

CHECKING UP

If you have trouble with these two tests, you can find help in lessons 74, 75, and 76.

Test 82

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

1 Both multiplication and addition of rational numbers are binary operations.

2 If a is replaced by 4, b by 7, c by 10, and d by 3, the product of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{12}{70}$.

3 Addition of rational numbers is a many-to-one mapping.

4 The difference of any two members of R_a is also a member of R_a.

5 $(\frac{2}{3}, \frac{1}{2})$ and $(\frac{6}{6}, \frac{1}{6})$ can be mapped onto the same rational number by multiplication.

6 The condition $\frac{4}{3} - x = \frac{1}{3}$ has the same solution as the condition $\frac{4}{3} + \frac{1}{3} = x$.

7 The product of $\frac{4}{5}$ and a given rational number is the same as the product of $\frac{8}{10}$ and the given number.

Test 83

For each of exercises 8 through 15, write "T" if the sentence expresses a true statement. Write "F" if it expresses a false statement.

8 $\frac{7}{8} - \frac{3}{4} = \frac{4}{32}$.

9 $\frac{5}{3} + \frac{3}{2} = \frac{15}{6}$.

10 $\frac{7}{12} + \frac{4}{12} = \frac{132}{144}$.

11 $\frac{5}{9} \times \frac{3}{7} = \frac{8}{63}$.

12 $\frac{0}{1} + \frac{0}{6} = \frac{0}{6}$.

13 $\frac{2}{7} \times \frac{1}{4} = \frac{15}{28}$.

14 $\frac{5}{6} - \frac{5}{7} = \frac{65}{42}$.

15 $\frac{8}{9} \times \frac{3}{6} = \frac{24}{54}$.

Find the sum of the two rational numbers named in each of exercises 16, 17, and 18.

16 $\frac{5}{9}, \frac{8}{9}$ **17** $\frac{7}{12}, \frac{1}{4}$ **18** $\frac{3}{11}, \frac{0}{5}$

Find the product of the two rational numbers named in each of exercises 19, 20, and 21.

19 $\frac{4}{7}, \frac{2}{9}$ **20** $\frac{2}{1}, \frac{1}{15}$ **21** $\frac{3}{4}, \frac{9}{10}$

For each ordered pair named in exercises 22, 23, and 24, find the difference of the components.

22 $(\frac{7}{2}, \frac{7}{2})$ **23** $(\frac{6}{7}, \frac{5}{6})$ **24** $(\frac{11}{8}, \frac{1}{6})$

End-of-block tests on sum, difference, and product of rational numbers

7 77 Exploring ideas

Common factors and common multiples

You have learned how to use the reduction property to obtain basic fractions. In this lesson you will learn to obtain basic fractions by using some of the ideas about factorization that you have studied. Then you will learn how to find the least common multiple of two numbers.

A Is $\frac{36}{60} = \frac{18}{30}$? Are 36 and 60 relatively prime? Is $\frac{36}{60}$ a basic fraction? Are 18 and 30 relatively prime? Is $\frac{18}{30}$ a basic fraction?

> REMINDER
> Two numbers are relatively prime if they have no common factor other than 1.
> A basic fraction is a fraction whose numerator and denominator are relatively prime.
> *See lesson 72, pages 314 and 315.*

B Is $\frac{36}{60} = \frac{6}{10}$? Are 6 and 10 relatively prime? Is $\frac{6}{10}$ a basic fraction? Is $\frac{36}{60} = \frac{3}{5}$? Are 3 and 5 relatively prime? Is $\frac{3}{5}$ a basic fraction?

Notice that, if you divide both the numerator and the denominator of $\frac{36}{60}$ by 12, you obtain the equivalent basic fraction $\frac{3}{5}$, but if you divide by any other common factor, you do not obtain a basic fraction. In the work that follows, you will learn how to find the number by which you can divide both the numerator and the denominator of a given fraction to get the equivalent basic fraction.

$$\frac{36}{60} = \frac{2 \cdot 2 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 3 \cdot 5}.$$

D1

$$\frac{\overset{5}{\cancel{70}}}{\underset{4}{\cancel{56}}} = \frac{5}{4}.$$

D2

C Look at the sentence in D1. Is $2 \cdot 2 \cdot 3 \cdot 3$ the complete factorization of 36? What is the complete factorization of 60?

> REMINDER
> The complete factorization of a given number is the number expressed as the product of prime numbers.
> *See lesson 70, page 300.*

D What two numbers are the only common prime factors of 36 and 60?

E The greatest power of 2 that is a factor of both 36 and 60 is 2^2. What is the greatest power of 3 that is a factor of both 36 and 60?

F Is $2^2 \times 3^1$, or 12, the greatest common factor of 36 and 60?

G If you divide both the numerator and the denominator of $\frac{36}{60}$ by their greatest common factor, do you obtain the basic fraction $\frac{3}{5}$?

H Look at D2. To obtain the basic fraction equivalent to $\frac{70}{56}$, divide both 70 and 56 by their greatest common factor. What is their greatest common factor? Is the basic fraction $\frac{5}{4}$ equivalent to $\frac{70}{56}$?

Notice the marks through the numerals 70 and 56 in D2. The numeral 5 is written above the numeral 70 to express the quotient of 70 and 14. The numeral 4 is written below the numeral 56 to express the quotient of 56 and 14.

I If you divide the numerator and the denominator of any fraction by their greatest common factor, do you always obtain a basic fraction?

A fraction is named in each of exercises J through O. First find the greatest common factor of the numerator and the denominator of the given fraction. Then find the basic fraction that is equivalent to the given fraction.

J $\frac{13}{39}$ **K** $\frac{27}{36}$ **L** $\frac{35}{49}$ **M** $\frac{30}{75}$ **N** $\frac{66}{88}$ **O** $\frac{72}{24}$

P What is the greatest common factor of the numerator and the denominator of any basic fraction?

Q The greatest common factor is sometimes called the *greatest common divisor*. What is the greatest common divisor of 10 and 100? Of 16 and 24?

Now you know how to obtain the basic fraction equivalent to a given fraction. You will often use this knowledge to obtain basic fractions for sums, differences, or products of rational numbers.

A Does sentence A in D3 express a true statement?

B What basic fraction is equivalent to $\frac{28}{24}$?

C Decide whether each of the other sentences in D3 expresses a true statement. Then, for each statement, find the basic fraction for the sum, difference, or product.

You have already learned how to find the product of two rational numbers and how to find the basic fraction for this product. Next you will learn a shorter way to find the basic fraction for a product.

D Does sentence I in D4 express a true statement? Use the definition of the product of two rational numbers to explain your answer.

E Does sentence J express a true statement? Use the definition of complete factorization to explain your answer.

D3

A $\frac{2}{4}+\frac{4}{6}=\frac{28}{24}$.

B $\frac{6}{8}-\frac{1}{5}=\frac{22}{40}$.

C $\frac{8}{3}\times\frac{7}{14}=\frac{56}{42}$.

D $\frac{5}{6}+\frac{6}{9}=\frac{81}{54}$.

E $\frac{15}{9}\times\frac{5}{10}=\frac{75}{90}$.

F $\frac{5}{2}-\frac{9}{12}=\frac{42}{24}$.

G $\frac{14}{8}+\frac{6}{10}=\frac{188}{80}$.

H $\frac{11}{7}\times\frac{21}{22}=\frac{231}{154}$.

D4

I $\frac{10}{3}\times\frac{9}{14}=\frac{10\cdot 9}{3\cdot 14}$.

J $\frac{10\cdot 9}{3\cdot 14}=\frac{5\cdot 2\cdot 3\cdot 3}{3\cdot 7\cdot 2}$.

K $\frac{5\cdot \overset{1}{\cancel{2}}\cdot \overset{1}{\cancel{3}}\cdot 3}{\underset{1}{\cancel{3}}\cdot 7\cdot \underset{1}{\cancel{2}}}=\frac{15}{7}$.

F Look at sentence K. Both the numerator and the denominator of $\frac{5\cdot 2\cdot 3\cdot 3}{3\cdot 7\cdot 2}$ have been divided by 2 and also by 3. How is this shown in sentence K?

G If you divide a number by 2 and also by 3, do you get the same result as when you divide the number by 6? Is 6 the greatest common factor of the numerator and the denominator of $\frac{5\cdot 2\cdot 3\cdot 3}{3\cdot 7\cdot 2}$? Explain your answers.

H How does sentence K in D4 show that the numerator and the denominator of $\frac{5\cdot 2\cdot 3\cdot 3}{3\cdot 7\cdot 2}$ have been divided by their greatest common factor?

I Is $5\cdot 1\cdot 1\cdot 3=15$? Is $1\cdot 7\cdot 1=7$? Does sentence K express a true statement?

When you divide two numbers (expressed as the product of their prime factors) by all of their common prime factors, the result is the same as when you divide the numbers by their greatest common factor. Also, when you divide both the numerator and the denominator of a fraction by their greatest common factor, you obtain the equivalent basic fraction. Therefore,

when you divide both the numerator and the denominator by all their common prime factors, you also obtain the equivalent basic fraction.

You know that the basic fraction $\frac{15}{7}$ indicates the product $\frac{10}{3} \times \frac{9}{14}$. Now you will learn another method of finding the basic fraction for the product $\frac{10}{3} \times \frac{9}{14}$.

J Look at sentence L in D5. Does the sentence express a true statement?

K Look at sentences M and N. The numerator of $\frac{5 \cdot 2}{3}$ and the denominator of $\frac{3 \cdot 3}{7 \cdot 2}$ in the product $\frac{5 \cdot 2}{3} \times \frac{3 \cdot 3}{7 \cdot 2}$ have been divided by what number? The numerator of $\frac{3 \cdot 3}{7 \cdot 2}$ and the denominator of $\frac{5 \cdot 2}{3}$ in the product $\frac{5 \cdot 2}{3} \times \frac{3 \cdot 3}{7 \cdot 2}$ have been divided by what number? Is the result the same as when the numerator and the denominator in $\frac{5 \cdot 2 \cdot 3 \cdot 3}{3 \cdot 7 \cdot 2}$ are divided by 2 and also by 3?

L How do you know that sentences M and N express true statements?

M You have used two methods to find the basic fraction for the product $\frac{10}{3} \times \frac{9}{14}$. Describe these two methods.

A product is expressed in each of exercises N through V. Find the basic fraction for each product.

N $\frac{2 \cdot 5}{3 \cdot 5} \times \frac{2 \cdot 7}{3 \cdot 7}$

O $\frac{2 \cdot 2}{3 \cdot 3 \cdot 3} \times \frac{3 \cdot 3}{2 \cdot 2 \cdot 2}$

P $\frac{6}{8} \times \frac{2}{9}$

Q $\frac{2}{12} \times \frac{18}{14}$

R $\frac{39}{13} \times \frac{17}{34}$

S $\frac{7}{11} \times \frac{44}{21}$

T $\frac{28}{12} \times \frac{3}{14}$

U $\frac{5}{7} \times \frac{35}{10}$

V $\frac{48}{5} \times \frac{11}{21}$

So far in this lesson you have learned how to use the greatest common factor of two numbers to obtain a basic fraction. Now you will learn what is meant by the least common multiple of two numbers. This knowledge will help you to add or subtract rational numbers.

A Look at sentence A in D6. Is 12 a factor of 12? Is 12 divisible by 12?

B Study sentences B through E. Is 12 a factor of 24? Of 36? Of 48? Of 60? Is each of these numbers divisible by 12?

L $\frac{10}{3} \times \frac{9}{14} = \frac{5 \cdot 2}{3} \times \frac{3 \cdot 3}{7 \cdot 2}$.

M $\frac{\overset{1}{5 \cdot \cancel{2}}}{\cancel{3}} \times \frac{\overset{1}{3 \cdot \cancel{3}}}{7 \cdot \cancel{2}} = \frac{15}{7}$.
$\quad\quad 1 \quad\quad\quad 1$

N $\frac{\overset{1}{5 \cdot \cancel{2}}}{\cancel{3}} \times \frac{\overset{1}{3 \cdot \cancel{3}}}{7 \cdot \cancel{2}} = \frac{5 \cdot \overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{3}} \cdot 3}{\underset{1}{\cancel{3}} \cdot 7 \cdot \underset{1}{\cancel{2}}}$.
$\quad\quad 1 \quad\quad\quad 1 \quad\quad\quad\quad 1 \quad\quad 1$

D5

A $12 = 1(12)$.
B $24 = 2(12)$.
C $36 = 3(12)$.
D $48 = 4(12)$.
E $60 = 5(12)$.

F $15 = 1(15)$.
G $30 = 2(15)$.
H $45 = 3(15)$.
I $60 = 4(15)$.
J $75 = 5(15)$.

D6

mul ti ple (mul′tə pəl). A number of which a given number is a factor. A multiple of any number is divisible by that number. 100 is a multiple of 4; 35 is a multiple of 7; 50 is a multiple of 10.

The numbers 12, 24, 36, 48, and 60 are all *multiples* of 12. For any two numbers, the second number is a multiple of the first number if the first number is a factor of the second.

C Name five other multiples of 12. Is the set of multiples of 12 an infinite set?

D Look at sentences F through J in D6. Is 15 a multiple of 15? Is 30 a multiple of 15? Are 45, 60, and 75 all multiples of 15? Explain your answers.

E Which numbers named below are multiples of 18?

36 6 9 18 54 24

F Think of the numbers 12 and 15. Is 60 a multiple of both these numbers? Is 60 the least number that is a multiple of both 12 and 15?

60 is the *least common multiple* of 12 and 15. The least common multiple of two numbers is the least number that is a multiple of both numbers.

Next you will learn how to find the least common multiple of two numbers.

G The sentences in D7 express complete factorizations of the numbers 14 and 4. What are the prime factors of 14? Of 4?

H Is 7^1 the greatest power of 7 that is a factor of either 14 or 4 or of both 14 and 4?

I Now find the greatest power of 2 that is a factor of either 14 or 4 or of both. Is 2^1 this power? Is 2^2 this power?

J 7^1 and 2^2 are the greatest powers of 7 and 2 that are factors of either 14 or 4 or of both. Is $7 \cdot 2 \cdot 2$, or 28, the product of these factors?

K Is 28 a common multiple of 14 and 4? Is 28 the least common multiple of 14 and 4? Explain your answers.

You know that 28 is the least common multiple of 14 and 4. Notice that, to find the least common multiple of two numbers, you can first find the prime factors of each of the numbers. Next you choose the greatest power of each prime factor in either one or the other of the numbers or in both of the numbers. Then you find the product of these powers.

For each exercise below, find the least common multiple of the two numbers expressed in the exercise.

L 8, 6 **N** 14, 7 **P** 20, 25
M 4, 12 **O** 6, 9 **Q** 4, 8

R Explain why you can obtain the least common multiple of two numbers by dividing the product of the numbers by their greatest common factor.

You can use what you have learned about the least common multiple of two numbers to learn more about finding common denominators.

$14 = 2 \cdot 7.$
$4 = 2 \cdot 2.$

D7

K $\frac{1}{2} + \frac{3}{4}$ **O** $\frac{15}{16} - \frac{1}{12}$
L $\frac{1}{6} + \frac{8}{9}$ **P** $\frac{3}{14} - \frac{5}{28}$
M $\frac{7}{8} - \frac{11}{20}$ **Q** $\frac{10}{15} - \frac{4}{12}$
N $\frac{13}{30} + \frac{4}{25}$ **R** $\frac{9}{45} + \frac{6}{12}$

D8

least common multiple of two numbers. The least number that is a multiple of both numbers. 18 is the least common multiple of 6 and 9; 42 is the least common multiple of 7 and 6.

S Look at numeral K in D8. Is $2 \cdot 4$, or 8, a common denominator for $\frac{1}{2}$ and $\frac{3}{4}$? Is $2 \cdot 4$, or 8, also a common multiple of 2 and 4?

T You have already learned that you can use the product of the denominators of two fractions as a common denominator for the fractions. Is the product of two numbers always a common multiple of the numbers? Is the product always the *least* common multiple? Explain your answers.

Because the product of the denominators of two fractions is not always their least common multiple, it is not always the most convenient number to use as a common denominator. From now on, you may want to use the least common multiple of the denominators of two fractions as a common denominator for the fractions.

The least common multiple of the denominators of two fractions is the *least common denominator* for the fractions.

U Is 4 the least common denominator for $\frac{1}{2}$ and $\frac{3}{4}$? Explain your answer.

v Is $\frac{1}{2}+\frac{3}{4}=\frac{2}{4}+\frac{3}{4}$? Find the basic fraction that indicates the sum of $\frac{1}{2}$ and $\frac{3}{4}$.

w Now look at numerals L through P in D8. Find the basic fraction that indicates the sum or difference named by each numeral.

x Now look at numeral Q. Instead of finding a common denominator for $\frac{10}{15}$ and $\frac{4}{12}$, you can use the reduction property to obtain the basic fractions $\frac{2}{3}$ and $\frac{1}{3}$ from $\frac{10}{15}$ and $\frac{4}{12}$. What is the denominator of $\frac{2}{3}$ and $\frac{1}{3}$? What is the difference of $\frac{2}{3}$ and $\frac{1}{3}$? Is this difference the same as the difference of $\frac{10}{15}$ and $\frac{4}{12}$? How do you know?

y Look at numeral R. Is $\frac{1}{5}$ the basic fraction that is equivalent to $\frac{9}{45}$? Is $\frac{1}{2}$ the basic fraction that is equivalent to $\frac{6}{12}$? What is the least common denominator for $\frac{1}{5}$ and $\frac{1}{2}$? What is the sum of $\frac{1}{5}$ and $\frac{1}{2}$? Is this sum also the sum of $\frac{9}{45}$ and $\frac{6}{12}$?

Sometimes you will be asked to find the sum or the difference of two rational numbers when one or both of the numbers are *not* indicated by a basic fraction. You can see from exercises X and Y that, when this is the case, you can first find a basic fraction to indicate each of the rational numbers. Then you can find the least common denominator for the basic fractions.

Now you know how to use the greatest common factor and the least common multiple of two numbers in your work with rational numbers. You also know that the methods used in this lesson depend on definitions and properties that you have previously studied.

On your own

In each of exercises 1 through 10, a sum, difference, or product is named by a fraction numeral. For each exercise, first find the greatest common factor of the numerator and the denominator of the fraction named. Then find the basic fraction that indicates the sum, difference, or product.

1 $\frac{2}{6}+\frac{4}{8}=\frac{40}{48}$.
2 $\frac{12}{8}-\frac{4}{8}=\frac{8}{8}$.
3 $\frac{9}{3}-\frac{10}{12}=\frac{78}{36}$.
4 $\frac{7}{4}\times\frac{6}{14}=\frac{42}{56}$.
5 $\frac{21}{12}\times\frac{4}{5}=\frac{84}{60}$.
6 $\frac{2}{8}+\frac{9}{6}=\frac{84}{48}$.
7 $\frac{10}{12}\times\frac{4}{6}=\frac{40}{72}$.
8 $\frac{16}{8}-\frac{4}{16}=\frac{224}{128}$.
9 $\frac{3}{10}\times\frac{4}{12}=\frac{12}{120}$.
10 $\frac{7}{21}+\frac{8}{4}=\frac{196}{84}$.

Find the basic fraction that indicates the product expressed in each of exercises 11 through 19.

11 $\frac{7}{5}\times\frac{9}{14}$
12 $\frac{6}{4}\times\frac{8}{9}$
13 $\frac{3}{4}\times\frac{8}{15}$
14 $\frac{5}{6}\times\frac{7}{10}$
15 $\frac{2}{3}\times\frac{11}{4}$
16 $\frac{11}{22}\times\frac{6}{2}$
17 $\frac{7}{5}\times\frac{20}{35}$
18 $\frac{30}{36}\times\frac{12}{15}$
19 $\frac{25}{8}\times\frac{16}{15}$

For each of exercises 20 through 28, find the least common denominator for the two fractions named. Then find the basic fraction that indicates the sum or difference expressed in the exercise.

20 $\frac{5}{6}+\frac{2}{9}$
21 $\frac{3}{10}+\frac{7}{15}$
22 $\frac{5}{2}-\frac{5}{4}$
23 $\frac{8}{11}-\frac{7}{33}$
24 $\frac{17}{24}-\frac{2}{3}$
25 $\frac{3}{4}+\frac{11}{14}$
26 $\frac{7}{3}+\frac{5}{12}$
27 $\frac{8}{21}-\frac{5}{28}$
28 $\frac{19}{20}-\frac{7}{12}$

For each exercise below, first find the sum of the numbers named in the exercise. Then find the product of the numbers named. Use a name of a basic fraction to express each sum and product.

29 $\frac{1}{3}, \frac{6}{5}$
30 $\frac{1}{4}, \frac{4}{12}$
31 $\frac{2}{5}, \frac{5}{10}$
32 $\frac{4}{2}, \frac{6}{8}$
33 $\frac{3}{6}, \frac{2}{9}$
34 $\frac{3}{4}, \frac{7}{14}$
35 $\frac{8}{9}, \frac{18}{27}$
36 $\frac{9}{11}, \frac{7}{3}$
37 $\frac{5}{7}, \frac{14}{35}$
38 $\frac{12}{15}, \frac{4}{9}$
39 $\frac{10}{7}, \frac{7}{10}$
40 $\frac{2}{12}, \frac{18}{14}$

KEEPING SKILFUL

For each of exercises 1 through 9, find the product of the numbers named in the exercise.

1 $\frac{2}{3}, \frac{1}{4}$
2 $\frac{9}{7}, \frac{8}{5}$
3 $\frac{12}{1}, \frac{4}{4}$
4 $\frac{7}{3}, \frac{7}{3}$
5 $\frac{7}{8}, \frac{2}{5}$
6 $\frac{0}{11}, \frac{13}{7}$
7 $\frac{9}{3}, \frac{6}{2}$
8 $\frac{5}{7}, \frac{15}{7}$
9 $\frac{0}{10}, \frac{1}{2}$

For each of exercises 10 through 18, find the sum of the numbers named in the exercise.

10 $\frac{3}{16}, \frac{7}{16}$
11 $\frac{3}{5}, \frac{1}{4}$
12 $\frac{7}{12}, \frac{2}{3}$
13 $\frac{9}{2}, \frac{2}{3}$
14 $\frac{15}{1}, \frac{8}{1}$
15 $\frac{3}{8}, \frac{11}{9}$
16 $\frac{0}{8}, \frac{7}{11}$
17 $\frac{5}{7}, \frac{1}{12}$
18 $\frac{16}{27}, \frac{9}{35}$

The natural numbers related to certain rational numbers

In this lesson you will learn how the set of natural numbers is related to a subset of the set of rational numbers of arithmetic.

A Study the pictures in D1. Picture A represents a natural-number line. Picture B represents a rational-number line. Is each natural number mapped onto a rational number? Is each rational number mapped onto a natural number?

B If we were to continue the mapping represented in D1, would each of the natural numbers be mapped onto a rational number? Would each of the rational numbers be mapped onto a natural number?

C Now look at the tabulation in D2. Set S is a set of rational numbers, each of which is indicated by a basic fraction. Notice that each of these basic fractions has a denominator of 1. What is the numerator of the first fraction? Is the numerator of each of the other fractions the successor of the numerator of the preceding fraction?

D1

$S = \{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{5}{1}, \frac{6}{1}, \ldots\}.$

D2

D Is set S a subset of the set of rational numbers? Is set S an infinite set? Explain your answers.

E Think about the mapping represented in D1. In this mapping, is each natural number mapped onto a member of set S? Is each member of set S mapped onto a natural number?

F Are the set of natural numbers and set S in one-to-one correspondence?

> REMINDER
> A one-to-one correspondence is two one-to-one mappings that map a first set onto a second set and the second set onto the first set.
> *See lesson 57, page 246.*

You have seen that the set of natural numbers and a particular subset of the set of rational numbers can be put in one-to-one correspondence. You know that 0 and $\frac{0}{1}$ are mapped onto each other; that 1 and $\frac{1}{1}$ are mapped onto each other, and so on. We call 0 and $\frac{0}{1}$ *mates*. 1 and $\frac{1}{1}$ are also mates.

In the correspondence that we have been discussing, any two numbers that map onto each other are mates. Each number is the mate of the other. One of the mates is a natural number, and the other is a rational number.

G What natural number is the mate of $\frac{4}{1}$? Of $\frac{105}{1}$? Of $\frac{17}{1}$? Of $\frac{1056}{1}$?

H You know that $\frac{7}{1}$ is the mate of 7. Is $\frac{14}{2}$ also the mate of 7? Is $\frac{35}{5}$ also the mate of 7? For each replacement of n, is $\frac{7n}{n}$ the mate of 7? The universe for n is C. Explain your answers.

I Name the mate of 20. Of 45.

J What is the mate of a? The universe for a is N.

Notice that 0 and $\frac{0}{1}$ are not the same number. 0 is a natural number, and $\frac{0}{1}$ is a rational number. These different kinds of numbers belong to different sets, but they are mates in the correspondence that we have been discussing.

K Is $2+4=6$? Is $\frac{2}{1}+\frac{4}{1}=\frac{6}{1}$?

L You know that 2 and $\frac{2}{1}$ are mates and that 4 and $\frac{4}{1}$ are mates also. Are 6 and $\frac{6}{1}$ mates?

M Are 5 and $\frac{5}{1}$ mates? Are 8 and $\frac{8}{1}$ mates? Is the sum of 5 and 8 the mate of the sum of $\frac{5}{1}$ and $\frac{8}{1}$?

N Think of a and $\frac{a}{1}$ and b and $\frac{b}{1}$ as any two pairs of mates. The universe for the variables is N. The sum of a and b is $a+b$. The sum of $\frac{a}{1}$ and $\frac{b}{1}$ is $\frac{a}{1}+\frac{b}{1}$. For each replacement of a and b, is $\frac{a}{1}+\frac{b}{1}=\frac{a+b}{1}$? Are $a+b$ and $\frac{a+b}{1}$ mates? Are $a+b$ and $\frac{a}{1}+\frac{b}{1}$ mates?

If you add two natural numbers and then add the rational numbers that are their mates, the sum of the two natural numbers is the mate of the sum of the two rational numbers.

O Is $5 \cdot 6=30$? Is $\frac{5}{1} \cdot \frac{6}{1}=\frac{30}{1}$?

P You know that 5 and $\frac{5}{1}$ and 6 and $\frac{6}{1}$ are two pairs of mates. Are 30 and $\frac{30}{1}$ mates?

Q Are 2 and $\frac{2}{1}$ and 9 and $\frac{9}{1}$ two pairs of mates? Is the product of 2 and 9 the mate of the product of $\frac{2}{1}$ and $\frac{9}{1}$?

R Think of a and $\frac{a}{1}$ and b and $\frac{b}{1}$ as any two pairs of mates. The universe for the variables is N. The product of a and b is $a \cdot b$. The product of $\frac{a}{1}$ and $\frac{b}{1}$ is $\frac{a}{1} \cdot \frac{b}{1}$. For each replacement of a and b, is $\frac{a}{1} \cdot \frac{b}{1}=\frac{a \cdot b}{1}$? Are $a \cdot b$ and $\frac{a \cdot b}{1}$ mates? Are $a \cdot b$ and $\frac{a}{1} \cdot \frac{b}{1}$ mates?

If you multiply two natural numbers and then multiply the rational numbers that are their mates, the product of the two natural numbers is the mate of the product of the two rational numbers.

Because the natural number a and the rational number $\frac{a}{1}$ behave in much the same way in addition and multiplication, it is often convenient to use a to indicate $\frac{a}{1}$. You will learn how to do this in the next lesson.

Now you know that the set of natural numbers and $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots\}$ can be put in one-to-one correspondence. You know what it means to say that a natural number a and a rational number $\frac{a}{1}$ are mates. You also know that, if a and b are natural numbers, then $a+b$ and $\frac{a}{1}+\frac{b}{1}$ are mates and $a \cdot b$ and $\frac{a}{1} \cdot \frac{b}{1}$ are mates.

On your own

To answer the questions in the following exercises, you will have to keep in mind the way in which we put the set of natural numbers and $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots\}$ in one-to-one correspondence.

Name the natural number that maps onto the rational number named in each of exercises 1 through 6.

1 $\frac{9}{1}$ **2** $\frac{36}{1}$ **3** $\frac{11}{1}$ **4** $\frac{50}{1}$ **5** $\frac{100}{1}$ **6** $\frac{175}{1}$

Name the rational number that maps onto the natural number named in each of exercises 7 through 12.

7 15 **8** 6 **9** 38 **10** 59 **11** 505 **12** 0

Tell what natural number is the mate of the rational number named in each exercise below.

13 $\frac{8}{2}$ **14** $\frac{7}{7}$ **15** $\frac{0}{4}$ **16** $\frac{10}{5}$ **17** $\frac{12}{4}$ **18** $\frac{15}{3}$

The sentences in exercises 19 through 24 express true statements about the sums or products of natural numbers. For each statement, replace the natural numbers by the rational numbers that are their mates. Then write a sentence to express this statement. For exercise 19, you should write the following sentence: $\frac{12}{1} \cdot \frac{0}{1}=\frac{0}{1}$.

19 $12 \cdot 0=0$.

20 $0+1=1$.

21 $15 \cdot 20=300$.

22 $5+505=510$.

23 $3 \cdot 1=3$.

24 $10000+1=10001$.

The sentences in exercises 25 through 30 express true statements about the sums or products of rational numbers. For each statement, replace the rational numbers by the natural numbers that are their mates. Then write a sentence to express this statement. For exercise 25, you should write the following sentence: $62 + 70 = 132$.

25 $\frac{62}{1} + \frac{70}{1} = \frac{132}{1}$.
26 $\frac{14}{1} \cdot \frac{0}{1} = \frac{0}{1}$.
27 $\frac{25}{1} \cdot \frac{1}{1} = \frac{25}{1}$.
28 $\frac{101}{1} + \frac{1}{1} = \frac{102}{1}$.
29 $\frac{0}{1} + \frac{15}{1} = \frac{15}{1}$.
30 $\frac{222}{1} \cdot \frac{3}{1} = \frac{666}{1}$.

KEEPING SKILFUL

For each condition expressed in exercises 1 through 20, tabulate the solution set. U = C.

1 $7/5 \sim x/100$.
2 $c + 11(13) < 150$.
3 $27/37 \sim 108/m$.
4 18 is 3% of n.
5 $115 - x > 97$.
6 $26(29 + 36) = s$.
7 $813 + x < 27(34)$.
8 $3/y \sim 18/120$.
9 $(13 + 18)x < 4(8)$.
10 $16 - 9 + 18 = m$.
11 $110/100 \sim r/40$.
12 $x = 23(15) + 9(72)$.
13 273 is a% of 364.
14 $439 - 49(3) = m$.
15 $x/92 \sim 19/23$.
16 $4y = y$.
17 $c = 48 \div 3 \cdot 25$.
18 $x - 98 > 172$.
19 $k < 4(49) \div 2$.
20 84 is s% of 400.

For each ordered pair named below, first name the natural number onto which you can map the ordered pair by addition. Then name the natural number onto which you can map the ordered pair by multiplication.

21 (30, 78)
22 (45, 92)
23 (56, 234)
24 (457, 18)
25 (109, 83)
26 (211, 314)
27 (314, 560)
28 (199, 307)
29 (605, 470)

For each ordered pair of rational numbers named in exercises 30 through 41, find the difference of the first and second components.

30 $(\frac{9}{7}, \frac{3}{7})$
31 $(\frac{1}{2}, \frac{1}{3})$
32 $(\frac{3}{4}, \frac{5}{12})$
33 $(\frac{8}{9}, \frac{4}{7})$
34 $(\frac{3}{5}, \frac{1}{6})$
35 $(\frac{4}{7}, \frac{4}{7})$
36 $(\frac{13}{14}, \frac{6}{7})$
37 $(\frac{5}{6}, \frac{2}{9})$
38 $(\frac{15}{4}, \frac{3}{2})$
39 $(\frac{6}{7}, \frac{3}{6})$
40 $(\frac{11}{4}, \frac{16}{9})$
41 $(\frac{72}{2}, \frac{27}{1})$

Mixed numerals used in computation

In this lesson you will use the idea of "mates" to help you understand better the "mixed numbers" you used in earlier work in arithmetic.

A Read sentence A in D1. From your work in arithmetic you know that $4\frac{1}{2}$ means $4 + \frac{1}{2}$. Does sentence A express a true statement?

B Now look at sentence B. Is the natural number 4 the mate of the rational number $\frac{4}{1}$?

You know that for each natural number there is a rational number that is its mate. You also know that {0, 1, 2, . . .} and $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots\}$ behave in the same way with respect to the operations of addition and multiplication. Consequently, we can, if we wish, use the same numeral to name a member of {0, 1, 2, . . .} and a member of $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots\}$. For example, in the expression $4 + \frac{1}{2}$ in sentence B, the numeral 4 is used as a name for the rational number $\frac{4}{1}$.

C Does sentence C in D1 express a true statement? Does sentence D express a true statement?

The number $4\frac{1}{2}$ is the rational number $\frac{9}{2}$. The numeral $4\frac{1}{2}$, which is a name of the number $4\frac{1}{2}$, is a *mixed numeral*. Notice that a mixed numeral is made up of a standard name of a counting number and a fraction numeral. Mixed numerals are names of rational numbers.

Now you will learn a convenient way to obtain a fraction for the rational number that is named by a mixed numeral.

Mixed numerals; converting mixed numerals to fraction numerals and fraction numerals to mixed numerals; computation involving mixed numerals

A $4\frac{1}{2} = 4 + \frac{1}{2}$.

B $4 + \frac{1}{2} = \frac{4}{1} + \frac{1}{2}$.

C $\frac{4}{1} + \frac{1}{2} = \frac{9}{2}$.

D $4\frac{1}{2} = \frac{9}{2}$.

D1

E $8\frac{1}{3} = \frac{8}{1} + \frac{1}{3}$.

F $\frac{8}{1} + \frac{1}{3} = \frac{8(3)+1(1)}{1(3)}$.

G $6\frac{5}{8} = \frac{6}{1} + \frac{5}{8}$.

H $\frac{6}{1} + \frac{5}{8} = \frac{6(8)+1(5)}{1(8)}$.

I $4\frac{3}{4} = \frac{4}{1} + \frac{3}{4}$.

J $\frac{4}{1} + \frac{3}{4} = \frac{4(4)+1(3)}{1(4)}$.

D2

K $\frac{16}{5} = \frac{15+1}{5}$.

L $\frac{15+1}{5} = \frac{15}{5} + \frac{1}{5}$.

M $\frac{15}{5} + \frac{1}{5} = 3 + \frac{1}{5}$.

N $3 + \frac{1}{5} = 3\frac{1}{5}$.

O $\frac{16}{5} = 3\frac{1}{5}$.

D3

In earlier grades you may have worked with rational numbers without quite understanding why you did your work in certain ways. The work that follows explains why you can use these ways.

D Does sentence E in D2 express a true statement? How do you know?

E Now look at sentence F. Use the sum property of rational numbers to explain why the statement expressed by this sentence is true.

F Is $\frac{8(3)+1(1)}{1(3)} = \frac{25}{3}$? Is $8\frac{1}{3} = \frac{25}{3}$?

G You know that $\frac{25}{3}$ and $8\frac{1}{3}$ are the same number. What is the product of 8 and the denominator of $\frac{1}{3}$? Does the sum of this product and the numerator of $\frac{1}{3}$ equal the numerator of $\frac{25}{3}$? Is the denominator of $\frac{1}{3}$ the same as the denominator of $\frac{25}{3}$?

H Does each of sentences G and H express a true statement? Is $6\frac{5}{8} = \frac{53}{8}$?

I Does each of sentences I and J express a true statement? Is $4\frac{3}{4} = \frac{19}{4}$?

J Now find a fraction for $5\frac{3}{4}$. What number will you use as the denominator of this fraction? To find the numerator, first find the product of 5 and 4. What should you do then?

In each of exercises K through O, find a fraction for the rational number named by the mixed numeral.

K $2\frac{1}{8}$ L $3\frac{3}{5}$ M $8\frac{1}{2}$ N $4\frac{4}{9}$ O $5\frac{3}{7}$

In the following exercises you will learn to obtain a mixed numeral that names a given rational number.

A Look at sentence K in D3. What number is the denominator of $\frac{16}{5}$? Is 15 the greatest multiple of 5 that is less than 16? Is the numerator of $\frac{16}{5}$ the sum of 15 and 1? Does sentence K express a true statement?

Look at sentence L. From the definition of the sum of two rational numbers, you know that $\frac{15}{5} + \frac{1}{5} = \frac{15+1}{5}$ and that $\frac{15+1}{5} = \frac{15}{5} + \frac{1}{5}$. Therefore, sentence L expresses a true statement.

B Is $\frac{15}{5}$ the same rational number as $\frac{3}{1}$? Is the ? rational number $\frac{15}{5}$ the mate of the natural number 3? Explain why sentence M in D3 expresses a true statement.

C Does sentence N express a true statement? Does sentence O express a true statement?

In exercises A, B, and C you found a mixed numeral that names the rational number $\frac{16}{5}$. The first step in finding this numeral was to express the numerator of $\frac{16}{5}$ as the sum of two numbers. One of these numbers was the greatest multiple of the denominator that was less than the numerator.

D Look at sentence P in D4. Is 7 the greatest multiple of 7 that is less than 12?

E Look at sentence Q. What is the greatest multiple of 2 that is less than 13?

F Look at sentence R. What is the greatest multiple of 8 that is less than 25?

G Look at sentence S. What is the greatest multiple of 9 that is less than 35?

H What mixed numeral can be used instead of the numeral $\frac{12}{7}$? The numeral $\frac{13}{2}$? The numeral $\frac{25}{8}$? The numeral $\frac{35}{9}$?

For each of the fraction numerals below, find a mixed numeral that expresses the same rational number.

I $\frac{14}{5}$	L $\frac{29}{9}$	O $\frac{41}{12}$	R $\frac{16}{3}$	U $\frac{101}{7}$
J $\frac{17}{13}$	M $\frac{37}{6}$	P $\frac{63}{20}$	S $\frac{21}{11}$	V $\frac{91}{10}$
K $\frac{10}{3}$	N $\frac{50}{7}$	Q $\frac{39}{5}$	T $\frac{84}{5}$	W $\frac{80}{3}$

Now you will use the ideas you have studied in this lesson to learn some methods of computing with mixed numerals.

A Look at sentence T in D5. Are the natural number 4 and the rational number $\frac{4}{1}$ mates? Explain why sentence T expresses a true statement.

B Now add 4 and $\frac{3}{2}$. First use a fraction numeral to express your answer. Then use a mixed numeral to express your answer.

C Look at sentence U. Are the natural number 6 and the rational number $\frac{6}{1}$ mates? Explain why sentence U expresses a true statement.

D Find the difference of 6 and $\frac{4}{3}$. First use a fraction numeral to express your answer. Then use a mixed numeral to express your answer.

E Look at sentence V in D6. Does sentence V express a true statement?

A short cut may be used to find a fraction for the product $5 \times \frac{3}{4}$. You can multiply 5 and 3 to obtain the numerator and use 4 as the denominator.

P $\frac{12}{7} = \frac{7+5}{7}$.

Q $\frac{13}{2} = \frac{12+1}{2}$.

R $\frac{25}{8} = \frac{24+1}{8}$.

S $\frac{35}{9} = \frac{27+8}{9}$.

D4

T $4 + \frac{3}{2} = \frac{4}{1} + \frac{3}{2}$.

U $6 - \frac{4}{3} = \frac{6}{1} - \frac{4}{3}$.

D5

V $5 \times \frac{3}{4} = \frac{5}{1} \times \frac{3}{4}$.

W $2\frac{5}{8} \times 3\frac{1}{9} = \frac{21}{8} \times \frac{28}{9}$.

X $12 \times 1\frac{1}{7} = 13\frac{5}{7}$.

Y $6 \times \frac{2}{3} = 4$.

D6

F Is $5 \times \frac{3}{4} = \frac{15}{4}$? Explain why the short cut gives you the correct answer.

For each of exercises G through O, find a fraction numeral for the product expressed. Then find a mixed numeral for the product.

G $6 \times \frac{7}{8}$	J $10 \times \frac{2}{3}$	M $\frac{10}{13} \times 11$
H $9 \times \frac{1}{4}$	K $8 \times \frac{4}{9}$	N $42 \times \frac{1}{9}$
I $\frac{6}{5} \times 12$	L $13 \times \frac{3}{10}$	O $16 \times \frac{15}{7}$

P Study sentence W in D6. How do you obtain $\frac{21}{8} \times \frac{28}{9}$ from $2\frac{5}{8} \times 3\frac{1}{9}$? What mixed numeral names this product?

Q Does sentence X in D6 express a true statement? Explain your answer.

R Does sentence Y express a true statement? Explain your answer.

Now you know that mixed numerals are numerals that name rational numbers. You also know some ways to use mixed numerals in computation.

On your own

For each of exercises 1 through 6, find a mixed numeral that names the sum expressed.

1 $5+\frac{7}{8}$ 3 $6+\frac{2}{3}$ 5 $10+\frac{6}{8}$

2 $3+\frac{4}{5}$ 4 $\frac{7}{15}+11$ 6 $6+\frac{3}{4}$

For each of exercises 7 through 18, find a fraction numeral that expresses the rational number named.

7 $8\frac{6}{7}$ 10 $4\frac{1}{16}$ 13 $7\frac{7}{8}$ 16 $15\frac{2}{5}$

8 $10\frac{3}{4}$ 11 $12\frac{5}{9}$ 14 $11\frac{1}{2}$ 17 $13\frac{5}{6}$

9 $9\frac{2}{3}$ 12 $6\frac{1}{5}$ 15 $16\frac{9}{10}$ 18 $10\frac{11}{12}$

For each of exercises 19 through 26, find a mixed numeral to express the rational number named.

19 $\frac{11}{4}$ 21 $\frac{13}{8}$ 23 $\frac{25}{11}$ 25 $\frac{35}{4}$

20 $\frac{12}{5}$ 22 $\frac{16}{7}$ 24 $\frac{26}{9}$ 26 $\frac{38}{3}$

For each of exercises 27 through 38, first find a fraction numeral that expresses the sum, difference, or product. Then use a mixed numeral to express the sum, difference, or product.

27 $7+\frac{8}{3}$ 31 $5\times 1\frac{1}{4}$ 35 $10+\frac{8}{9}$

28 $12-\frac{4}{5}$ 32 $11+\frac{3}{8}$ 36 $5\frac{2}{5}\times 1\frac{1}{8}$

29 $2\frac{1}{2}\times 3\frac{2}{3}$ 33 $4-\frac{2}{7}$ 37 $6\frac{3}{5}\times 2\frac{1}{10}$

30 $7\times\frac{5}{6}$ 34 $16-\frac{7}{12}$ 38 $7\times\frac{5}{12}$

KEEPING SKILFUL

For each exercise below, find the least common denominator for the two fractions named. Then find the basic fraction that indicates the sum or difference expressed in the exercise.

1 $\frac{2}{3}+\frac{1}{4}$ 5 $\frac{6}{7}-\frac{2}{3}$ 9 $\frac{3}{9}+\frac{7}{12}$

2 $\frac{8}{9}-\frac{1}{3}$ 6 $\frac{3}{6}+\frac{2}{7}$ 10 $\frac{28}{4}-\frac{12}{9}$

3 $\frac{17}{16}-\frac{3}{4}$ 7 $\frac{1}{5}+\frac{5}{8}$ 11 $\frac{15}{17}-\frac{5}{6}$

4 $\frac{5}{6}+\frac{1}{12}$ 8 $\frac{18}{14}-\frac{3}{5}$ 12 $\frac{7}{11}+\frac{2}{12}$

For each exercise below, simplify the expression.

13 $176+398-496$ 16 $432\div(9\cdot 6)-8$

14 $9(36-7)+28\div 2$ 17 $(15+13)(23-16)$

15 $68+15\cdot 13-21$ 18 $123(9)+4(96)$

CHECKING UP

If you have trouble with these two tests, you should refer to lessons 77, 78, and 79.

Test 84

1 Which of the numbers named below is the greatest common factor of 72 and 126?

a $2^3\cdot 3^2$ c $2\cdot 3^2$

b $2^3\cdot 3^2\cdot 7$ d $2\cdot 3$

2 Which of the numbers named below is the least common denominator for the fractions $\frac{7}{60}$ and $\frac{5}{162}$?

a $2\cdot 3^4$ c $2\cdot 3$

b $2^3\cdot 3^5\cdot 5$ d $2^2\cdot 3^4\cdot 5$

3 Which of the following symbols is not a name of a rational number?

a $\frac{0}{5}$ b $\frac{5}{5}$ c $\frac{5}{0}$ d $\frac{5}{1}$

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

4 $\frac{25}{11}\times\frac{33}{5}=15.$ 7 $\frac{18}{4}\times\frac{24}{27}=4.$

5 $\frac{5}{6}\times\frac{33}{35}=\frac{11}{14}.$ 8 $\frac{8}{21}\times\frac{5}{12}=\frac{5}{63}.$

6 $\frac{4}{15}\times\frac{24}{5}=2.$ 9 $\frac{12}{13}\times\frac{3}{8}=\frac{3}{26}.$

For each of exercises 10 through 13, write a mixed numeral that expresses the rational number named.

10 $\frac{31}{7}$ 11 $\frac{47}{8}$ 12 $\frac{87}{12}$ 13 $\frac{130}{30}$

For each of exercises 14 through 21, write a fraction numeral that expresses the rational number named.

14 $1\frac{7}{10}$ 16 $2\frac{4}{7}$ 18 $4\frac{7}{8}$ 20 $8\frac{2}{5}$

15 $3\frac{5}{12}$ 17 $12\frac{3}{5}$ 19 $1\frac{1}{12}$ 21 $18\frac{7}{12}$

Test 85

For each of exercises 22 through 36, find the basic fraction for the sum, product, or difference expressed.

22 $\frac{10}{12}+\frac{12}{15}$ 27 $\frac{3}{15}+\frac{2}{21}$ 32 $\frac{15}{16}-\frac{5}{6}$

23 $\frac{27}{9}\times\frac{14}{35}$ 28 $3+\frac{2}{3}$ 33 $3\frac{1}{8}\times 4\frac{4}{5}$

24 $3\frac{4}{7}\times 2\frac{4}{5}$ 29 $5-\frac{7}{9}$ 34 $\frac{20}{25}+\frac{3}{5}$

25 $2-\frac{15}{16}$ 30 $\frac{16}{15}\times\frac{27}{24}$ 35 $2\frac{1}{3}\times 5\frac{1}{4}$

26 $\frac{19}{3}\times\frac{4}{6}$ 31 $\frac{24}{36}+\frac{1}{3}$ 36 $11-\frac{19}{24}$

The commutative properties

In this lesson you will first study an important property of equality. Then you will use this property to help you understand the commutative properties of multiplication and addition of rational numbers.

A Read sentences A and B in D1. Does each of these sentences express a true statement?

In your work so far, you have often used the idea that, if a first number is the same as a second number and the second number is the same as a third number, then the first number is the same as the third number.

B You know that $\frac{1}{4}+\frac{5}{4}$ is the same number as $\frac{6}{4}$ and that $\frac{6}{4}$ is the same number as $\frac{3}{4}+\frac{3}{4}$. Study sentence C in D1. Does it also express a true statement?

The fact that sentence C can be obtained from sentences A and B is an example of the *transitive* (tran′sə tiv) *property of equality*. The property is expressed below.

The universe for x, y, *and* z *is* R_a. *For each replacement of* x, y, *and* z, *if* x = y *and* y = z, *then* x = z.

C To obtain the statement expressed by sentence A from $x = y$, you replace x by the number $\frac{1}{4}+\frac{5}{4}$ and y by the number $\frac{6}{4}$. What replacements do you make in $y = z$ to obtain the statement expressed by sentence B? How do you obtain the statement expressed by sentence C from $x = z$? Notice that you use expressions to name the replacements for x and z.

D Look at D2. What is the universe for the variables? To obtain the statement expressed by sentence D, x was replaced by $\frac{7}{8}-\frac{1}{2}$, and y was replaced by $\frac{3}{8}$. Does sentence D express a true statement?

E To obtain the statement expressed by sentence E, y was again replaced by $\frac{3}{8}$, and z was replaced by $\frac{1}{4}+\frac{1}{8}$. Does sentence E express a true statement?

F Think about the statement expressed by sentence F. x was replaced by $\frac{7}{8}-\frac{1}{2}$ and z by $\frac{1}{4}+\frac{1}{8}$. Is the statement true? Upon what property does your answer depend?

D1

A $\frac{1}{4}+\frac{5}{4}=\frac{6}{4}$.

B $\frac{6}{4}=\frac{3}{4}+\frac{3}{4}$.

C $\frac{1}{4}+\frac{5}{4}=\frac{3}{4}+\frac{3}{4}$.

D2

Universe for x, y, and $z = R_a$.

D $x = y$. $\quad \frac{7}{8}-\frac{1}{2}=\frac{3}{8}$.

E $y = z$. $\quad \frac{3}{8}=\frac{1}{4}+\frac{1}{8}$.

F $x = z$. $\quad \frac{7}{8}-\frac{1}{2}=\frac{1}{4}+\frac{1}{8}$.

In unit 6 you studied the commutative properties of multiplication and addition of natural numbers. You learned that the order in which you multiply two natural numbers does not affect the product and that the order in which you add two natural numbers does not affect the sum. Now you will develop the commutative properties of multiplication and addition of rational numbers.

A Does each sentence in D3 express a true statement about the product of two rational numbers?

B Look at sentences G and H. Use the transitive property to explain why $\frac{1}{5} \times \frac{2}{1} = \frac{2}{1} \times \frac{1}{5}$.

C Does the order in which you multiply $\frac{1}{5}$ and $\frac{2}{1}$ affect the product?

D Read sentences I and J. Is $\frac{3}{8} \times \frac{1}{6} = \frac{1}{6} \times \frac{3}{8}$? Explain your answer.

E Does the order in which you multiply $\frac{3}{8}$ and $\frac{1}{6}$ affect the product?

F Read sentences K and L in D3. Is $\frac{3}{16} \times \frac{3}{4} = \frac{3}{4} \times \frac{3}{16}$? How do you know?

G Does the order in which you multiply $\frac{3}{16}$ and $\frac{3}{4}$ affect the product?

H Do you think that the operation of multiplication of rational numbers has the commutative property?

You can use variables to help you decide whether or not the order in which you multiply any two rational numbers affects the product.

I Look at D4. Think of $\frac{a}{b}$ and $\frac{c}{d}$ as any two rational numbers. What is the universe for each of the variables?

J Read sentence M in D4. For each replacement of the variables, is $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$? Upon what definition does your answer depend?

K Read sentence N. For each replacement of the variables, is $\frac{c}{d} \times \frac{a}{b} = \frac{ca}{db}$? Upon what definition does your answer depend?

L Read sentence O. For each replacement of the variables, is $\frac{ac}{bd} = \frac{ca}{db}$? Upon what property of multiplication of natural numbers does your answer depend?

M Read sentence P. You know that, for each replacement of the variables, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ and $\frac{ac}{bd} = \frac{ca}{db}$. For each replacement of the variables, is $\frac{a}{b} \times \frac{c}{d} = \frac{ca}{db}$? Upon what property does your answer depend?

N Read sentence Q in D4. You know that, for each replacement of the variables, $\frac{a}{b} \times \frac{c}{d} = \frac{ca}{db}$ and $\frac{c}{d} \times \frac{a}{b} = \frac{ca}{db}$. For each replacement of the variables, is $\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$? Upon what property does your answer depend?

G $\frac{1}{5} \times \frac{2}{1} = \frac{2}{5}$.

H $\frac{2}{1} \times \frac{1}{5} = \frac{2}{5}$.

I $\frac{3}{8} \times \frac{1}{6} = \frac{3}{48}$.

J $\frac{1}{6} \times \frac{3}{8} = \frac{3}{48}$.

K $\frac{3}{16} \times \frac{3}{4} = \frac{9}{64}$.

L $\frac{3}{4} \times \frac{3}{16} = \frac{9}{64}$.

D3

Universe for a and c = N.
Universe for b and d = C.

M $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

N $\frac{c}{d} \times \frac{a}{b} = \frac{ca}{db}$.

O $\frac{ac}{bd} = \frac{ca}{db}$.

P $\frac{a}{b} \times \frac{c}{d} = \frac{ca}{db}$.

Q $\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$.

D4

You have just discovered that the order in which you multiply two rational numbers does not affect the product. This important property is the *commutative property of multiplication of rational numbers*. The property is expressed below.

The universe for x *and* y *is* R_a. *For each replacement of the variables*, xy = yx.

Notice that you have developed the commutative property of multiplication of rational numbers from information that you already have. In earlier work you accepted the definition of product, the commutative property of multiplication of natural numbers, and the transitive property of equality. From this information you *derived* the commutative property of multiplication of rational numbers. To derive a property means to use information that you already have to show that a condition like $\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$ is actually a property. After you have derived a property, you can say that you have *proved* that the condition is a property. When you have proved that a certain condition is a property, you know that you will obtain a true statement from the condition for each replacement of the variables.

R $\frac{3}{8}+\frac{3}{4}=\frac{9}{8}$.

S $\frac{3}{4}+\frac{3}{8}=\frac{9}{8}$.

T $\frac{7}{10}+\frac{1}{5}=\frac{9}{10}$.

U $\frac{1}{5}+\frac{7}{10}=\frac{9}{10}$.

V $\frac{22}{3}+\frac{11}{12}=\frac{99}{12}$.

W $\frac{11}{12}+\frac{22}{3}=\frac{99}{12}$.

D5

Universe for x and $y = R_a$.

$x - y = y - x$.

X $\frac{7}{10}-\frac{3}{10}=\frac{3}{10}-\frac{7}{10}$.

Y $\frac{3}{4}-\frac{1}{2}=\frac{1}{2}-\frac{3}{4}$.

Z $\frac{11}{12}-\frac{0}{3}=\frac{0}{3}-\frac{11}{12}$.

D6

Decide whether each of the following sentences expresses a true statement or a false statement.

O $\frac{3}{2}\times\frac{2}{3}=\frac{2}{3}\times\frac{3}{2}$.

P $\frac{1}{5}\times\frac{5}{5}=\frac{5}{1}\times\frac{5}{5}$.

Q $\frac{3}{10}\times\frac{6}{25}=\frac{6}{25}\times\frac{3}{10}$.

R $\frac{4}{2}\times\frac{6}{1}=\frac{6}{1}\times\frac{4}{2}$.

S $\frac{3}{4}\times\frac{1}{19}=\frac{1}{19}\times\frac{4}{3}$.

T $\frac{30}{1}\times\frac{11}{12}=\frac{11}{1}\times\frac{12}{30}$.

Next you will decide whether or not the order in which you add two rational numbers affects the sum.

A Does each sentence in D5 express a true statement about the sum of two rational numbers?

B Look again at sentences R and S. Use the transitive property to explain why $\frac{3}{8}+\frac{3}{4}=\frac{3}{4}+\frac{3}{8}$.

C Does the order in which you add $\frac{3}{8}$ and $\frac{3}{4}$ affect the sum?

D Read sentences T and U. Is $\frac{7}{10}+\frac{1}{5}=\frac{1}{5}+\frac{7}{10}$? Upon what property does your answer depend?

E Does the order in which you add $\frac{7}{10}$ and $\frac{1}{5}$ affect the sum?

F Read sentences V and W in D5. Is $\frac{22}{3}+\frac{11}{12}=\frac{11}{12}+\frac{22}{3}$? How do you know?

G Does the order in which you add $\frac{22}{3}$ and $\frac{11}{12}$ affect the sum?

H Do you think that the operation of addition of rational numbers has the commutative property?

If you continue to study statements like those expressed in D5, you will see that the order in which you add any two rational numbers does not affect the sum. We will accept without proof the *commutative property of addition of rational numbers*. The property is expressed below.

The universe for x *and* y *is* R_a. *For each replacement of the variables,* $x + y = y + x$.

Decide whether each of the following sentences expresses a true statement or a false statement.

I $\frac{7}{8}+\frac{2}{3}=\frac{2}{3}+\frac{7}{8}$.

J $\frac{4}{5}+\frac{5}{4}=\frac{4}{4}+\frac{5}{5}$.

K $\frac{0}{8}+\frac{9}{10}=\frac{9}{10}+\frac{0}{8}$.

L $\frac{33}{10}+\frac{16}{3}=\frac{16}{3}+\frac{10}{33}$.

Now you know that both multiplication and addition of rational numbers have the commutative property. Next you will decide whether or not subtraction of rational numbers has the commutative property.

M Explain in words the condition expressed in D6. What is the universe for the variables?

N To obtain each statement expressed in D6, what replacements were made for the variables?

O Look again at sentence X. You know that $\frac{7}{10}-\frac{3}{10}=\frac{4}{10}$. How do you know that $\frac{3}{10}-\frac{7}{10}$ is not equal to $\frac{4}{10}$?

P Do any of the sentences in D6 express true statements?

Q Does subtraction of rational numbers have the commutative property? Explain your answer.

In this lesson you have learned to use the transitive property of equality. You also have learned that both addition and multiplication of rational numbers have the commutative property and that subtraction of rational numbers does not have the commutative property.

On your own

Two true statements about rational numbers are expressed in each of exercises 1 through 6. Use these statements and the transitive property of equality to write a sentence that expresses another true statement. For exercise 1 you could write the sentence $\frac{1}{6}+\frac{2}{3}=\frac{1}{3}\times\frac{5}{2}$.

1 $\frac{1}{6}+\frac{2}{3}=\frac{5}{6}$. $\frac{5}{6}=\frac{1}{3}\times\frac{5}{2}$.

2 $\frac{3}{4}+\frac{1}{4}=\frac{4}{4}$. $\frac{4}{4}=\frac{2}{4}+\frac{2}{4}$.

3 $\frac{5}{8}+\frac{1}{2}=\frac{9}{8}$. $\frac{9}{8}=\frac{10}{4}-\frac{11}{8}$.

4 $\frac{11}{20}-\frac{1}{4}=\frac{6}{20}$. $\frac{6}{20}=\frac{3}{5}-\frac{6}{20}$.

5 $\frac{7}{10}-\frac{7}{10}=\frac{0}{10}$. $\frac{0}{10}=\frac{2}{5}\times\frac{0}{2}$.

6 $\frac{2}{9}+\frac{1}{4}=\frac{17}{12}\times\frac{1}{3}$. $\frac{17}{12}\times\frac{1}{3}=\frac{17}{36}-\frac{0}{12}$.

Decide whether each of the following sentences expresses a true statement or a false statement.

7 $\frac{1}{2}+\frac{3}{4}=\frac{3}{4}+\frac{1}{2}$.

8 $\frac{7}{10}\times\frac{2}{3}=\frac{7}{2}\times\frac{10}{3}$.

9 $\frac{1}{8}+\frac{3}{8}=\frac{1}{3}+\frac{1}{8}$.

10 $\frac{0}{6}\times\frac{1}{11}=\frac{1}{11}\times\frac{0}{6}$.

11 $\frac{9}{21}\times\frac{7}{7}=\frac{7}{7}\times\frac{9}{21}$.

12 $\frac{0}{15}+\frac{3}{29}=\frac{3}{29}+\frac{0}{13}$.

13 $\frac{3}{5}+\frac{1}{10}=\frac{3}{10}+\frac{5}{1}$.

14 $\frac{17}{1}\times\frac{1}{1}=\frac{1}{17}\times\frac{17}{1}$.

15 $\frac{4}{6}+\frac{3}{2}=\frac{3}{2}+\frac{6}{4}$.

16 $\frac{11}{9}\times\frac{9}{11}=\frac{9}{11}\times\frac{11}{9}$.

For each ordered pair of rational numbers named in exercises 17 through 28, first write a sentence about the product of the components. Then use the same numbers and the commutative property of multiplication to write another sentence that expresses a true statement about the same product. For exercise 17 you could write the following sentences: $\frac{1}{3}\times\frac{7}{8}=\frac{7}{24}$. $\frac{7}{8}\times\frac{1}{3}=\frac{7}{24}$.

17 $(\frac{1}{3}, \frac{7}{8})$ **18** $(\frac{4}{9}, \frac{1}{2})$ **19** $(\frac{3}{3}, \frac{7}{6})$ **20** $(\frac{8}{10}, \frac{9}{9})$ **21** $(\frac{5}{2}, \frac{0}{15})$ **22** $(\frac{11}{12}, \frac{1}{11})$ **23** $(\frac{4}{5}, \frac{5}{4})$ **24** $(\frac{0}{8}, \frac{8}{8})$ **25** $(\frac{4}{9}, \frac{2}{5})$ **26** $(\frac{7}{2}, \frac{3}{4})$ **27** $(\frac{2}{6}, \frac{0}{11})$ **28** $(\frac{15}{8}, \frac{7}{7})$

For each ordered pair of rational numbers named in exercises 29 through 40, first write a sentence about the sum of the components. Then use the same numbers and the commutative property of addition to write another sentence that expresses a true statement about the same sum. For exercise 29 you could write the following sentences: $\frac{1}{4}+\frac{1}{12}=\frac{1}{3}$. $\frac{1}{12}+\frac{1}{4}=\frac{1}{3}$.

29 $(\frac{1}{4}, \frac{1}{12})$ **30** $(\frac{7}{5}, \frac{1}{8})$ **31** $(\frac{6}{3}, \frac{0}{5})$ **32** $(\frac{1}{10}, \frac{2}{3})$ **33** $(\frac{7}{7}, \frac{6}{6})$ **34** $(\frac{5}{4}, \frac{7}{12})$ **35** $(\frac{10}{2}, \frac{2}{10})$ **36** $(\frac{0}{8}, \frac{1}{6})$ **37** $(\frac{4}{5}, \frac{1}{2})$ **38** $(\frac{12}{4}, \frac{1}{3})$ **39** $(\frac{8}{9}, \frac{5}{18})$ **40** $(\frac{2}{7}, \frac{11}{10})$

KEEPING SKILFUL

Use the following information to answer the questions asked in exercises 1 through 14: Point B is between points A and C. Points B, C, and D are noncollinear. Point B is between points D and E. Make a sketch if you need to.

1 How many planes contain points A and C? Points B and E? Points B, C, and D?

2 How many segments determined by points A, B, C, D, and E contain point B?

3 Suppose that point R is between A and C. Write a sentence that expresses a true statement comparing $\overline{AR}$ and $\overline{AC}$.

4 Suppose that C is between A and R. Write a sentence that expresses a true statement comparing $\overline{AR}$ and $\overline{AC}$.

5 Suppose that C and R are the same point. Write a sentence that expresses a true statement comparing $\overline{AR}$ and $\overline{AC}$.

6 Describe the union of $\overline{AC}$, $\overline{CD}$, and $\overline{DA}$.

7 Describe the union of $\overrightarrow{EA}$ and $\overrightarrow{EC}$.

8 Describe the intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{DE}$.

9 Name the linear pairs of angles formed by $\overleftrightarrow{AC}$ and $\overleftrightarrow{DE}$.

10 Name the sides and the angles of polygon ADCE.

11 Describe the union of $\overline{EC}$, $\overline{CD}$, and $\overline{DE}$.

12 Describe the union of $\overline{EB}$, $\overline{BC}$, and $\overline{CE}$.

13 $\triangle BCD$ is the union of what segments?

14 How many triangles are determined by points A, B, C, D, and E? Name these triangles.

15 How many triangles are determined by five points, no three of which are collinear?

The associative properties

When you studied the associative properties of multiplication and addition of natural numbers, you learned that the way in which you group three natural numbers does not affect their product or their sum. In this lesson you will see whether or not multiplication and addition of rational numbers also have the associative properties.

Look at symbol A in D1. You know that multiplication is a binary operation. To find the product of the numbers named by symbol A, you must work with one pair of numbers at a time.

In your work with natural numbers you used parentheses to help you think about grouping numbers when you multiplied. Parentheses are also useful in helping you think about grouping rational numbers.

A Study sentence B in D1. In the numeral $(\frac{1}{2} \cdot \frac{2}{3})\frac{3}{4}$, the product of what pair of numbers is expressed within parentheses? Is $\frac{2}{6}$ the product of $\frac{1}{2}$ and $\frac{2}{3}$?

B What is the product of $\frac{2}{6}$ and $\frac{3}{4}$? Is $(\frac{1}{2} \cdot \frac{2}{3})\frac{3}{4} = \frac{6}{24}$?

C Look again at sentence B. In the numeral $\frac{1}{2}(\frac{2}{3} \cdot \frac{3}{4})$, the product of what pair of numbers is expressed within parentheses? What is the product of $\frac{2}{3}$ and $\frac{3}{4}$?

D What is the product of $\frac{1}{2}$ and the number you found for exercise C? Is $\frac{1}{2}(\frac{2}{3} \cdot \frac{3}{4}) = \frac{6}{24}$?

E Does sentence B in D1 express a true statement? Does the way in which you group $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$ when you multiply affect the product?

A $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4}$

B $(\frac{1}{2} \cdot \frac{2}{3})\frac{3}{4} = \frac{1}{2}(\frac{2}{3} \cdot \frac{3}{4})$.

D1

C $(\frac{1}{5} \cdot \frac{1}{2})\frac{2}{5} = \frac{1}{5}(\frac{1}{2} \cdot \frac{2}{5})$.

D $(\frac{3}{4} \cdot \frac{5}{8})\frac{4}{3} = \frac{3}{4}(\frac{5}{8} \cdot \frac{4}{3})$.

D2

F Now read sentence C in D2. What is the product of $\frac{1}{5}$ and $\frac{1}{2}$? Of $\frac{1}{10}$ and $\frac{2}{5}$?

G What is the product of $\frac{1}{2}$ and $\frac{2}{5}$? What is the product of $\frac{1}{5}$ and $\frac{2}{10}$?

H Does sentence C express a true statement? Does the way in which you group $\frac{1}{5}$, $\frac{1}{2}$, and $\frac{2}{5}$ when you multiply affect the product?

I Read sentence D in D2. How do you know that sentence D expresses a true statement? Does the way in which you group $\frac{3}{4}$, $\frac{5}{8}$, and $\frac{4}{3}$ when you multiply affect the product?

J Do you think that the operation of multiplication of rational numbers has the associative property?

As you continue to think about statements like those expressed in D2, you will see that the way in which you group any three rational numbers does not affect the product. We will accept without proof the *associative property of multiplication of rational numbers*. The property is expressed below.

The universe for x, y, *and* z *is* R_a. *For each replacement of the variables*, (xy)z = x(yz).

Decide whether each of the following sentences expresses a true statement or a false statement.

K $(\frac{1}{2} \cdot \frac{17}{5})\frac{4}{9} = \frac{1}{2}(\frac{17}{5} \cdot \frac{4}{9})$.

L $(\frac{8}{7} \cdot \frac{2}{11})\frac{3}{4} = \frac{8}{7} + (\frac{2}{11} \cdot \frac{3}{4})$.

M $(\frac{17}{3} \cdot \frac{22}{16})\frac{9}{43} = \frac{17}{3}(\frac{22}{16} \cdot \frac{9}{43})$.

N $\frac{4}{5}(\frac{1}{2} \cdot \frac{1}{10}) = (\frac{4}{5} \cdot \frac{1}{2})\frac{1}{10}$.

Now you will see if addition of rational numbers also has the associative property.

Look at symbol E in D3. You know that addition is a binary operation. To find the sum of the numbers named by symbol E, you must work with one pair of numbers at a time.

You can use parentheses to help you think about grouping rational numbers when you add.

A Study sentence F in D3. In the numeral $(\frac{3}{8}+\frac{1}{5})+\frac{2}{5}$, the sum of what pair of numbers is expressed within parentheses? Is $\frac{23}{40}$ the sum of $\frac{3}{8}$ and $\frac{1}{5}$?

B What is the sum of $\frac{23}{40}$ and $\frac{2}{5}$? Is $(\frac{3}{8}+\frac{1}{5})+\frac{2}{5}=\frac{195}{200}$? Is $\frac{195}{200}=\frac{39}{40}$?

C Look again at sentence F. In the numeral $\frac{3}{8}+(\frac{1}{5}+\frac{2}{5})$, the sum of what pair of numbers is expressed within parentheses? What is the sum of $\frac{1}{5}$ and $\frac{2}{5}$?

D What is the sum of $\frac{3}{8}$ and the number that you named for exercise C? Is $\frac{3}{8}+(\frac{1}{5}+\frac{2}{5})=\frac{39}{40}$?

E Does sentence F express a true statement? Does the way in which you group $\frac{3}{8}$, $\frac{1}{5}$, and $\frac{2}{5}$ affect the sum?

F Now read sentence G in D4. What is the sum of $\frac{2}{3}$ and $\frac{1}{4}$? What is the sum of $\frac{11}{12}$ and $\frac{3}{4}$?

G What is the sum of $\frac{1}{4}$ and $\frac{3}{4}$? What is the sum of $\frac{2}{3}$ and 1?

H Does sentence G express a true statement? Does the way in which you group $\frac{2}{3}$, $\frac{1}{4}$, and $\frac{3}{4}$ when you add affect the sum?

I Read sentence H in D4. How do you know that sentence H expresses a true statement? Does the way in which you group $\frac{5}{6}$, $\frac{1}{3}$, and $\frac{2}{4}$ when you add affect the sum?

As you continue to think about statements like those expressed in D4, you will see that the way in which you group three rational numbers does not affect the sum. We will accept without proof the *associative property of addition of rational numbers.* The property is expressed below.

E $\frac{3}{8}+\frac{1}{5}+\frac{2}{5}$

F $(\frac{3}{8}+\frac{1}{5})+\frac{2}{5}=\frac{3}{8}+(\frac{1}{5}+\frac{2}{5})$.

D3

G $(\frac{2}{3}+\frac{1}{4})+\frac{3}{4}=\frac{2}{3}+(\frac{1}{4}+\frac{3}{4})$.

H $(\frac{5}{6}+\frac{1}{3})+\frac{2}{4}=\frac{5}{6}+(\frac{1}{3}+\frac{2}{4})$.

D4

Universe for x, y, and $z = R_a$.

$(x-y)-z=x-(y-z)$.

I $(\frac{25}{2}-\frac{3}{4})-\frac{3}{4}=\frac{25}{2}-(\frac{3}{4}-\frac{3}{4})$.

D5

The universe for x, y, *and* z *is* R_a. *For each replacement of the variables,* $(x+y)+z=x+(y+z)$.

Decide whether each of the following sentences expresses a true statement or a false statement.

J $(\frac{4}{3}+\frac{10}{10})+\frac{0}{6}=\frac{4}{3}+(\frac{10}{10}+\frac{0}{6})$.

K $(\frac{2}{5}+\frac{5}{8})+\frac{1}{3}=\frac{2}{5}+(\frac{8}{5}+\frac{3}{1})$.

L $(\frac{9}{9}+\frac{3}{11})+\frac{2}{16}=\frac{9}{9}+(\frac{3}{11}+\frac{2}{16})$.

Now you know that both multiplication and addition of rational numbers have the associative property. Next you will decide if subtraction of rational numbers has the associative property.

M Explain in words the condition expressed in D5. What is the universe for the variables?

N What replacements were made for the variables to obtain the statement expressed by sentence I in D5? Is the statement true?

O From the false statement expressed in D5, how do you know that subtraction of ra-

J $3\frac{3}{4} + 2\frac{1}{8} = (3 + \frac{3}{4}) + (2 + \frac{1}{8})$.

K $(3 + \frac{3}{4}) + (2 + \frac{1}{8}) = 3 + (\frac{3}{4} + 2) + \frac{1}{8}$.

L $3 + (\frac{3}{4} + 2) + \frac{1}{8} = 3 + (2 + \frac{3}{4}) + \frac{1}{8}$.

M $3 + (2 + \frac{3}{4}) + \frac{1}{8} = (3 + 2) + (\frac{3}{4} + \frac{1}{8})$.

D6

A	B	C	D
$4\frac{7}{8}$	$5\frac{2}{3}$	$5\frac{8}{12}$	$4\frac{20}{12}$
$3\frac{1}{2}$	$2\frac{3}{4}$	$2\frac{9}{12}$	$2\frac{9}{12}$
$1\frac{3}{8}$			$2\frac{11}{12}$

D7

tional numbers does not have the associative property?

In your earlier arithmetic work, you learned to add rational numbers that were expressed by mixed numerals. To find the sum of $3\frac{3}{4}$ and $2\frac{1}{8}$, for example, you first found the sum of $\frac{3}{4}$ and $\frac{1}{8}$. Then you found the sum of 3 and 2.

Now you will use the transitive, associative, and commutative properties to explain why you can use this method when you work with rational numbers expressed by mixed numerals.

A Read sentence J in D6. Does sentence J express a true statement? How do you know?

B Does sentence K express a true statement? ? Upon what property of addition does your answer depend?

C Does sentence L express a true statement? Upon what property of addition does your answer depend?

D Does sentence M express a true statement? Upon what property of addition does your answer depend?

E Is $3\frac{3}{4} + 2\frac{1}{8} = (3 + 2) + (\frac{3}{4} + \frac{1}{8})$? Is $(3 + 2) + (\frac{3}{4} + \frac{1}{8}) = 5\frac{7}{8}$? Is $3\frac{3}{4} + 2\frac{1}{8} = 5\frac{7}{8}$? Explain your answers.

From the true statements expressed in D6 and from exercises A through E, you see why you can first find the sum of $\frac{1}{8}$ and $\frac{3}{4}$ and then find the sum of 2 and 3 to obtain the sum of $3\frac{3}{4}$ and $2\frac{1}{8}$. In arithmetic you arranged your work differently, but the method you used depended on properties that you have studied in this book.

To find the difference of two rational numbers such as $4\frac{7}{8}$ and $3\frac{1}{2}$, you were taught first to find the difference of $\frac{7}{8}$ and $\frac{1}{2}$. Then you found the difference of 4 and 3.

F Look at the computation labeled A in D7. What is the difference of $\frac{7}{8}$ and $\frac{1}{2}$? Of 4 and 3? Is $4\frac{7}{8} - 3\frac{1}{2} = 1\frac{3}{8}$?

G Now look at the computation labeled B, C, and D in D7. To find the difference of $5\frac{2}{3}$ and $2\frac{3}{4}$, you first find a common denominator for the fractions $\frac{2}{3}$ and $\frac{3}{4}$. Is 12 a common denominator for $\frac{2}{3}$ and $\frac{3}{4}$? Is $\frac{2}{3} = \frac{8}{12}$? Is $\frac{3}{4} = \frac{9}{12}$?

H Is $5\frac{2}{3} = 5\frac{8}{12}$? Is $2\frac{3}{4} = 2\frac{9}{12}$?

I Before you can find the difference of $5\frac{8}{12}$ and $2\frac{9}{12}$, you will think of $5\frac{8}{12}$ as $4\frac{20}{12}$. Why is this necessary? Is $5\frac{8}{12} = 4\frac{20}{12}$?

J What is the difference of $\frac{20}{12}$ and $\frac{9}{12}$? Of 4 and 2? Is $5\frac{2}{3} - 2\frac{3}{4} = 2\frac{11}{12}$?

In this lesson you learned that both multiplication and addition of rational numbers have the associative property, but that subtraction of rational numbers does not have this property. The associative property enables you to change the grouping when you add or multiply.

On your own

First decide whether each of the sentences in exercises 1 through 14 expresses a true statement or a false statement. Then, for each true statement (or group of statements), tell which of the following properties applies:

Transitive property of equality

Associative property of addition

Commutative property of addition
Associative property of multiplication
Commutative property of multiplication

1 $\frac{3}{8}+\frac{1}{2}=\frac{1}{2}+\frac{3}{8}$.

2 $\frac{1}{3}(\frac{1}{9}\cdot\frac{1}{6})=(\frac{1}{3}\cdot\frac{1}{9})\frac{1}{6}$.

3 $\frac{4}{5}-\frac{1}{4}=\frac{1}{4}-\frac{4}{5}$.

4 $\frac{7}{2}\times\frac{2}{7}=\frac{2}{7}\times\frac{7}{2}$.

5 $\frac{1}{4}+\frac{3}{12}=\frac{6}{12}$.
$\frac{6}{12}=\frac{7}{12}-\frac{0}{12}$.
$\frac{1}{4}+\frac{3}{12}=\frac{7}{12}-\frac{0}{12}$.

6 $\frac{2}{3}\times\frac{1}{5}=\frac{2}{15}$.
$\frac{2}{15}=\frac{1}{3}-\frac{1}{5}$.
$\frac{2}{3}\times\frac{1}{5}=\frac{1}{3}-\frac{1}{5}$.

7 $\frac{3}{10}\times\frac{4}{5}=\frac{3}{4}\times\frac{10}{5}$.

8 $\frac{3}{1}(\frac{1}{8}\cdot\frac{1}{2})=(\frac{3}{1}\cdot\frac{1}{8})\frac{1}{2}$.

9 $\frac{8}{3}+\frac{3}{8}=\frac{8}{8}+\frac{3}{3}$.

10 $\frac{12}{4}\times\frac{6}{8}=\frac{6}{8}\times\frac{12}{4}$.

11 $\frac{1}{8}+(\frac{7}{8}+\frac{1}{10})=(\frac{1}{8}+\frac{7}{8})+\frac{1}{10}$.

12 $\frac{6}{2}-(\frac{3}{2}-\frac{4}{3})=(\frac{6}{2}-\frac{3}{2})-\frac{4}{3}$.

13 $\frac{1}{9}+(\frac{6}{6}+\frac{2}{7})=(\frac{1}{9}+\frac{6}{6})+\frac{2}{7}$.

14 $\frac{7}{8}+\frac{8}{7}=\frac{8}{7}+\frac{7}{8}$.

For each of exercises 15 through 22, first show two ways in which the numbers expressed in the exercise can be grouped without affecting the product. Then use a fraction numeral to name the product.

15 $\frac{3}{4}\times\frac{3}{8}\times\frac{1}{5}$

16 $\frac{9}{10}\times\frac{10}{9}\times\frac{1}{3}$

17 $\frac{6}{5}\times\frac{1}{2}\times\frac{1}{4}$

18 $\frac{0}{6}\times\frac{3}{7}\times\frac{5}{8}$

19 $\frac{1}{25}\times\frac{25}{2}\times\frac{1}{16}$

20 $\frac{4}{4}\times\frac{2}{3}\times\frac{1}{2}$

21 $\frac{4}{11}\times\frac{1}{7}\times\frac{0}{5}$

22 $\frac{3}{3}\times\frac{4}{5}\times\frac{2}{3}$

For each of exercises 23 through 28, first show two ways in which the numbers expressed can be grouped without affecting the sum. Then use a fraction numeral or a mixed numeral to name the sum.

23 $\frac{1}{6}+\frac{2}{3}+\frac{5}{2}$

24 $\frac{5}{8}+\frac{3}{4}+\frac{1}{5}$

25 $\frac{3}{7}+\frac{2}{2}+\frac{6}{7}$

26 $\frac{3}{6}+\frac{13}{13}+\frac{7}{6}$

27 $\frac{2}{3}+\frac{1}{4}+\frac{7}{12}$

28 $\frac{6}{20}+\frac{3}{10}+\frac{4}{5}$

For each ordered pair named in exercises 29 through 37, first find the sum of the components. Then find the difference of the first and second components.

29 $(10\frac{3}{8}, 7\frac{5}{8})$

30 $(19\frac{5}{6}, 13\frac{3}{8})$

31 $(20, 3\frac{1}{2})$

32 $(14\frac{4}{5}, 13\frac{7}{8})$

33 $(36\frac{3}{4}, 18)$

34 $(4\frac{2}{3}, 1\frac{2}{5})$

35 $(8\frac{1}{4}, 6\frac{3}{8})$

36 $(101, 11\frac{1}{2})$

37 $(17\frac{7}{10}, 14\frac{9}{10})$

7 82 Exploring ideas

The distributive property and the system of rational numbers

In unit 6 you learned that multiplication distributes over addition in the set of natural numbers. In this lesson you will see whether or not multiplication distributes over addition in the set of rational numbers.

A Look at sentence A in D1. What product is expressed at the left of the symbol for equality? What sum is expressed at the right of the symbol for equality?

Now you will decide if sentence A expresses a true statement. To do this, you will first work with $\frac{1}{2}(\frac{2}{3}+\frac{1}{4})$.

B Look at sentence B in D2. Is $\frac{11}{12}$ the sum of $\frac{2}{3}$ and $\frac{1}{4}$? Does sentence B express a true statement?

C What is the product of $\frac{1}{2}$ and $\frac{11}{12}$? Does sentence C express a true statement?

D Does sentence D express a true statement? Upon what property does your answer depend?

A $\frac{1}{2}(\frac{2}{3}+\frac{1}{4})=\frac{1}{2}(\frac{2}{3})+\frac{1}{2}(\frac{1}{4})$.

D1

B $\frac{1}{2}(\frac{2}{3}+\frac{1}{4})=\frac{1}{2}(\frac{11}{12})$.

C $\frac{1}{2}(\frac{11}{12})=\frac{11}{24}$.

D $\frac{1}{2}(\frac{2}{3}+\frac{1}{4})=\frac{11}{24}$.

D2

E Look again at sentence A in D1. Is the product expressed at the left of the symbol for equality equal to $\frac{11}{24}$?

Now you will work with $\frac{1}{2}(\frac{2}{3}) + \frac{1}{2}(\frac{1}{4})$.

F Look at D3. Is $\frac{1}{3}$ the product of $\frac{1}{2}$ and $\frac{2}{3}$? Is $\frac{1}{8}$ the product of $\frac{1}{2}$ and $\frac{1}{4}$? Does sentence E express a true statement?

G Is $\frac{11}{24}$ the sum of $\frac{1}{3}$ and $\frac{1}{8}$? Does sentence F express a true statement?

H How do you know that sentence G expresses a true statement?

I Look again at sentence A in D1. Is the sum expressed at the right of the symbol for equality equal to $\frac{11}{24}$? Does sentence A express a true statement?

J Now read the sentences in D4. For each sentence, first find the sum and then the product expressed at the left of the symbol for equality. Next, find the sum of the products expressed at the right of the symbol for equality. Then decide if the sentence expresses a true statement.

Look again at sentence H. Notice that, when you multiply the sum of $\frac{1}{5}$ and $\frac{2}{5}$ by $\frac{2}{3}$, you obtain the same number as when you add the product of $\frac{2}{3}$ and $\frac{1}{5}$ and the product of $\frac{2}{3}$ and $\frac{2}{5}$. In each sentence in D4, the product expressed at the left of the symbol for equality is the same number as the sum expressed at the right. Each statement expressed in D4 is an example of the *distributive property of multiplication over addition of rational numbers*. We will accept the property without proof. The property is expressed below.

The universe for x, y, *and* z *is* R_a. *For each replacement of the variables*, $x(y + z) = xy + xz$.

K Look at D5. What replacements were made for the variables in $x(y + z) = xy + xz$ to obtain each of the statements expressed in the display? How do you know that each statement is true?

A $\frac{1}{2}(\frac{2}{3} + \frac{1}{4}) = \frac{1}{2}(\frac{2}{3}) + \frac{1}{2}(\frac{1}{4})$.

D1

E $\frac{1}{2}(\frac{2}{3}) + \frac{1}{2}(\frac{1}{4}) = \frac{1}{3} + \frac{1}{8}$.

F $\frac{1}{3} + \frac{1}{8} = \frac{11}{24}$.

G $\frac{1}{2}(\frac{2}{3}) + \frac{1}{2}(\frac{1}{4}) = \frac{11}{24}$.

D3

H $\frac{2}{3}(\frac{1}{5} + \frac{2}{5}) = \frac{2}{3}(\frac{1}{5}) + \frac{2}{3}(\frac{2}{5})$.

I $\frac{5}{6}(\frac{2}{3} + \frac{1}{9}) = \frac{5}{6}(\frac{2}{3}) + \frac{5}{6}(\frac{1}{9})$.

J $\frac{1}{4}(\frac{3}{4} + \frac{1}{2}) = \frac{1}{4}(\frac{3}{4}) + \frac{1}{4}(\frac{1}{2})$.

K $\frac{3}{4}(\frac{1}{6} + \frac{1}{3}) = \frac{3}{4}(\frac{1}{6}) + \frac{3}{4}(\frac{1}{3})$.

D4

Universe for x, y, and $z = R_a$.
$x(y + z) = xy + xz$.

L $\frac{3}{4}(\frac{1}{3} + \frac{2}{7}) = \frac{3}{4}(\frac{1}{3}) + \frac{3}{4}(\frac{2}{7})$.

M $\frac{1}{8}(\frac{9}{10} + \frac{2}{5}) = \frac{1}{8}(\frac{9}{10}) + \frac{1}{8}(\frac{2}{5})$.

N $\frac{4}{3}(\frac{1}{8} + \frac{3}{7}) = \frac{4}{3}(\frac{1}{8}) + \frac{4}{3}(\frac{3}{7})$.

D5

L Look at D6. What condition is expressed? What is the universe for the variables?

M What replacements were made for the variables in $(y + z)x = yx + zx$ to obtain each of the statements expressed in D6?

N Does each sentence in D6 express a true statement?

O In D6 there are three sentences that express true statements obtained from $(y + z)x = yx + zx$. Try to prove that, for each replacement of the variables, $(y + z)x = yx + zx$. The universe for the variables is R_a. For your proof

you can use the property $xy = yx$, the property $x(y + z) = xy + xz$, and the transitive property of equality.

To find the product of two numbers like 25 and $104\frac{1}{3}$, it is often convenient to arrange your work in either of the ways shown in D7.

P Study the computation shown in D7. What number is the product of 25 and 104? Is $\frac{25}{3}$, or $8\frac{1}{3}$, the product of 25 and $\frac{1}{3}$? Is the product of 25 and $104\frac{1}{3}$ the same number as the sum of 2600 and $8\frac{1}{3}$?

Now you will see that you have been using the distributive property to obtain the product of 25 and $104\frac{1}{3}$.

Q Look at D8. Does sentence R express a true statement? How do you know?

R Does sentence S express a true statement? Upon what property of rational numbers does your answer depend?

S Do sentences T and U express true statements?

From the true statements expressed in D8, you see how to use the distributive property to find the product of two numbers when one of them is expressed by a mixed numeral. Use this property to help you find the product of the two numbers named in each exercise below.

T 4, $2\frac{1}{3}$ **U** $80\frac{1}{2}$, 6 **V** $14\frac{2}{7}$, 35 **W** 47, $11\frac{1}{9}$

You have learned that multiplication distributes over addition in the set of rational numbers. Next you will decide if addition distributes over multiplication.

A Look at D9. What condition is expressed?

B What replacements were made for the variables in $x + (yz) = (x + y)(x + z)$ to obtain the statement expressed by sentence V?

If the statement expressed in D9 is true, then the number that you obtain when you add the product $\frac{2}{3} \cdot \frac{3}{4}$ to $\frac{1}{2}$ should be equal to the number that you obtain when you add each of the numbers, $\frac{2}{3}$ and $\frac{3}{4}$, to $\frac{1}{2}$ and then multiply the sums.

C Does sentence V express a true statement? How do you know?

D Is the condition expressed in D9 a property?

? Does addition distribute over multiplication of rational numbers?

Universe for x, y, and $z = R_a$.
$(y + z)x = yx + zx$.

O $(\frac{4}{6} + \frac{1}{5})\frac{2}{3} = \frac{4}{6}(\frac{2}{3}) + \frac{1}{5}(\frac{2}{3})$.

P $(\frac{10}{3} + \frac{1}{4})\frac{5}{8} = \frac{10}{3}(\frac{5}{8}) + \frac{1}{4}(\frac{5}{8})$.

Q $(\frac{8}{7} + \frac{1}{9})\frac{2}{5} = \frac{8}{7}(\frac{2}{5}) + \frac{1}{9}(\frac{2}{5})$.

D6

$$\begin{array}{r} 104\frac{1}{3} \\ 25 \\ \hline 520 \\ 2080 \\ \hline 2600 \\ 8\frac{1}{3} \\ \hline 2608\frac{1}{3} \end{array} \qquad \begin{array}{r} 104\frac{1}{3} \\ 25 \\ \hline 8\frac{1}{3} \\ 520 \\ 2080 \\ \hline 2608\frac{1}{3} \end{array}$$

D7

R $25(104\frac{1}{3}) = 25(104 + \frac{1}{3})$.

S $25(104 + \frac{1}{3}) = 25(104) + 25(\frac{1}{3})$.

T $25(104) + 25(\frac{1}{3}) = 2600 + 8\frac{1}{3}$.

U $2600 + 8\frac{1}{3} = 2608\frac{1}{3}$.

D8

Universe for x, y, and $z = R_a$.
$x + (yz) = (x + y)(x + z)$.

V $\frac{1}{2} + (\frac{2}{3} \cdot \frac{3}{4}) = (\frac{1}{2} + \frac{2}{3})(\frac{1}{2} + \frac{3}{4})$.

D9

Universe for x, y, and $z = R_a$.

$x(y - z) = xy - xz$.

W $\frac{4}{5}(\frac{2}{3} - \frac{1}{8}) = \frac{4}{5}(\frac{2}{3}) - \frac{4}{5}(\frac{1}{8})$.

X $\frac{4}{5}(\frac{1}{4} - \frac{5}{2}) = \frac{4}{5}(\frac{1}{4}) - \frac{4}{5}(\frac{5}{2})$.

D10

E Look at D10. What condition is expressed? What is the universe for the variables?

F What replacements were made for the variables in $x(y - z) = xy - xz$ to obtain the statement expressed by sentence W? The statement expressed by sentence X?

G Does sentence W express a true statement? Can you decide if sentence X expresses a true statement?

H If the replacement for y is greater than or equal to the replacement for z, do you think that multiplication distributes over subtraction in the set of rational numbers?

In unit 6 you learned that the set of natural numbers, together with the operations of addition and multiplication and some of their properties, forms the natural-number system. Now you are prepared to decide if the set of rational numbers of arithmetic, together with the operations of addition and multiplication, forms a number system.

REMINDER

A number system consists of a set of objects and two operations. The two operations must be closed, commutative, and associative. One of the operations must distribute over the other.

See lesson 66, page 282.

A Study D11. What set of numbers is the universe for the variables?

B Which operations are named in D11? Is the set of rational numbers closed under each of these operations?

C To which operation does the property $x + y = y + x$ apply? To which operation does the property $xy = yx$ apply?

D To which operation does the property $(x + y) + z = x + (y + z)$ apply? To which operation does $(xy)z = x(yz)$ apply?

E Which property expressed in D11 relates multiplication and addition?

F Does the set of rational numbers of arithmetic, together with the operations of addition and multiplication, form a number system?

Now you will study a subset of the set of rational numbers of arithmetic to decide if it

Universe for x, y, and $z = R_a$.		
Operations ⟶	Addition	Multiplication
Closure properties ⟶	$x + y$ is a member of R_a.	xy is a member of R_a.
Commutative properties ⟶	$x + y = y + x$.	$xy = yx$.
Associative properties ⟶	$(x + y) + z = x + (y + z)$.	$(xy)z = x(yz)$.
Distributive property ⟶	$x(y + z) = xy + xz$.	

D11

$M = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\}$.

D12

forms a number system with respect to the operations of addition and multiplication.

G Describe set M, tabulated in D12.

H Is set M closed under multiplication? Give two examples that help to explain your answer.

I Is set M closed under addition? Give two examples that help to explain your answer.

J Does set M, together with the operations of addition and multiplication, form a number system? Explain your answer.

In this lesson you learned that multiplication distributes over addition of rational numbers. You also learned that the set of rational numbers of arithmetic, together with the operations of addition and multiplication and some of their properties, forms a number system.

On your own

For each of exercises 1 through 9, decide whether the sentence expresses a true statement or a false statement. For each true statement, name the property that you used to make your decision.

1 $\frac{1}{4}(\frac{1}{3} \times \frac{2}{5}) = \frac{1}{4}(\frac{2}{5} \times \frac{1}{3})$.

2 $\frac{7}{8} - (\frac{1}{3} \times \frac{1}{4}) = (\frac{7}{8} - \frac{1}{3})(\frac{7}{8} - \frac{1}{4})$.

3 $\frac{1}{12}(\frac{1}{3} - \frac{2}{3}) = \frac{1}{12}(\frac{1}{3}) - \frac{1}{12}(\frac{2}{3})$.

4 $(\frac{5}{8} + \frac{2}{5}) + \frac{1}{9} = (\frac{2}{5} + \frac{5}{8}) + \frac{1}{9}$.

5 $(\frac{2}{3} + \frac{1}{9}) + \frac{1}{8} = \frac{2}{3} + (\frac{1}{9} + \frac{1}{8})$.

6 $(\frac{8}{5} + \frac{3}{5})\frac{1}{10} = \frac{8}{5}(\frac{1}{10}) + \frac{3}{5}(\frac{1}{10})$.

7 $\frac{4}{1}(\frac{3}{1} + \frac{1}{3}) = \frac{4}{1}(\frac{3}{1}) + \frac{4}{1}(\frac{1}{3})$.

8 $\frac{4}{7}(\frac{1}{6}) + \frac{3}{8}(\frac{1}{6}) = (\frac{4}{7} + \frac{3}{8})\frac{1}{6}$.

9 $(\frac{3}{11} + \frac{1}{4})\frac{8}{9} = \frac{3}{11} + (\frac{1}{4} \times \frac{8}{9})$.

The sentences in exercises 10 through 17 express conditions. For each condition, tell what replacement you could make for the variable to obtain a true statement. $U = R_a$.

10 $(\frac{1}{3} + z)\frac{2}{5} = \frac{1}{3}(\frac{2}{5}) + \frac{4}{9}(\frac{2}{5})$.

11 $\frac{3}{4}(y) + \frac{3}{4}(\frac{8}{9}) = \frac{3}{4}(\frac{8}{9} + \frac{11}{12})$.

12 $\frac{1}{12}(y + \frac{3}{8}) = \frac{1}{12}(\frac{4}{7}) + \frac{1}{12}(\frac{3}{8})$.

13 $x(\frac{7}{10} + \frac{6}{5}) = \frac{3}{4}(\frac{7}{10}) + \frac{3}{4}(\frac{6}{5})$.

14 $\frac{2}{9}(\frac{1}{5}) + \frac{2}{9}(\frac{8}{7}) = \frac{2}{9}(y + \frac{1}{5})$.

15 $\frac{2}{7}(\frac{2}{7}) + \frac{1}{4}(\frac{2}{7}) = (\frac{2}{7} + \frac{1}{4})x$.

16 $x(\frac{4}{5} + \frac{4}{5}) = \frac{3}{8}(\frac{4}{5}) + \frac{3}{8}(\frac{4}{5})$.

17 $\frac{11}{12}(\frac{3}{13}) + \frac{5}{9}(x) = (\frac{11}{12} + \frac{5}{9})\frac{3}{13}$.

Use the distributive property to help you find the product of the two numbers named in each of exercises 18 through 26.

18 13, $2\frac{1}{2}$	**21** $3\frac{1}{4}$, 4	**24** 32, $5\frac{1}{4}$
19 $28\frac{1}{3}$, 5	**22** $20\frac{1}{5}$, 7	**25** $7\frac{2}{3}$, 54
20 16, $41\frac{7}{8}$	**23** 2, $6\frac{3}{5}$	**26** 15, $12\frac{2}{3}$

Use the distributive property to obtain a product from each of the sums expressed in exercises 27 through 35. For exercise 27 you could write these sentences: $\frac{6}{12} + \frac{15}{16} = \frac{3}{4}(\frac{2}{3}) + \frac{3}{4}(\frac{5}{4})$. $\frac{3}{4}(\frac{2}{3}) + \frac{3}{4}(\frac{5}{4}) = \frac{3}{4}(\frac{2}{3} + \frac{5}{4})$. $\frac{3}{4}(\frac{2}{3} + \frac{5}{4}) = \frac{3}{4}(\frac{23}{12})$.

27 $\frac{6}{12} + \frac{15}{16}$	**30** $\frac{8}{12} + \frac{8}{27}$	**33** $\frac{17}{2} + \frac{34}{3}$
28 $\frac{2}{12} + \frac{16}{21}$	**31** $\frac{24}{5} + \frac{3}{10}$	**34** $\frac{8}{15} + \frac{1}{10}$
29 $\frac{3}{7} + \frac{5}{14}$	**32** $\frac{5}{18} + \frac{20}{21}$	**35** $\frac{12}{32} + \frac{24}{56}$

KEEPING SKILFUL

For each exercise below, first find a fraction that indicates the sum, difference, or product. Then use a mixed numeral or a numeral for a basic fraction to express your answer.

1 $9 + \frac{8}{9}$	**7** $4 \times 1\frac{1}{6}$	**13** $6\frac{2}{5} + 2\frac{1}{2}$
2 $\frac{9}{12} - \frac{6}{12}$	**8** $1\frac{1}{9} + \frac{1}{6}$	**14** $7\frac{4}{5} - 4\frac{1}{3}$
3 $\frac{2}{3} \times 7$	**9** $3 - 1\frac{3}{4}$	**15** $2\frac{1}{3} \times 3\frac{3}{4}$
4 $\frac{3}{8} \times \frac{20}{12}$	**10** $\frac{5}{9} \times \frac{12}{17}$	**16** $8\frac{1}{4} - 5\frac{1}{2}$
5 $\frac{4}{8} + \frac{4}{6}$	**11** $9 - \frac{23}{11}$	**17** $1\frac{1}{2} - \frac{6}{7}$
6 $11\frac{3}{7} + 6$	**12** $6\frac{2}{5} \times 7\frac{3}{4}$	**18** $7\frac{4}{5} + 6\frac{2}{3}$

The identity elements

In unit 6 you learned that the sum of 0 and any given natural number is the given natural number. You also learned that the product of 1 and any given natural number is the given natural number. Zero is the identity element for addition of natural numbers; 1 is the identity element for multiplication of natural numbers. Now you will learn what the identity elements are for addition and multiplication of rational numbers.

A Does sentence A in D1 express a true statement? To what rational number has $\frac{2}{5}$ been added? What is the sum of $\frac{0}{5}$ and $\frac{2}{5}$? Is $\frac{0}{5}+\frac{2}{5}=0+\frac{2}{5}$? Explain your answer.

B Does each of the other sentences in D1 express a true statement? What is the sum of $\frac{0}{6}$ and $\frac{5}{7}$? Of $\frac{0}{3}$ and $2\frac{1}{2}$? Of $\frac{0}{1}$ and 8?

You can see that, if you add any one of the rational numbers $\frac{2}{5}$, $\frac{5}{7}$, $2\frac{1}{2}$, or 8 to the rational number 0, you obtain the rational number that you added. Now you will use variables to help you decide if the sum of 0 and any given rational number is the given rational number.

C Look at D2. Think of $\frac{0}{b}$ as the rational number that is indicated by any fraction with a numerator of 0. Think of $\frac{c}{d}$ as any rational number. What is the universe for each of the variables b, c, and d? For each replacement of b, is the rational number $\frac{0}{b}$ the same as the rational number 0?

D For each replacement of b, c, and d, do you obtain a true statement from the condition expressed by sentence E? How do you know?

A $\frac{0}{5}+\frac{2}{5}=\frac{2}{5}$.
B $\frac{0}{6}+\frac{5}{7}=\frac{5}{7}$.
C $\frac{0}{3}+2\frac{1}{2}=2\frac{1}{2}$.
D $\frac{0}{1}+8=8$.

D1

Universe for c = N.
Universe for b and d = C.

E $\frac{0}{b}+\frac{c}{d}=\frac{0\cdot d+bc}{bd}$
F $\frac{0}{b}+\frac{c}{d}=\frac{bc}{bd}$.
G $\frac{0}{b}+\frac{c}{d}=\frac{c}{d}$.

D2

$U=R_a$.
$0+x=x$.

H $0+\frac{4}{5}=\frac{4}{5}$.
I $0+\frac{1}{4}=\frac{1}{4}$.
J $0+8\frac{1}{3}=8\frac{1}{3}$.

D3

E What is the product of 0 and any natural number? What is the sum of 0 and any natural number? For each replacement of the variables, is $\frac{0\cdot d+bc}{bd}=\frac{bc}{bd}$?

F How do you know that, for each replacement of the variables, you obtain a true statement from the condition expressed by sentence F?

G How do you know that, for each replacement of the variables, $\frac{bc}{bd}=\frac{c}{d}$?

H How do you know that, for each replacement of the variables, you obtain a true statement from the condition expressed by sentence G?

You have learned that the sum of 0 and any given rational number is the given rational number. We say that the rational number 0 is

the *identity element for addition of rational numbers*. The *identity-element property of addition of rational numbers* is expressed below.

The universe for x *is* R_a. *For each replacement of the variable*, $0 + x = x$.

I Look at D3. What replacement was made for the variable in $0 + x = x$ to obtain each of the statements expressed in the display? Is each statement true? How do you know?

J Do you think there is a number, other than zero, that can be added to a given rational number to obtain the given rational number?

K What condition is expressed in D4? For each replacement of x, is $x + 0 = x$? How do you know?

L Which of the conditions expressed in D5 are properties of rational numbers? In answering this question, you may find it helpful to relate subtraction to addition.

Now you will see whether or not there is an identity element for multiplication of rational numbers.

A Look at D6. Does sentence N express a true statement? Is the product of $\frac{1}{1}$ and $\frac{2}{7}$ the same as $\frac{2}{7}$? Is the product of $\frac{1}{1}$ and $\frac{2}{7}$ the same as the product of 1 and $\frac{2}{7}$?

B Study sentences O, P, and Q in D6. Explain how to obtain each of the products $\frac{9}{10}$, $3\frac{1}{2}$, and 6. In each statement, is the product of the rational number $\frac{1}{1}$ and the given rational number the same as the given number?

Next you will use variables to decide if the product of $\frac{1}{1}$ and any given rational number is the same as the given rational number. Think of $\frac{c}{d}$ as any rational number. The universe for c is N. The universe for d is C.

C Look at D7. How do you know that, for each replacement of the variables, you obtain a true statement from the condition expressed by sentence R?

$U = R_a$.
$x + 0 = x$.

D4

$U = R_a$.

K $x - x = 0$.
L $x - 0 = x$.
M $0 - x = x$.

D5

N $\frac{1}{1} \times \frac{2}{7} = \frac{2}{7}$.
O $\frac{1}{1} \times \frac{9}{10} = \frac{9}{10}$.
P $\frac{1}{1} \times 3\frac{1}{2} = 3\frac{1}{2}$.
Q $\frac{1}{1} \times 6 = 6$.

D6

Universe for c = N.
Universe for d = C.

R $\frac{1}{1} \times \frac{c}{d} = \frac{1 \times c}{1 \times d}$.
S $\frac{1}{1} \times \frac{c}{d} = \frac{c}{d}$.

D7

D For each replacement of the variables, is $1 \times c = c$? Is $1 \times d = d$? Upon what property of natural numbers do your answers depend?

E For each replacement of the variables, is $\frac{1 \times c}{1 \times d} = \frac{c}{d}$?

F Read sentence S. For each replacement of the variables, is $\frac{1}{1} \times \frac{c}{d} = \frac{c}{d}$? Explain your answer.

G Is $\frac{1}{1}$ the same rational number as $\frac{2}{2}$? As $\frac{3}{3}$? As $\frac{4}{4}$? As $\frac{257}{257}$? Is $\frac{1}{1}$ the same rational number as 1?

You have learned that the product of 1 and any given rational number is the given rational number. We say that the number 1 is the *identity element for multiplication of rational num-*

$U = R_a$.
$1 \cdot x = x$.

T $1 \times \frac{5}{6} = \frac{5}{6}$.

U $1 \times \frac{10}{13} = \frac{10}{13}$.

V $1 \times 1 = 1$.

D8

$U = R_a$.
$x \cdot 1 = x$.

D9

W $0 \times \frac{9}{7} = 0$.

X $0 \times \frac{1}{3} = 0$.

Y $0 \times 4\frac{1}{2} = 0$.

D10

bers. The *identity-element property of multiplication of rational numbers* is expressed below.

The universe for x *is* R_a. *For each replacement of the variable*, $1 \cdot x = x$.

H Look at D8. What replacement was made for the variable in $1 \cdot x = x$ to obtain each statement expressed in D8? Is each statement true? How do you know?

I Is $\frac{2}{5} \times \frac{3}{4} = \frac{3}{4}$? Is $\frac{8}{8} \cdot 7\frac{3}{8} = 7\frac{3}{8}$? Explain your answers.

J What condition is expressed in D9? For each replacement of x, is $x \cdot 1 = x$? How do you know?

In unit 6 you learned that the product of zero and any natural number is the number 0. Now you will investigate the rôle of 0 in multiplication of rational numbers.

K Look at D10. Does each sentence express a true statement? For each statement, what is the product of 0 and the given number?

L For each replacement of b, c, and d, is $\frac{0}{b} \cdot \frac{c}{d} = \frac{0}{bd}$? Explain your answer. The universe for b and d is C. The universe for c is N.

M Is 0 the product of 0 and any rational number? Explain your answer.

You have learned that the product of 0 and any rational number is 0. This property is the *zero property of multiplication of rational numbers*. The property is expressed below.

The universe for x *is* R_a. *For each replacement of the variable*, $0 \cdot x = 0$.

N For each replacement of x, is $x \cdot 0 = 0$? Explain your answer.

In this lesson you have studied properties of addition and multiplication of rational numbers that involve zero and one.

On your own

For each of exercises 1 through 14, decide whether the condition produces a true statement or a false statement for each replacement of the variable. $U = R_a$.

1 $x + 0 = 0$.
2 $x = x \cdot 1$.
3 $x + x = 0$.
4 $0 + x = x$.
5 $1 + x = x$.
6 $x \cdot x = 0$.
7 $0 + x = 0$.
8 $x \cdot 0 = x$.
9 $x = x + 0$.
10 $0 - x = x$.
11 $0 = x - x$.
12 $x - 0 = x$.
13 $1 \cdot x = x$.
14 $1 - x = x$.

For each of exercises 15 through 30, tell whether the sentence expresses a true statement or a false statement.

15 $\frac{5}{3} - 0 = \frac{5}{3}$.
16 $0 - \frac{1}{2} = \frac{1}{2}$.
17 $0 = \frac{5}{8} - \frac{5}{8}$.
18 $\frac{0}{5} + \frac{0}{5} = 0$.
19 $\frac{2}{3} + \frac{2}{3} = 0$.
20 $\frac{7}{10} = \frac{7}{10} \cdot 1$.
21 $1 \times \frac{7}{8} = \frac{7}{8}$.
22 $0 \times \frac{5}{5} = 0$.
23 $\frac{5}{6} + 1 = \frac{5}{6}$.
24 $\frac{0}{6} + \frac{5}{7} = \frac{5}{42}$.
25 $\frac{1}{1} + \frac{0}{1} = \frac{0}{1}$.
26 $\frac{0}{7} + \frac{0}{6} = \frac{0}{42}$.
27 $\frac{0}{6} \times 1 = \frac{0}{6}$.
28 $2\frac{1}{2} \cdot \frac{3}{3} = \frac{5}{2}$.
29 $\frac{41}{7} = 5\frac{6}{7} + \frac{0}{9}$.
30 $\frac{4}{4} \times \frac{6}{6} = 1$.

Reciprocals and the reciprocal property

In lesson 83 you learned that the product of 1 and any given rational number is the given rational number. In this lesson you will learn about pairs of rational numbers, each of which has a product of 1.

A Look at D1. Does each sentence express a true statement? How do you know?

B The product of $\frac{7}{8}$ and $\frac{8}{7}$ is 1. When the product of two rational numbers is 1, each of the numbers is the *reciprocal* of the other. Is $\frac{8}{7}$ the reciprocal of $\frac{7}{8}$? Is $\frac{7}{8}$ the reciprocal of $\frac{8}{7}$?

C Look at sentence B in D1. Why is $\frac{12}{5}$ the reciprocal of $\frac{5}{12}$? What number is the reciprocal of $\frac{12}{5}$?

D Look at sentence C. Is $\frac{47}{2}$ the reciprocal of $\frac{2}{47}$? How do you know? Is $\frac{2}{47}$ the reciprocal of $\frac{47}{2}$?

E Look at sentence D. Is $\frac{10}{6}$ the reciprocal of $\frac{6}{10}$? Is $\frac{6}{10}$ the reciprocal of $\frac{10}{6}$? Explain your answers.

F Look again at sentence A in D1. You know that $\frac{7}{8}$ and $\frac{8}{7}$ are reciprocals of each other. Is the denominator of the fraction $\frac{7}{8}$ the same as the numerator of the fraction $\frac{8}{7}$? Is the numerator of $\frac{7}{8}$ the same as the denominator of $\frac{8}{7}$?

G What number is the numerator of $\frac{12}{5}$? Is this number the denominator of $\frac{5}{12}$? What number is the denominator of $\frac{12}{5}$ and the numerator of $\frac{5}{12}$?

You can see that two fractions that indicate reciprocals are related in a special way. To obtain a fraction that indicates the reciprocal of a given rational number, you can interchange the numerator and the denominator of the fraction that indicates the given number.

A $\frac{7}{8} \times \frac{8}{7} = 1.$ **C** $\frac{47}{2} \times \frac{2}{47} = 1.$

B $\frac{12}{5} \times \frac{5}{12} = 1.$ **D** $\frac{6}{10} \times \frac{10}{6} = 1.$

D1

E $2\frac{2}{3} \times \frac{3}{8} = 1.$

F $5 \times \frac{1}{5} = 1.$

D2

re cip ro cal (ri sip′rə kəl). If the product of two numbers is 1, each number is the reciprocal of the other. For each replacement of x and y, if $x \cdot y = 1$, then each of the numbers, x and y, is the reciprocal of the other. The universe for x and y is R_a.

For each rational number named below, name a fraction that indicates the reciprocal.

H $\frac{4}{7}$ **I** $\frac{100}{9}$ **J** $\frac{16}{23}$ **K** $\frac{5}{20}$

L Look at sentence E in D2. What fraction indicates the rational number $2\frac{2}{3}$?

M Does sentence E express a true statement? Are $\frac{8}{3}$ and $\frac{3}{8}$ reciprocals? Are $2\frac{2}{3}$ and $\frac{3}{8}$ reciprocals? Explain your answers.

N Does sentence F in D2 express a true statement? Is 5 the reciprocal of $\frac{1}{5}$? Is $\frac{1}{5}$ the reciprocal of 5? Is $\frac{2}{10}$ the reciprocal of 5? Explain your answers.

O What is the product of 0 and any rational number? How do you know?

P Does 0 have a reciprocal? Is 0 the reciprocal of any number? Explain your answers.

Now you will learn how to find the reciprocal of any rational number $\frac{a}{b}$ when the universe for a and b is C.

Universe for a and b = C.

$\frac{a}{b} \cdot \frac{b}{a} = 1.$

G $\frac{5}{9} \times \frac{9}{5} = 1.$

H $\frac{11}{12} \times \frac{12}{11} = 1.$

I $\frac{5}{20} \times \frac{20}{5} = 1.$

D3

J $\frac{6}{10} \times \frac{10}{6} = 1.$

K $\frac{3}{5} \times \frac{10}{6} = 1.$

L $\frac{6}{10} \times \frac{5}{3} = 1.$

D4

A What condition is expressed in D3? To obtain the statement expressed by sentence G, a was replaced by 5, and b by 9. Does sentence G express a true statement?

B What replacements were made for a and b in $\frac{a}{b} \cdot \frac{b}{a} = 1$ to obtain each of the other statements expressed in D3? Is each of the statements true?

C What is the reciprocal of $\frac{9}{5}$? Of $\frac{11}{12}$? Of $\frac{20}{5}$?

D For each replacement of a and b, is $\frac{a}{b} \cdot \frac{b}{a} = ?\ \frac{ab}{ba}$? Is $\frac{ab}{ba} = 1$? Is $\frac{a}{b} \cdot \frac{b}{a} = 1$? Explain your answers. Remember that the universe for a and b is C.

E What is the reciprocal of the rational number $\frac{a}{b}$? What is the reciprocal of the rational number $\frac{b}{a}$?

Now you know how to find the reciprocal of any rational number that is not 0. The property that enables you to do this is the *reciprocal property of rational numbers*. The property is expressed below.

The universe for a *and* b *is C. For each replacement of* a *and* b, *the reciprocal of* $\frac{a}{b}$ *is* $\frac{b}{a}$. *The reciprocal of* $\frac{b}{a}$ *is* $\frac{a}{b}$.

F Does sentence J in D4 express a true statement? What is the reciprocal of $\frac{10}{6}$? Of $\frac{6}{10}$?

G In the statement expressed by sentence J, ? can $\frac{6}{10}$ be replaced by $\frac{3}{5}$? Upon what property of rational numbers does your answer depend?

H Does sentence K express a true statement? Is $\frac{3}{5}$ the reciprocal of $\frac{10}{6}$? Is $\frac{10}{6}$ the reciprocal of $\frac{3}{5}$? Explain your answers.

I In the statement expressed by sentence J, can $\frac{10}{6}$ be replaced by $\frac{5}{3}$? How do you know?

J Does sentence L express a true statement? Are $\frac{6}{10}$ and $\frac{5}{3}$ reciprocals of each other? Explain your answer.

K Think about the rational number $\frac{6}{10}$. Which ? of the following are its reciprocals: $\frac{5}{3}$, $\frac{10}{6}$, $\frac{15}{9}$, $\frac{20}{12}$? How many different rational numbers are reciprocals of $\frac{6}{10}$? Explain your answer.

L What rational number is the reciprocal of ? itself? Explain your answer.

Now you know that two rational numbers whose product is 1 are reciprocals of each other. You also know how to find the reciprocal of a given rational number.

On your own

For each of exercises 1 through 12, decide whether or not the two numbers named are reciprocals of each other. Explain each of your answers.

1 $\frac{5}{2}, \frac{2}{5}$
2 $\frac{1}{12}$, 12
3 $\frac{19}{2}, 9\frac{1}{2}$
4 $\frac{4}{17}, \frac{71}{4}$
5 $5\frac{3}{4}, 5\frac{4}{3}$
6 $6\frac{2}{3}, \frac{3}{20}$
7 $\frac{23}{12}, \frac{21}{32}$
8 $\frac{7}{3}, \frac{21}{9}$
9 $\frac{5}{5}, \frac{5}{5}$
10 $\frac{12}{10}, \frac{5}{6}$
11 $\frac{0}{3}, \frac{0}{4}$
12 $\frac{9}{10}, 1\frac{1}{9}$

A number is named in each of exercises 13 through 27. For each exercise, tell by what number you can multiply the given number to obtain the product 1.

13 $4\frac{2}{3}$
14 $\frac{7}{8}$
15 $7\frac{1}{2}$
16 21
17 $\frac{652}{1895}$
18 $\frac{14}{3}$
19 327
20 $7\frac{1}{9}$
21 $\frac{27}{61}$
22 $\frac{1}{100}$
23 $\frac{6}{12}$
24 $3\frac{1}{8}$
25 5000
26 $10\frac{1}{2}$
27 $\frac{1}{9}$

Find the sum of the numbers named in each of exercises 28 through 35. Then find the reciprocal of the sum.

28 $\frac{7}{10}, \frac{3}{5}$ 30 $7\frac{1}{10}, \frac{6}{5}$ 32 $\frac{0}{5}, \frac{3}{7}$ 34 $\frac{11}{30}, \frac{2}{12}$

29 $\frac{5}{72}, 1\frac{1}{3}$ 31 $\frac{15}{7}, 2\frac{1}{3}$ 33 $8, 2\frac{3}{4}$ 35 $4\frac{3}{7}, 2$

Find the product of the numbers named in each of exercises 36 through 43. Then find the reciprocal of the product.

36 $\frac{8}{9}, \frac{7}{6}$ 38 $\frac{2}{5}, \frac{3}{100}$ 40 $\frac{8}{11}, 6\frac{1}{2}$ 42 $\frac{4}{9}, \frac{27}{12}$

37 $1\frac{1}{12}, \frac{4}{9}$ 39 $\frac{4}{17}, \frac{1}{2}$ 41 $9, \frac{5}{6}$ 43 $33, 2\frac{7}{11}$

KEEPING SKILFUL

Find the difference.

1 7000, 6938
2 9174, 4916
3 44286, 12590
4 129165, 40878
5 94345, 89346
6 210406, 92354
7 753849, 619389
8 343015, 291857

Divide

9 1764 by 37.
10 8084 by 86.
11 18135 by 45.
12 19653 by 62.
13 8928 by 182.
14 41067 by 507.
15 79026 by 218.
16 229140 by 380.

For each condition expressed in exercises 17 through 26, tabulate the solution set. U = C.

17 $6/7 \sim 78/v$.
18 114 is $c\%$ of 95.
19 $28(21 - 9) = x$.
20 166 is 4% of b.
21 $23/x \sim 17/34$.
22 w is 75% of 124.
23 $9(144 \div 3) > g$.
24 27 is $y\%$ of 450.
25 $m + 115 > 23(24)$.
26 $243/216 \sim d/8$.

For each exercise below, first find a fraction that indicates the sum, difference, or product. Then use a mixed numeral or a numeral for a basic fraction to express your answer.

27 $\frac{8}{12} + \frac{9}{6}$ 33 $1\frac{1}{4} - \frac{5}{8}$ 39 $11\frac{1}{2} - 7\frac{11}{12}$

28 $1\frac{1}{2} + \frac{3}{10}$ 34 $6 \times 2\frac{3}{8}$ 40 $2\frac{1}{3} \times 3\frac{1}{4}$

29 $\frac{3}{8} \times \frac{8}{9}$ 35 $2\frac{1}{3} + 4\frac{5}{9}$ 41 $9\frac{1}{5} \times 8\frac{1}{2}$

30 $\frac{3}{4} - \frac{1}{3}$ 36 $\frac{7}{8} \times 5$ 42 $6\frac{5}{6} + 11\frac{5}{8}$

31 $2\frac{2}{5} \times \frac{3}{14}$ 37 $7\frac{6}{7} - 3\frac{2}{3}$ 43 $8\frac{6}{7} + 9\frac{8}{12}$

32 $5 - \frac{6}{12}$ 38 $4\frac{3}{5} + \frac{1}{3}$ 44 $17\frac{5}{6} - 15\frac{8}{9}$

Division of rational numbers

You know that you use division to find the quotient of two natural numbers. In this lesson you will learn how to find the quotient of two rational numbers.

A You know that the quotient of 15 and 3 is 5. You know that $15 \div 3 = 5$ also means $15 = 5 \times 3$. What is the quotient of 15 and 5? Does $15 \div 5 = 3$ also mean $15 = 3 \times 5$?

B Does $80 \div 17 = x$ also mean $80 = x \cdot 17$? Tabulate $\{x | 80 = x \cdot 17\}$. U = N.

C Is division closed in the set of natural numbers? Explain your answer.

D Look at sentence A in D1. The number $\frac{2}{3}$ is the dividend. Which number is the divisor?

E Now look at sentence B. Does $\frac{2}{3} \div \frac{3}{4} = y$ also mean $\frac{2}{3} = y \cdot \frac{3}{4}$?

You want to find a replacement for y that will make the product $y \cdot \frac{3}{4}$ equal to $\frac{2}{3}$. You can use what you have learned about the reciprocal of a rational number to help you develop a condition that has the same solution set as the solution set of $\frac{2}{3} = y \cdot \frac{3}{4}$. The condition that you will develop will contain $y \cdot 1$ instead of $y \cdot \frac{3}{4}$. Remember that the universe for the variable is R_a.

D1

$U = R_a$.

A $\frac{2}{3} \div \frac{3}{4} = y$.

B $\frac{2}{3} = y \cdot \frac{3}{4}$.

F Look at D2. Sentence C in D2 is the same as sentence B in D1. Now look at sentence D. $(y \cdot \frac{3}{4})$ and what number have been multiplied?

From sentence C, you know that $\frac{2}{3}$ equals $y \cdot \frac{3}{4}$. Therefore, you can use $\frac{2}{3}$ to replace $y \cdot \frac{3}{4}$ in the product $(y \cdot \frac{3}{4})\frac{4}{3}$. You can do this because of the well-defined property of multiplication of rational numbers.

> REMINDER
> The product of two rational numbers remains the same when either or both of the numbers used in forming the product are replaced by equal rational numbers.
> *See lesson 74, page 326.*

G Now compare the expressions at the right of the symbols for equality in sentences D and E in D2. For each replacement of y, is $y(\frac{3}{4} \cdot \frac{4}{3}) = (y \cdot \frac{3}{4})\frac{4}{3}$? Upon what property of multiplication does your answer depend?

H Compare the expressions at the right of the symbols for equality in sentences E and F. Is $\frac{4}{3}$ the reciprocal of $\frac{3}{4}$? What is the product of $\frac{3}{4}$ and $\frac{4}{3}$? For each replacement of y, is $y \cdot 1 = y(\frac{3}{4} \cdot \frac{4}{3})$?

I Now compare the expressions at the right of the symbols for equality in sentences F and G. For each replacement of y, is $y = y \cdot 1$? How do you know?

Notice that we have obtained several conditions that have the same solution as the solution of $\frac{2}{3} = y \cdot \frac{3}{4}$. Each of the conditions expressed by sentences D through G was obtained from the preceding condition by first using a property of multiplication and then using the transitive property of equality.

Now look at D3. Sentence H is the same as sentence A in D1, and sentence I is the same as sentence G in D2. The condition $(\frac{2}{3})\frac{4}{3} = y$ was obtained from $\frac{2}{3} \div \frac{3}{4} = y$ by relating division to multiplication and then using the procedure that was followed in D2. Thus, $\frac{2}{3} \div \frac{3}{4} = y$ and $(\frac{2}{3})\frac{4}{3} = y$ both have the same solution, and $\frac{2}{3} \div \frac{3}{4} = (\frac{2}{3})\frac{4}{3}$ is a true statement.

J You know that $\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3}$ and that $\frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$. Is $\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$? Upon what property does your answer depend?

K You have learned that the solution of $\frac{2}{3} \div \frac{3}{4} = y$ is $\frac{8}{9}$. Does $\frac{8}{9}$ satisfy $\frac{2}{3} = y \cdot \frac{3}{4}$? Explain your answer.

From the preceding work, you can see that the quotient of two rational numbers is equal to the product of the dividend and the reciprocal of the divisor. This property is the *quotient property of rational numbers*. The property is expressed below.

The universe for a *is N. The universe for* b, c, *and* d *is C. For each replacement of* a, b, c, *and* d, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

$U = R_a$.

A $\frac{2}{3} \div \frac{3}{4} = y$.

B $\frac{2}{3} = y \cdot \frac{3}{4}$.

D1

$U = R_a$.

C $\frac{2}{3} = y \cdot \frac{3}{4}$.

D $(\frac{2}{3})\frac{4}{3} = (y \cdot \frac{3}{4})\frac{4}{3}$.

E $(\frac{2}{3})\frac{4}{3} = y(\frac{3}{4} \cdot \frac{4}{3})$.

F $(\frac{2}{3})\frac{4}{3} = y \cdot 1$.

G $(\frac{2}{3})\frac{4}{3} = y$.

D2

$U = R_a$.

H $\frac{2}{3} \div \frac{3}{4} = y$.

I $(\frac{2}{3})\frac{4}{3} = y$.

J $\frac{2}{3} \div \frac{3}{4} = (\frac{2}{3})\frac{4}{3}$.

D3

L Does the rational number 0 have a reciprocal? In the quotient property expressed on page 366, why is C, rather than N, the universe for the variable c?

M Follow the steps given in D1 and D2 to show that $\frac{5}{6} \div \frac{1}{8} = \frac{5}{6} \times \frac{8}{1}$.

N Follow the steps given in D1 and D2 to show that, for each replacement of the variables, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$. The universe for a is N. The universe for b, c, and d is C. The following three sentences will help you.

$\frac{a}{b} \div \frac{c}{d} = y.$ $\quad \frac{a}{b} = y \cdot \frac{c}{d}.$ $\quad \frac{a}{b} \times \frac{d}{c} = (y \cdot \frac{c}{d})\frac{d}{c}.$

O Look at sentence K in D4. Does sentence K express a true statement? What is the quotient of 2 and $\frac{4}{5}$?

P Now look at sentence L. What is the reciprocal of 4? Does sentence L express a true statement? How do you know that $\frac{7}{32}$ is the quotient of $\frac{7}{8}$ and 4?

Find the quotient of the first and second components of each ordered pair of rational numbers named below.

Q $(\frac{5}{2}, \frac{3}{8})$ T $(\frac{15}{4}, \frac{3}{7})$ W $(\frac{4}{4}, \frac{3}{3})$

R $(1, \frac{1}{5})$ U $(\frac{5}{6}, \frac{5}{6})$ X $(\frac{7}{2}, 3)$

S $(\frac{0}{6}, \frac{2}{3})$ V $(\frac{1}{9}, \frac{9}{1})$ Y $(\frac{9}{10}, \frac{9}{20})$

Now that you know how to find the quotient of two rational numbers, you will study some of the properties of division of rational numbers.

A A rational number that is not 0 is a *non-zero rational number*. For example, $\frac{3}{3}$, $\frac{9}{16}$, and $\frac{14}{7}$ are non-zero rational numbers. Do you think that the quotient of any two non-zero rational numbers is always a rational number?

B How can you use the quotient property of rational numbers to prove that the set of non-zero rational numbers is closed under division? Remember that, for each replacement of the variables, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$. The universe for the variables is C.

K $2 \div \frac{4}{5} = 2 \times \frac{5}{4}$.

L $\frac{7}{8} \div 4 = \frac{7}{8} \times \frac{1}{4}$.

D4

M $\frac{3}{8} \div \frac{1}{4} = \frac{3}{2}$. O $\frac{3}{8} \div \frac{2}{8} = \frac{3}{2}$.

N $\frac{6}{16} \div \frac{3}{12} = \frac{3}{2}$. P $\frac{12}{32} \div \frac{1}{4} = \frac{3}{2}$.

D5

C Look at D5. Does each sentence express a true statement about the quotient of two rational numbers?

D Compare sentences M and N. To obtain the statement expressed by sentence N from the statement expressed by sentence M, $\frac{3}{8}$ was replaced by $\frac{6}{16}$ and $\frac{1}{4}$ was replaced by $\frac{3}{12}$. Is $\frac{6}{16} = \frac{3}{8}$? Is $\frac{3}{12} = \frac{1}{4}$? Is $\frac{3}{8} \div \frac{1}{4} = \frac{6}{16} \div \frac{3}{12}$? How do you know?

E Compare sentences M and O. To obtain the statement expressed by sentence O from the statement expressed by sentence M, the divisor, $\frac{1}{4}$, was replaced by $\frac{2}{8}$. Is $\frac{3}{8} \div \frac{1}{4} = \frac{3}{8} \div \frac{2}{8}$?

F To obtain the statement expressed by sentence P from the statement expressed by sentence M, the dividend, $\frac{3}{8}$, was replaced by what number? Is $\frac{3}{8} \div \frac{1}{4} = \frac{12}{32} \div \frac{1}{4}$?

G Do you think that the quotient of two rational numbers remains the same when either the dividend or the divisor, or both, are replaced by equal rational numbers?

H You have learned that the product of a given rational number and 1 is the given number. Show that, for each replacement of a and b, $\frac{a}{b} \div 1 = \frac{a}{b}$. The universe for a is N. The universe for b is C. Then express this property of division in words.

I You also have learned that the product of 0 and a given rational number is the number 0. Show that, for each replacement of a and b, $0 \div \frac{a}{b} = 0$. The universe for a and b is C.

J Do you think that division of rational numbers has the commutative property? Use the following examples to explain your answer. $U = R_a$. $\frac{7}{8} \div \frac{1}{3} = x$. $\frac{1}{3} \div \frac{7}{8} = y$.

K Do you think that division of rational numbers has the associative property? Use the following examples to explain your answer. $U = R_a$. $(\frac{4}{5} \div \frac{2}{3}) \div \frac{1}{8} = m$. $\frac{4}{5} \div (\frac{2}{3} \div \frac{1}{8}) = n$.

Next you will see why you can think of any rational number as the quotient of two numbers. You will also study a new way of thinking about the reciprocal of a rational number.

A Is $\frac{3}{1} \div \frac{4}{1} = \frac{3}{4}$? Is $\frac{16}{1} \div \frac{9}{1} = \frac{16}{9}$?

B Suppose that the universe for a is N and that the universe for b is C. Use the quotient property of rational numbers to show that, for each replacement of the variables, $\frac{a}{1} \div \frac{b}{1} = \frac{a}{b}$.

The rational number $\frac{a}{b}$ is the quotient of the rational number $\frac{a}{1}$ and the rational number $\frac{b}{1}$. However, since a and b are the natural-number mates of $\frac{a}{1}$ and $\frac{b}{1}$, you can think of the rational number $\frac{a}{b}$ as the quotient of a and b, or as $a \div b$.

In lesson 79 you learned how to find a mixed numeral that names a given rational number. Now you can find mixed numerals by using division, because you can think of the number $\frac{a}{b}$ as $a \div b$.

C Look at D6. Does sentence Q express a true statement?

D How do you know that sentence R expresses a true statement?

E From your earlier work in arithmetic, you know that $17 \div 3$ is 5, remainder 2. You also know that you can express the quotient $17 \div 3$ by the mixed numeral $5\frac{2}{3}$. Does sentence S express a true statement?

F Does sentence T in D6 express a true statement? Upon what property does your answer depend?

Q $\frac{17}{3} = \frac{17}{1} \div \frac{3}{1}$.

R $\frac{17}{3} = 17 \div 3$.

S $17 \div 3 = 5\frac{2}{3}$.

T $\frac{17}{3} = 5\frac{2}{3}$.

D6

U $\frac{4}{5} \times \frac{5}{4} = 1$.

V $\frac{5}{4} = \frac{1}{\frac{4}{5}}$.

W $\frac{4}{5} \times \frac{1}{\frac{4}{5}} = 1$.

D7

Express each number named in exercises G through P by a mixed numeral. Think of each number as the quotient of two numbers.

G $\frac{15}{4}$ **H** $\frac{12}{5}$ **I** $\frac{23}{3}$ **J** $\frac{42}{8}$ **K** $\frac{19}{5}$ **L** $\frac{9}{8}$ **M** $\frac{27}{10}$ **N** $\frac{30}{4}$ **O** $\frac{101}{16}$ **P** $\frac{49}{20}$

You will recall that the reciprocal of the rational number $\frac{a}{b}$ is $\frac{b}{a}$. The universe for a and b is C. Now you will see that you can also think of the reciprocal of a rational number as the quotient of 1 and that number.

Q Does sentence U in D7 express a true statement? Is $\frac{5}{4}$ the reciprocal of $\frac{4}{5}$?

R Look at sentence V. The expression at the right of the symbol for equality is another way of naming the quotient of 1 and $\frac{4}{5}$. Now look at the two expressions at the right. Each expression names the quotient of 1 and $\frac{4}{5}$. Read each expression "1 divided by $\frac{4}{5}$." $1/\frac{4}{5}$ means the same as the expression $1 \div \frac{4}{5}$. You know that $1 \div \frac{4}{5}$ also means $1 \times \frac{5}{4}$. Does sentence V express a true statement? Explain your answer.

$\frac{1}{\frac{4}{5}}$ $1/\frac{4}{5}$

S Does sentence W express a true statement? Is $1/\frac{4}{5}$ the reciprocal of $\frac{4}{5}$?

T For each replacement of the variables, is $\frac{a}{b} \times \frac{b}{a} = 1$? The universe for a and b is C. For each replacement of the variables, is $\frac{a}{b} \times 1/\frac{a}{b} = 1$? For each replacement of the variables, is $1/\frac{a}{b}$ the reciprocal of $\frac{a}{b}$?

Now we can express the reciprocal property in another way:

The universe for x *is the set of non-zero rational numbers. For each replacement of* x, *the reciprocal of* x *is* $1/x$.

Now you know that the reciprocal of the non-zero rational number x is $1/x$. As a result, the quotient property can be expressed in another way:

The universe for x *is* R_a. *The universe for* y *is the set of non-zero rational numbers of arithmetic. For each replacement of the variables,* $x \div y = x \cdot 1/y$.

In this lesson you have learned how to find the quotient of two rational numbers. You have also learned that the set of non-zero rational numbers of arithmetic is closed under division.

On your own

Find the quotient of the first and second components of each ordered pair of rational numbers named in exercises 1 through 15.

1 $(\frac{14}{3}, \frac{6}{5})$ **2** $(\frac{7}{8}, \frac{1}{4})$ **3** $(\frac{15}{2}, \frac{1}{3})$ **4** $(\frac{3}{4}, \frac{3}{4})$ **5** $(\frac{25}{12}, \frac{6}{6})$ **6** $(\frac{0}{6}, \frac{1}{8})$ **7** $(\frac{7}{12}, \frac{3}{5})$ **8** $(\frac{9}{10}, \frac{9}{2})$ **9** $(\frac{5}{6}, \frac{6}{5})$ **10** $(7, \frac{2}{3})$ **11** $(\frac{11}{2}, \frac{3}{4})$ **12** $(\frac{1}{3}, \frac{1}{3})$ **13** $(\frac{6}{1}, \frac{8}{2})$ **14** $(\frac{5}{5}, \frac{7}{7})$ **15** $(\frac{0}{8}, \frac{3}{8})$

Express each quotient named in exercises 16 through 25 by a mixed numeral.

16 $\frac{27}{5}$ **17** $\frac{14}{3}$ **18** $\frac{6}{4}$ **19** $\frac{110}{25}$ **20** $\frac{8}{5}$ **21** $\frac{39}{4}$ **22** $\frac{13}{11}$ **23** $\frac{45}{7}$ **24** $\frac{75}{50}$ **25** $\frac{100}{12}$

Decide whether each of the sentences below expresses a true statement or a false statement.

26 $\frac{3}{2} \times \frac{1}{\frac{2}{3}} = 1.$ **27** $\frac{1}{\frac{7}{4}} \times \frac{7}{4} = 1.$ **28** $\frac{1}{\frac{10}{2}} \times \frac{10}{2} = 1.$ **29** $\frac{1}{5} \times \frac{1}{\frac{5}{1}} = 1.$ **30** $\frac{11}{12} \times \frac{1}{\frac{12}{1}} = 1.$ **31** $\frac{1}{\frac{2}{16}} \times \frac{2}{16} = 1.$ **32** $\frac{7}{8} \times \frac{1}{\frac{8}{7}} = 1.$ **33** $\frac{2}{3} \times \frac{1}{\frac{4}{6}} = 1.$ **34** $\frac{1}{\frac{5}{6}} \times \frac{6}{5} = 1.$

CHECKING UP

If you have trouble with these two tests, you should refer to lessons 80 through 85.

Test 86

Give the reciprocal of each number named in exercises 1 through 5.

1 $\frac{15}{11}$ **2** $\frac{5}{9}$ **3** $8\frac{2}{5}$ **4** 14 **5** $\frac{1}{70}$

6 By what number should you multiply $\frac{3}{10}$ to get the product 1?

7 By what number should you multiply $\frac{3}{10}$ to get the product $\frac{3}{10}$?

8 What number should you add to $\frac{3}{10}$ to get the sum $\frac{3}{10}$?

9 By what rational number should you divide $\frac{3}{10}$ to get the quotient $\frac{9}{20}$?

Each of exercises 10 through 16 expresses a property of addition or multiplication of rational numbers. For each exercise, give the name of the property. $U = R_a$.

10 $0 + x = x.$

11 $xy = yx.$

12 $(x + y) + z = x + (y + z).$

13 $x(y + z) = xy + xz.$

14 $1 \cdot x = x.$

15 $x + y = y + x.$

16 $x(yz) = (xy)z.$

17 Using the transitive property of equality, what condition can you obtain from the two conditions expressed below? $U = R_a$.

$m = 2x - 5.$ $\qquad 2x - 5 = n.$

Test 87

For each of exercises 18 through 21, write a mixed numeral that expresses the quotient named.

18 $\frac{26}{3}$ **19** $\frac{107}{12}$ **20** $\frac{27}{14}$ **21** $\frac{59}{6}$

An ordered pair of rational numbers is named in each of exercises 22 through 25. Find the quotient of the first and second components of each ordered pair.

22 $(\frac{4}{5}, \frac{2}{3})$ **23** $(\frac{3}{40}, 6)$ **24** $(3, \frac{2}{7})$ **25** $(4\frac{1}{2}, \frac{7}{24})$

7 86 Exploring ideas

Decimal numerals

In unit 5 you learned about the decimal numeration system. In this lesson you will learn more about this system.

A Look at D1. Is numeral A the expanded form of the numeral 972? Remember that exponential notation is used in the expanded form of a numeral. What powers of 10 are expressed in numeral A?

> REMINDER
> When you write the numeral 10^2 to express 100, you are using exponential notation. A power is a number that is obtained by using the same factor one or more times.
> *See lesson 53, page 227.*

B Write the numeral 48 in expanded form. Write the numeral 36,728 in expanded form.

C Now think about the number $972\frac{564}{1000}$. Is $972\frac{564}{1000} = 972 + \frac{564}{1000}$? The denominator of the fraction $\frac{564}{1000}$ is the third power of what number?

You already know how to write the numeral 972 in expanded form. Now you will learn how to write the numeral $\frac{564}{1000}$ in expanded form.

D Look at D2. Numeral B expresses the sum of three numbers. What are the numbers? Is the sum the same as $\frac{564}{1000}$?

E Now look at numeral C. How was numeral C obtained from numeral B? Each denominator expressed in numeral C is a power of what number?

D1

972

A $9(10^2) + 7(10^1) + 2(1)$

D2

$\frac{564}{1000}$

B $\frac{500}{1000} + \frac{60}{1000} + \frac{4}{1000}$

C $\frac{5}{10} + \frac{6}{100} + \frac{4}{1000}$

D $\frac{5}{10^1} + \frac{6}{10^2} + \frac{4}{10^3}$

E $5(\frac{1}{10^1}) + 6(\frac{1}{10^2}) + 4(\frac{1}{10^3})$

F What property of addition tells you that numeral C names the same number as numeral B?

G How do you know that numeral D names the same number as numeral C?

H Look at numeral E. This is the expanded form of the numeral $\frac{564}{1000}$. Is the product of 5 and $\frac{1}{10^1}$ the same as $\frac{5}{10^1}$? Which symbol in numeral E expresses the product of 5 and $\frac{1}{10^1}$? Is $6(\frac{1}{10^2}) = \frac{6}{10^2}$? Is $4(\frac{1}{10^3}) = \frac{4}{10^3}$?

You know that each of the denominators 10^1, 10^2, and 10^3 is a power of 10. Now you will see that each of the digits 5, 6, and 4 is associated with the reciprocal of a power of 10.

I You know that the reciprocal of 10 is $\frac{1}{10}$. Is the reciprocal of 10^1 also $\frac{1}{10}$? Is it also $\frac{1}{10^1}$? $\frac{1}{10^1}$ is the reciprocal of what power of 10? $\frac{1}{10^2}$ is the reciprocal of what power of 10? $\frac{1}{10^3}$ is the reciprocal of what power of 10?

J Look again at numeral E. The digit 5 is associated with $\frac{1}{10^1}$. With what number is the digit 6 associated? With what number is the digit 4 associated? Is each of the numbers that you named the reciprocal of a power of 10?

Now look at D3. Numeral F is the expanded form of the numeral $972\frac{564}{1000}$. Notice that each

$$972\tfrac{564}{1000}$$

F $9(10^2) + 7(10^1) + 2(1) + 5(\frac{1}{10^1}) + 6(\frac{1}{10^2}) + 4(\frac{1}{10^3})$

D3

G 972.564

G	9	7	2.	5	6	4
H	10^2	10^1	1	$\frac{1}{10^1}$	$\frac{1}{10^2}$	$\frac{1}{10^3}$
	hundreds	tens	ones	tenths	hundredths	thousandths

D4

of the digits 9 and 7 is associated with a power of 10, and the digit 2 is associated with the number 1. Notice also that each of the digits 5, 6, and 4 is associated with the reciprocal of a power of 10.

K In numeral F, with what power of 10 is the digit 9 associated? With what power of 10 is the digit 7 associated? Is $\frac{1}{10}(10^2) = 10^1$? The number associated with the digit 7 is one tenth times the number associated with the digit 9.

L The digit 2 is associated with 1. Is $\frac{1}{10}(10^1) = 1$? What number is one tenth times 10^1?

M The digit 5 is associated with $\frac{1}{10^1}$. Is $\frac{1}{10}(1) = \frac{1}{10^1}$? What number is one tenth times 1?

N Is the number associated with the digit 6 one tenth times the number associated with the digit 5? The number associated with the digit 4 is one tenth times the number associated with which digit?

O In the expanded form of the numeral $972\frac{564}{1000}$, is the number that is associated with each digit one tenth times the number that is associated with the digit to its left?

Look at D4. Numeral G names the same rational number as the mixed numeral $972\frac{564}{1000}$. Notice the dot to the right of the digit 2. This dot, which is placed at the right of the digit associated with 1, is a *decimal point*. Row H in the chart in D4 tells you what number is associated with each digit in the numeral 972.564.

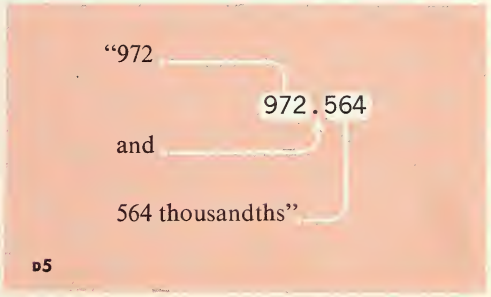
P Which digits in the numeral 972.564 are to the left of the digit associated with 1? Each of the digits that you named is associated with a power of what number?

Q Which digits in the numeral 972.564 are to the right of the digit associated with 1? How do you know that each digit you named is associated with a number less than 1?

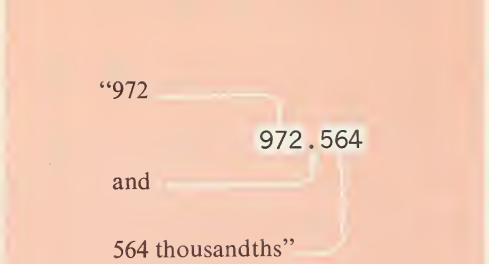
In your work in unit 5 you learned how to name the positions, or places, associated with the digits to the left of the decimal point in a decimal numeral. The names of the first three positions to the left and to the right of the decimal point are given in the chart in D4.

R For each of the digits to the right of the decimal point in the numeral 972.564, give the name of its position.

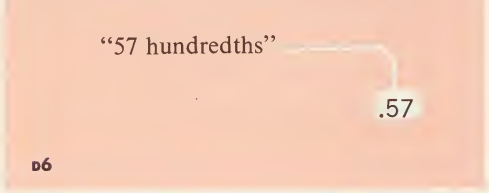
S D5 shows how to read the numeral 972.564. You read the digits to the left of the decimal point as you would read the numeral 972. Then you say "and" for the decimal point. Next, you read the part of the numeral to the right of the



D5



D5



D6

decimal point as you would read the numeral 564. Finally, you say "thousandths" for the name of the position of the digit 4, the last digit. Read the numeral 972.564.

T D6 shows how to read a numeral that has no digits to the left of the decimal point. Notice that you do not say "and" for the decimal point. You read the numeral as you would read the numeral 57. Then you read the name of the position of the last digit. Read the numeral .57.

Now read each of the numerals in exercises U through Z.

U 92.6	**W** .39	**Y** .010
V 42.005	**X** 9742.81	**Z** 3330.33

In unit 5 you learned that decimal numerals are written in a base-ten numeration system with the property of place value. Numerals such as 4, 92, 37.82, .7, and .315 are decimal numerals. In this book, when we say "decimal," we will mean a decimal numeral. Numerals like 37.82, .7, and .315 are sometimes called "decimal fraction numerals." Numerals such as $\frac{1}{4}$, $\frac{5}{1}$, $\frac{32}{73}$, and $\frac{7}{10}$ are sometimes called "common fraction numerals."

A What common fraction numeral names the rational number nine tenths?

B Use a decimal fraction numeral to name the rational number nine tenths.

You have already learned about the numbers associated with some of the digits in a decimal. Now look at the chart in D7. You will use this chart to help you learn about the numbers associated with other digits in a decimal.

Row A in the chart shows the digits in the numeral 698,510.34,297. The number associated with each of the digits is named in row B.

C Study row B. Is 10^4 one tenth times 10^5? What number is one tenth times 1? What number is one tenth times $\frac{1}{10^3}$?

D Remember that each number named in row B is associated with a digit in the numeral in row A. Is each number named in row B one tenth times the number named to its left?

E Now look at row C in the chart. Each number named in row C is the same number as that named in row B directly above it. Which numeral in row C expresses 10^3? $\frac{1}{10^1}$?

F Is $\frac{1}{100} = \frac{1}{10^2}$? Is $\frac{1}{1000} = \frac{1}{10^3}$? Which numeral in row C expresses $\frac{1}{100}$? $\frac{1}{1000}$?

G How do you know that $.0001 = \frac{1}{10000}$ is a true statement? Is $.00001 = \frac{1}{100000}$?

H Which digit in row A is associated with 10^5? Which digit is associated with 10^2? Is $600{,}000 = 6(10^5)$? Is $500 = 5(10^2)$?

I Which digit in row A is associated with $\frac{1}{10^1}$? Is $\frac{3}{10} = 3(\frac{1}{10^1})$? Is $.3 = \frac{3}{10}$?

J Which digit in row A is associated with $\frac{1}{10^2}$? Is $\frac{4}{100} = 4(\frac{1}{10^2})$? Is $.04 = \frac{4}{100}$?

K Which digit in row A is associated with $\frac{1}{1000}$? Is $.002 = \frac{2}{1000}$?

L Which digit in row A is associated with $\frac{1}{10000}$? Is $.0009 = \frac{9}{10000}$? Is $.00007 = \frac{7}{100000}$?

Next look at the names of the positions in the chart in D7. In unit 5 (page 238) you learned that the digits to the left of the decimal

point in a decimal numeral are written in groups of three. You remember that these groups of three are called periods.

M The first period to the left of the decimal point has no special name. Name the second period to the left of the decimal point.

N Give the names of the next two periods to the left of the thousands' period.

O Now give the name of the position of each of the first twelve digits to the left of the decimal point. Start with the ones.

P Give the name of the position of each of the first five digits to the right of the decimal point. Start with the tenths.

Q When you have a decimal fraction numeral with six digits to the right of the decimal point, you read the numeral as usual and then say "millionths." How do you read the numeral 5.346,892?

R In a decimal fraction numeral, what is the name of the seventh position to the right? Of the eighth position to the right? Of the ninth position to the right?

Now read each numeral below.

S 698,510.34,297

T 92,840.62,011

U 243.000,028,451

V 73,006.010,101

From the chart in D7 you have learned that a number is associated with each digit in a decimal. You have also learned about the names of the positions in a decimal. In the following exercises, you will learn more about decimals for rational numbers.

A In the decimal numeral .25, with what number is the digit 2 associated? With what number is the digit 5 associated? Is $.25 = \frac{2}{10} + \frac{5}{100}$? Is .25 the same number as $\frac{25}{100}$?

B Does the decimal .037 express $\frac{37}{1000}$? Use a common fraction numeral to express each of the numbers below.

.425 .0217 .003 .76511

C Does the decimal .09 express 9 hundredths? Can you think of 10 hundredths as the sum of 1 tenth and 0 hundredths? Is $.10 = .1$?

D Why can you think of 12 hundredths as the sum of 1 tenth and 2 hundredths? Does the decimal .12 express 12 hundredths?

E Does the decimal .009 express 9 thousandths? Suppose that you want to write a decimal for 10 thousandths. Is 10 thousandths the same as the sum of 0 tenths, 1 hundredth, and 0 thousandths? Is $.010 = .01$?

698,510.34,297

A	6	9	8	5	1	0.	3	4	2	9	7
B	10^5	10^4	10^3	10^2	10^1	1	$\frac{1}{10^1}$	$\frac{1}{10^2}$	$\frac{1}{10^3}$	$\frac{1}{10^4}$	$\frac{1}{10^5}$
C	100,000	10,000	1000	100	10	1	.1	.01	.001	.0001	.00001
	hundred thousands	ten thousands	thousands	hundreds	tens	ones	tenths	hundredths	thousandths	ten-thousandths	hundred-thousandths

D7

F Why can you think of 47 thousandths as the sum of 0 tenths, 4 hundredths, and 7 thousandths? Is $\frac{47}{1000} = \frac{0}{10} + \frac{4}{100} + \frac{7}{1000}$?

G Is 106 hundredths less than 1? Is 106 hundredths greater than 1? Why can you think of 106 hundredths as the sum of 1 one, 0 tenths, and 6 hundredths? Is $\frac{106}{100} = \frac{1}{1} + \frac{0}{10} + \frac{6}{100}$?

H Is $.100 = .10$? Is $.100 = .1$?

I Which of the four numerals below expresses 35 thousandths?

.350 .035 .0035 35,000

Use a decimal to express each rational number named in exercises J through Q.

J $\frac{99}{100}$ **L** $\frac{172}{1000}$ **N** $\frac{90}{10}$ **P** $\frac{7}{10000}$

K $2\frac{58}{1000}$ **M** $\frac{648}{100}$ **O** $27\frac{3}{10}$ **Q** $45\frac{34}{10000}$

What fraction whose denominator is a power of 10 indicates each of the rational numbers named below? Use common fraction numerals to express your answers.

R .4 **S** 5.00 **T** .0002 **U** 6.3 **V** .00045

Now you know the meaning of decimals like 1.68 and .592. You also know how to write decimals for certain rational numbers.

On your own

For each of exercises 1 through 5, use digits and the base-ten numeration system to write the numeral.

1 twenty-seven hundredths

2 eight hundred two and eight thousandths

3 four thousand one and five ten-thousandths

4 thirty-five and six hundredths

5 five hundred seventy-six and two thousand fifty-four ten-thousandths

For each of exercises 6 through 9, write the words that you say in reading the numeral.

6 4.6 **7** .007 **8** 70.0501 **9** 2789.56,358

For each of exercises 10, 11, and 12, write the numeral in expanded form.

10 33.00,654 **11** 982.080,812 **12** 6005.4708

Write a common fraction numeral or a mixed numeral to express each rational number named in exercises 13 through 24.

13 .047 **17** 356.7 **21** 7.001

14 2.36 **18** .8383 **22** .00041

15 .0721 **19** .4080 **23** .000009

16 4.0009 **20** 34.05 **24** 32.9710

Use a decimal to express each rational number named below.

25 $\frac{15}{10000}$ **27** $\frac{225}{10}$ **29** $\frac{205}{100000}$ **31** $\frac{376}{10000}$

26 $\frac{96}{1000}$ **28** $1\frac{7}{1000}$ **30** $\frac{10000}{100}$ **32** $\frac{4879}{10000}$

KEEPING SKILFUL

Tabulate the solution set of each condition expressed in exercises 1 through 11. The universe for (m, n) is $C \times C$.

1 $19n = 798 \wedge 1096 - m = n$.

2 $7/3 \sim m/n \wedge 30 - m = n$.

3 $m + n = 19 \wedge n - m = 3$.

4 $m/n \sim 36/24 \wedge m - n < 8$.

5 $m + n = 9 \wedge n - 1 > m$.

6 $n + 2072 = m \wedge m = 39(72)$.

7 $13/9 \sim m/n \wedge m + n < 70$.

8 $m - n > 115 \wedge m + 98 = 221$.

9 $m/n \sim 27/63 \wedge n - m = 28$.

10 $m/103 \sim 372/n \wedge 155/65 \sim m/13$.

11 $23 - n > m \wedge 6n = 102$.

For each exercise below, first find a fraction that indicates the sum, difference, product, or quotient. Then use a mixed numeral or a numeral for a basic fraction to express this number.

12 $\frac{6}{9} + \frac{10}{19}$ **18** $6\frac{7}{9} + 8\frac{1}{3}$ **24** $3 - 2\frac{1}{11}$

13 $\frac{2}{3} \times 1\frac{3}{4}$ **19** $\frac{4}{5} \div 6$ **25** $\frac{2}{5} \div \frac{9}{8}$

14 $8 - 1\frac{2}{5}$ **20** $4\frac{1}{5} - 2\frac{1}{6}$ **26** $9\frac{1}{2} - 4\frac{3}{4}$

15 $1\frac{2}{3} \times 1\frac{1}{3}$ **21** $\frac{4}{5} + \frac{5}{4}$ **27** $8\frac{1}{6} \times 3\frac{1}{7}$

16 $18 \div \frac{4}{9}$ **22** $1\frac{3}{4} \div \frac{1}{5}$ **28** $7\frac{1}{3} - 6\frac{4}{5}$

17 $\frac{1}{2} \div \frac{7}{8}$ **23** $2\frac{1}{3} \times 1\frac{1}{6}$ **29** $5\frac{1}{2} \div 3\frac{2}{3}$

Computing with decimals

You have learned how to use decimals to name rational numbers. Now you are ready to compute with decimals.

A Read sentence A in D1. The sum of two numbers is expressed in sentence A. What are the two numbers? Is $2.36 = \frac{236}{100}$? Is $.85 = \frac{85}{100}$? Does sentence A express a true statement?

B Does sentence B in D1 express a true statement? Upon what definition does your answer depend?

C Does sentence C express a true statement? How do you obtain the numerator of the fraction $\frac{321}{100}$ from $\frac{236}{100}$ and $\frac{85}{100}$?

D Does sentence D express a true statement? What decimal names the sum of 2.36 and .85?

E Now look at D2. The difference of two numbers is expressed in sentence E. What are the two numbers? Is $5.604 = \frac{5604}{1000}$? Is $3.972 = \frac{3972}{1000}$? Does sentence E express a true statement?

F Does sentence F express a true statement? Upon what definition does your answer depend?

G Read sentence G in D2. How do you obtain the numerator of $\frac{1632}{1000}$ from $\frac{5604}{1000}$ and $\frac{3972}{1000}$?

H Does sentence H express a true statement? What decimal names the difference of 5.604 and 3.972?

In the preceding exercises you used common fraction numerals to find a decimal for a sum and a decimal for a difference. Examples I and J in D3 show a more convenient way to do these computations.

A $2.36 + .85 = \frac{236}{100} + \frac{85}{100}$.

B $\frac{236}{100} + \frac{85}{100} = \frac{236 + 85}{100}$.

C $\frac{236 + 85}{100} = \frac{321}{100}$.

D $\frac{321}{100} = 3.21$.

D1

E $5.604 - 3.972 = \frac{5604}{1000} - \frac{3972}{1000}$.

F $\frac{5604}{1000} - \frac{3972}{1000} = \frac{5604 - 3972}{1000}$.

G $\frac{5604 - 3972}{1000} = \frac{1632}{1000}$.

H $\frac{1632}{1000} = 1.632$.

D2

Add.	Subtract.
I 2.3 6	**J** 5.6 0 4
.8 5	3.9 7 2
3.2 1	1.6 3 2

D3

I Study example I in D3. To find the sum of 2.36 and .85 by using decimals, you can first add just as you did to find the numerator of $\frac{321}{100}$. What is the sum of 236 and 85?

J Next you must determine the position of the decimal point in the decimal for the sum. Is the denominator of $\frac{236}{100}$ the same as the denominator of $\frac{85}{100}$? Each of the decimals, 2.36 and .85, expresses a number in hundredths. Therefore, the decimal obtained from the computation should also express the sum in hundredths. Does the decimal 3.21 express the sum in hundredths? Is the computation shown in example I correct?

K Study example J in D3. To find the difference of 5.604 and 3.972 by using decimals, you can first subtract just as you did to find the

numerator of $\frac{1632}{1000}$. What is the difference of 5604 and 3972?

L Next you must determine the position of the decimal point in the decimal that names the difference. Is the denominator of $\frac{5604}{1000}$ the same as the denominator of $\frac{3972}{1000}$? Which do the decimals 5.604 and 3.972 express, numbers in tenths, in hundredths, or in thousandths? Does the decimal 1.632 express the difference in thousandths?

M Suppose that you want to use decimals to find the sum of 3.8 and .62. Which of the decimals, 3.8 or .62, expresses a number in tenths? Which decimal expresses a number in hundredths?

N When you use decimals to find a sum, the fractions should have the same denominator. So each of the decimals should have the same number of digits to the right of the decimal point. Is $3.80 = 3.8$? What is the sum of 380 and 62? Is $3.80 + .62 = 4.42$? Is $3.8 + .62 = 4.42$?

O Suppose that you want to use decimals to find the difference of 4.32 and 1.657. When you use decimals to find a difference, both decimals should have the same number of digits to the right of the decimal point. What decimal with three digits to the right of the decimal point expresses the same number as the decimal 4.32?

P What is the difference of 4320 and 1657? Is $4.320 - 1.657 = 2.663$? Is $4.32 - 1.657 = 2.663$?

An ordered pair of numbers is named in each of exercises Q through V. For each exercise, first find a decimal to express the sum of the components. Then find a decimal to express the difference of the first and second components.

Q (1.295, .369)
R (42.31, .5)
S (6.89, 5.763)
T (37.04, .917)
U (1.898, 1.609)
V (7.0, .631)

Add.

I
$$\begin{array}{r} 2.36 \\ .85 \\ \hline 3.21 \end{array}$$

Subtract.

J
$$\begin{array}{r} 5.604 \\ 3.972 \\ \hline 1.632 \end{array}$$

D3

K $4.8 \times .015 = \frac{48}{10} \times \frac{15}{1000}$.

L $\frac{48}{10} \times \frac{15}{1000} = \frac{48 \times 15}{10 \times 1000}$.

M $\frac{48 \times 15}{10 \times 1000} = \frac{720}{10000}$.

N $\frac{720}{10000} = .0720$.

D4

$$\begin{array}{r} 4.8 \\ .015 \\ \hline 240 \\ 480 \\ \hline .0720 \end{array}$$

D5

Next you will learn how to use decimals to find the product of two rational numbers.

A Read sentence K in D4. The product of two numbers is expressed in sentence K. What are the two numbers? Is $4.8 = \frac{48}{10}$? Is $.015 = \frac{15}{1000}$? Does sentence K express a true statement?

B Does sentence L in D4 express a true statement? Upon what definition does your answer depend?

C Does sentence M express a true statement? How do you obtain the numerator of $\frac{720}{10000}$ from $\frac{48}{10}$ and $\frac{15}{1000}$? How do you obtain the denominator of $\frac{720}{10000}$ from $\frac{48}{10}$ and $\frac{15}{1000}$?

D Does sentence N express a true statement? What decimal names the product of 4.8 and .015?

The work in D4 shows how to use common fraction numerals to find the product of 4.8 and .015. The work in D5 shows another way to find the product.

E To find the product of 4.8 and .015 by using decimals, you can first multiply just as you did to find the numerator of $\frac{720}{10000}$. What is the product of 48 and 15?

F Next you must determine the position of the decimal point in the decimal for the product. You know that $\frac{720}{10000}$ is the product of $\frac{48}{10}$ and $\frac{15}{1000}$. Is $.0720 = \frac{720}{10000}$?

G The decimal 4.8 expresses a number in tenths. Does the decimal .015 express a number in thousandths? When you are using decimal fraction numerals to find the product of 4.8 and .015, why should you express the product in ten-thousandths?

H In the decimal 4.8, how many digits are to the right of the decimal point? In the decimal .015, how many digits are to the right of the decimal point?

I In the two decimals 4.8 and .015, how many digits in all are to the right of the decimal points? How many digits are to the right of the decimal point in the numeral for the product, .0720?

J Suppose that you use decimals to find the ? product of 12.784 and .0509. How many digits are to the right of the decimal point in the decimal 12.784? In the decimal .0509? The decimal that you obtain for the product should have how many digits to the right of the decimal point? How do you know?

Find a decimal that expresses the product of the two numbers named in each exercise below.

K .5, .9 **M** 6.84, .236 **O** .47, 693.0

L 1.2, .14 **N** 35.1, .875 **P** 1.05, .0003

In the following exercises you will learn a short method of finding the product of a given number and a power of 10.

O $10 \times .253 = 2.53$.

P $100 \times 498 = 49800$.

Q $1000 \times 3.6543 = 3654.3$.

D6

Q Read sentence O in D6. What decimal names the product of 10 and .253? Does this decimal name the product of 10^1 and .253? Explain your answer.

R Does the decimal point have the same position in the decimal .253 as in the decimal 2.53? How many digits are to the left of the decimal point in the decimal .253? How many digits are to the left of the decimal point in the decimal 2.53?

S Read sentence P in D6. What decimal names the product of 100 and 498? Does this decimal name the product of 10^2 and 498?

T Does the decimal 498.00 name the same number as the decimal 498? Does the decimal 49,800. name the same number as the decimal 49,800?

U How does the number of digits to the left of the decimal point in the decimal 498.00 differ from the number of digits to the left of the decimal point in the decimal 49,800.?

V Read sentence Q in D6. What decimal names the product of 1000 and 3.6543? The product of 10^3 and 3.6543? How does the decimal 3654.3 differ from the decimal 3.6543?

W Use a decimal to express the product of 7.24 ? and 10^4. Use a decimal to express the product of 60 and 10^2.

X 36,490 is the product of .3649 and what ? power of 10? 364.9 is the product of .03649 and what power of 10? 3.649 is the product of .003649 and what power of 10?

Now you will learn to use decimals to find the quotient of two rational numbers.

Suppose that you want to find the quotient of .45 and 1.5. In your earlier work in arithmetic, you learned that it is easier to find the quotient of .45 and 1.5 if you express the divisor in ones, instead of in tenths. Therefore, you use the decimals 4.5 and 15 when you compute.

A How can you obtain 4.5 from .45? How can you obtain 15 from 1.5?

Before you continue with the division of 4.5 by 15, you will discover that, no matter what numbers are being divided, both the dividend and the divisor can be multiplied by the same number without affecting the quotient.

B Study sentences A and B in D7. What is the universe for a? For b, c, and d? What is the reciprocal of $\frac{c}{d}$? For each replacement of the variables, is $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$?

C Now study sentences C and D in D7. What is the universe for m and n? If you multiply $\frac{a}{b}$ by the rational number $\frac{m}{n}$, do you obtain $\frac{ma}{nb}$? What product do you obtain if you multiply $\frac{c}{d}$ by $\frac{m}{n}$? For each replacement of the variables, is $\frac{ma}{nb} \div \frac{mc}{nd} = \frac{ma}{nb} \cdot \frac{nd}{mc}$? How do you know?

D For each replacement of the variables, is $\frac{ma}{nb} \cdot \frac{nd}{mc} = \frac{ad}{bc}$? Explain your answer.

E For each replacement of the variables, is the quotient of $\frac{ma}{nb}$ and $\frac{mc}{nd}$ equal to $\frac{ad}{bc}$? For each replacement of the variables, what is the quotient of $\frac{a}{b}$ and $\frac{c}{d}$? Is the quotient of $\frac{ma}{nb}$ and $\frac{mc}{nd}$ the same as the quotient of $\frac{a}{b}$ and $\frac{c}{d}$?

Now you know why the quotient of .45 and 1.5 is the same as the quotient of 4.5 and 15.

F Look at sentence E in D8. Does sentence E express a true statement? How do you know?

G You can use common fraction numerals to find the quotient of 4.5 and 15. Look at sentence F. What common fraction numeral names the number 4.5? What common fraction numeral names the number 15? Does sentence F express a true statement?

Universe for a = N.
Universe for b, c, d, m, and n = C.

A $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$.

B $\frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.

C $\frac{ma}{nb} \div \frac{mc}{nd} = \frac{ma}{nb} \cdot \frac{nd}{mc}$.

D $\frac{ma}{nb} \cdot \frac{nd}{mc} = \frac{ad}{bc}$.

D7

E $.45 \div 1.5 = 4.5 \div 15$.

F $4.5 \div 15 = \frac{45}{10} \div \frac{15}{1}$.

G $\frac{45}{10} \div \frac{15}{1} = \frac{45}{10} \cdot \frac{1}{15}$.

H $\frac{\cancel{45}^{3}}{\cancel{10}} \cdot \frac{1}{\cancel{15}_{1}} = \frac{3}{10}$.

I $\frac{3}{10} = .3$.

D8

$.45 \div 1.5 = 4.5 \div 15$.

J

$$\begin{array}{r|l} 15\overline{)45} & 3 \\ \underline{45} & \\ 0 & 3 \end{array}$$

K

$$\begin{array}{r|l} 15\overline{)4.5} & 3 \\ \underline{45} & \\ 0 & .3 \end{array}$$

D9

H What is the reciprocal of $\frac{15}{1}$? Does sentence G express a true statement? Upon what property does your answer depend?

I Read sentences H and I. What common fraction numeral expresses the quotient of 4.5 and 15? What decimal expresses the quotient of 4.5 and 15? Does each of sentences H and I express a true statement?

Now you will learn to use decimals to find the quotient of 4.5 and 15.

J Look at example J in D9. What decimal numeral names the quotient of 45 and 15?

K Now look at example K in D9. What decimal names the quotient of 4.5 and 15? Does this decimal also name the quotient of .45 and 1.5?

L In example K, is the dividend expressed in tenths? How is the quotient expressed?

Now think of how you have used decimals to find the quotient of .45 and 1.5. First, you multiplied both the dividend and the divisor by 10 so that you could express the divisor in ones. This meant that, instead of finding the quotient of .45 and 1.5, you found the quotient of 4.5 and 15. Next, you divided 45 by 15. Finally, you expressed the quotient in tenths because the dividend was expressed in tenths.

M Look at the computation shown in D10. Suppose that you use decimals to find the quotient of 1.836 and .12. By what number should you multiply both 1.836 and .12 so that the divisor can be expressed in ones? Is $183.6 \div 12 = 1.836 \div .12$?

N What is the quotient of 1836 and 12? The decimal 183.6 expresses the dividend in tenths. What decimal expresses the quotient in tenths?

O Is $183.6 \div 12 = 15.3$? Is $1.836 \div .12 = 15.3$?

P Suppose that you use decimals to find the quotient of 98.28 and 42. Why do you not need to multiply both the dividend and the divisor by a power of 10?

Q Use decimals to find the quotient of 98.28 and 42. What is this quotient?

R In your computation for exercise Q, the decimal 98.28 expresses the dividend in hundredths. This decimal has how many digits to the right of the decimal point? The decimal that you obtained for the quotient has how many digits to the right of the decimal point? Does this decimal express the quotient in hundredths?

When you use decimals to find the quotient of two rational numbers, the remainder in your work is not always zero. When a remainder is a number other than zero, you usually round the quotient to the *nearer tenth*, the *nearer hundredth*, or the *nearer thousandth*.

Suppose, for example, that you want to find the quotient of 7 and .73 in hundredths. D11 shows the computation that you should use.

A Study D11. How do you know that the quotient of 700 and 73 is the same as the quotient of 7 and .73?

B Why do you use the decimal 700.00, rather than the decimal 700, to express the dividend?

The work that follows explains how to find the quotient of 700.00 and 73 to the nearer hundredth.

$1.836 \div .12 = 183.6 \div 12.$

$$
\begin{array}{r|l}
12\overline{)183.6} & 100 \\
1200 & \\
\hline
636 & 50 \\
600 & \\
\hline
36 & 3 \\
36 & \\
\hline
0 & 15.3
\end{array}
$$

D10

$7 \div .73 = 700 \div 73.$

$$
\begin{array}{r|l}
73\overline{)700.00} & 900 \\
65700 & \\
\hline
4300 & 50 \\
3650 & \\
\hline
650 & 8 \\
584 & \\
\hline
66 & 9.58 \\
 & 9.59
\end{array}
$$

D11

C Notice the red numeral 66 in the computation. How was the number 66 obtained? Is 66 the last remainder you obtain in your work when you divide 700.00 by 73?

D What is the product of $\frac{1}{2}$ and the divisor, 73? Is $\frac{1}{2} \times 73 = 36\frac{1}{2}$? Decide whether the last remainder, 66, is less than $\frac{1}{2}$ times the divisor or greater than $\frac{1}{2}$ times the divisor.

Since the last remainder is greater than $\frac{1}{2}$ times the divisor, round the quotient to the next greater hundredth. In this case, the next greater hundredth is the nearer hundredth.

E What is the quotient of 700.00 and 73 to the nearer hundredth? What is the quotient of 7 and .73 to the nearer hundredth?

F Now you will use decimals to find the quotient of 17.842 and 6.3 to the nearer thousandth. D12 shows the computation that you should use. By what number should you first multiply both the dividend and the divisor? Explain your answer.

G In the computation, why is the dividend expressed in thousandths?

H What is the last remainder you obtain when you divide 178.420 by 63? Decide whether 4 is less than or greater than $\frac{1}{2}$ times the divisor.

Since 4 is less than $\frac{1}{2}$ times the divisor, you should round the quotient to the next lesser thousandth. In this case, the next lesser thousandth is the nearer thousandth, or 2.832.

I What is the quotient of 178.420 and 63 to the nearer thousandth? What is the quotient of 17.842 and 6.3 to the nearer thousandth?

J Use decimals to find the quotient of 2.511 and 5.4 to the nearer hundredth. By what number do you first multiply both the dividend and the divisor? Why do you use this number? What decimal do you use to express the dividend in your computation?

K The last remainder when you divide 25.11 by 54 is 27. Is 27 equal to $\frac{1}{2}$ times the divisor?

$7 \div .73 = 700 \div 73.$

$$
\begin{array}{r|l}
73\overline{)700.00} & 9.58 \\
\underline{657} & \\
430 & \\
\underline{365} & \\
65 & \\
\underline{58} & \\
7 & 9.59
\end{array}
$$

D11

$17.842 \div 6.3 = 178.42 \div 63.$

$$
\begin{array}{r|l}
63\overline{)178.420} & 2.832 \\
\underline{126000} & \\
52420 & \\
\underline{50400} & \\
2020 & \\
\underline{1890} & \\
130 & \\
\underline{126} & \\
4 &
\end{array}
$$

D12

Since the last remainder is equal to $\frac{1}{2}$ times the divisor, you should round the quotient .46 to the nearer *even* hundredth. Therefore, .46, not .47, is the quotient of 2.511 and 5.4 to the nearer even hundredth.

L Use decimals to find the quotient of 95.10 and 20 to the nearer hundredth. What is the last remainder that you obtain when you divide? Is this remainder equal to $\frac{1}{2}$ times the divisor?

M Which number, 4.75 or 4.76, is the quotient of 95.10 and 20 to the nearer even hundredth?

Now you will use decimals and what you have learned about division of rational numbers to find reciprocals. In lesson 85 you learned that the reciprocal of a given non-zero rational number is the quotient of 1 and the given number. If x is any member of Z, then the reciprocal of x is $\frac{1}{x}$. If you wish, you may use the symbol $\frac{1}{x}$ to express the quotient of 1 and x.

N Is 2.5 a rational number? Is $2.5 = \frac{25}{10}$? What is the reciprocal of $\frac{25}{10}$?

O Is $\frac{10}{25} = \frac{40}{100}$? Is $\frac{40}{100} = \frac{4}{10}$? What decimal fraction numeral expresses the reciprocal of 2.5 in tenths?

P You know that .4 is the reciprocal of 2.5. You also know that $\frac{1}{2.5}$ is the reciprocal of 2.5. Now use decimal fraction numerals to find the quotient of 1 and 2.5 in tenths. What is the quotient of 1 and 2.5?

Q Use decimal fraction numerals to find the reciprocal of .5. Is $\frac{1}{.5} = 2$?

R Use decimal fraction numerals to find the reciprocal of 8. Is the quotient of 1 and 8 the reciprocal of 8? Find the quotient of 1 and 8 in thousandths. What decimal fraction numeral expresses the reciprocal of 8?

In the following exercises, you will learn a short method of finding the quotient of a given number and a power of 10.

S Read sentence L in D13. What decimal expresses the quotient of 52.8 and 10? Is 5.28 the quotient of 52.8 and 10^1? How does the decimal 5.28 differ from the decimal 52.8?

L $52.8 \div 10 = 5.28$.
M $21.04 \div 100 = .2104$.
N $.638 \div 1000 = .000638$.
O $97 \div 10000 = .0097$.

D13

T Read sentence M. What decimal expresses the quotient of 21.04 and 100? Is .2104 the quotient of 21.04 and 10^2? How does the decimal .2104 differ from the decimal 21.04?

U Read sentence N. What decimal expresses the quotient of .638 and 1000? What is the quotient of .638 and 10^3? How does the decimal .638 differ from the decimal .000638?

V Read sentence O. What decimal expresses the quotient of 97 and 10^4?

W Use a decimal to name the quotient of 248.36 and each of the following powers of 10:
10 100 1000 10,000 100,000

In this lesson you have learned to use decimals to find sums, differences, products, quotients, and reciprocals.

On your own

Find the sum.

1 2736.05, 576
2 .0008, .0003
3 2.457, 327.15
4 2.001, 9.9999
5 .5, .66
6 .75, 8.3
7 437, .162
8 1.8, .3007, .42
9 97.32, 18, 3.6
10 3.5876, 251.8, 25.91

Find the difference of the first and second components of each ordered pair named in exercises 11 through 20.

11 (38.57, 17.02)
12 (4.809, 3.005)
13 (.954, .42)
14 (11.5, 11)
15 (276.43, .8976)
16 (.06, .009)
17 (65, .98)
18 (153.007, 15.3007)
19 (6.7, 1.38)
20 (33, .502)

Find the product.

21 27.8, 100
22 11.3, 10
23 13.6, 2.4
24 9.2, 1.45
25 213.2, .029
26 1.007, .009
27 1000, 4.7
28 783, .0012
29 76.3, .08
30 .44, 1.5

Find the quotient of the first and second components of each ordered pair named in exercises 31 through 39. The word in paren-

theses tells you how to express the quotient. For exercise 31, for example, you should express the quotient to the nearer hundredth. From now on, a word in parentheses will be used to tell you how to express the quotient.

31 (5.737, 2.3) (Hundredths)
32 (2256, 1.3) (Tenths)
33 (463, 35) (Ones)
34 (3482, 11) (Tenths)
35 (26.734, 32.6) (Hundredths)
36 (14.81, 1.76) (Hundredths)
37 (1, 3) (Ten-thousandths)
38 (.1, .8) (Thousandths)
39 (.485, 10) (Thousandths)

For each of exercises 40 through 45, find a decimal for the reciprocal of the number named.

40 4 (Hundredths)
41 5 (Tenths)
42 .8 (Hundredths)
43 1.6 (Thousandths)
44 .025 (Ones)
45 .1 (Ones)

KEEPING SKILFUL

For each of exercises 1 through 5, first tabulate the solution set described. Then make a graph of each solution set. $U = N \times N$.

1 $\{(x, y) | x < y \wedge x + y < 6\}$
2 $\{(x, y) | y - 1 > x \wedge 101 - y = 96\}$
3 $\{(x, y) | x + y = 7 \wedge 2 + x < y\}$
4 $\{(x, y) | 27x = 162 \wedge x + 4 > y\}$
5 $\{(x, y) | x + y > 5 \wedge y + x < 8\}$

For each ordered pair named below, first name the natural number onto which you can map the ordered pair by addition. Then name the natural number onto which you can map the ordered pair by multiplication.

6 (39, 3)
7 (15, 86)
8 (49, 49)
9 (24, 103)
10 (368, 51)
11 (67, 890)
12 (50, 706)
13 (311, 22)
14 (400, 386)
15 (111, 111)
16 (306, 124)
17 (705, 430)
18 (690, 250)
19 (109, 804)
20 (520, 906)

Repeating decimals

In this lesson you will learn how to convert common fraction numerals to decimal fraction numerals and decimal fraction numerals to common fraction numerals. You will also learn why decimals are sometimes called repeating decimals.

A Look at D1. How do you know that the sentence in D1 expresses a true statement?

B Use the reduction property to find the basic fraction for $\frac{376}{1000}$.

C Is $.376 = \frac{47}{125}$?

D Is $2.8 = \frac{28}{10}$? Is $2.8 = \frac{14}{5}$? How do you know?

E Suppose that you are given a decimal like .376 or 2.8. Explain how to find the basic fraction for the rational number named by the decimal.

For each of exercises F through M, find the basic fraction for the rational number named.

F .77
G .55
H 3.6
I .013
J 25.72
K .0025
L .875
M .382

You know how to convert a decimal to a common fraction numeral. Next you will learn how to convert a common fraction numeral to a decimal.

N Study D2. The work in D2 shows how to find a decimal for $\frac{1}{4}$. You know that the quotient of the rational numbers 1 and 4 is the rational number $\frac{1}{4}$. What decimal is used to name the dividend in D2?

O In D2, what decimal expresses the quotient of 1 and 4? Is $\frac{1}{4} = .25$?

P Now suppose that you want to find a decimal for $\frac{3}{8}$. Is $\frac{3}{8}$ the same number as $3 \div 8$?

$.376 = \frac{376}{1000}$.

D1

$$
\begin{array}{r|l}
4\overline{)1.00} & 20 \\
\underline{80} & \\
20 & 5 \\
\underline{20} & \\
0 & .25
\end{array}
$$

D2

$$
\begin{array}{r|l}
8\overline{)3.000} & 300 \\
\underline{2400} & \\
600 & 70 \\
\underline{560} & \\
40 & 5 \\
\underline{40} & \\
0 & .375
\end{array}
$$

D3

A

$$
\begin{array}{r|l}
11\overline{)7.0} & 6 \\
\underline{66} & \\
4 &
\end{array}
$$

B

$$
\begin{array}{r|l}
11\overline{)7.0} & 6 \\
\underline{66} & \\
40 & 3 \\
\underline{33} & \\
7 &
\end{array}
$$

D4

Q The work in D3 shows how to find a decimal for $\frac{3}{8}$. What decimal is used to name the dividend?

R What decimal expresses the quotient of 3 and 8? Is $\frac{3}{8} = .375$?

For each of exercises S through X, use the method shown in D2 and D3 to find a decimal for the rational number named. The word in parentheses tells you how to express the quotient.

S $\frac{13}{20}$ (Hundredths)

T $\frac{7}{5}$ (Tenths)

U $\frac{1}{8}$ (Thousandths)

V $\frac{5}{16}$ (Ten-thousandths)

W $\frac{25}{4}$ (Hundredths)

X $\frac{5}{2}$ (Tenths)

Notice that the division examples that you worked for exercises S through X all "came out even." In other words, at some step in the division process, the remainder 0 was obtained. A decimal like .25 or .375 is a *terminating decimal*. If, at some step in the division process, the remainder 0 is obtained, the quotient can be named by a terminating decimal.

Now you will see that certain rational numbers cannot be named by terminating decimals.

A Suppose that you want to find a decimal for $\frac{7}{11}$. Is $\frac{7}{11}$ the same number as $7 \div 11$?

For exercises S through X, you were told to express the quotients in tenths, in hundredths, and so on. In each case, the division "came out even."

B Now look at example A in D4. This example shows the first step in dividing 7 by 11. What decimal expresses the dividend? What is the first digit in the decimal for the quotient? What remainder is obtained in the first step?

Since you did not obtain a remainder of 0 in the first step of the division process, you know that the division process can be continued. Look at example B, which is the second step in dividing 7 by 11. Notice the digits in white that were inserted after the first step so that the process could be continued.

C How is the dividend expressed in the second step? What is the second digit in the decimal for $\frac{7}{11}$? What is the second remainder?

You know that, in the division process, each remainder that you obtain is used to help you decide what further digits are in the decimal for the quotient.

D Now look at example C in D5. This is the third step in dividing 7 by 11. Where was the digit zero inserted for this step?

E For the first step, you divided 70 by 11. For the third step, what number do you divide by 11?

F Now look at the first digit and the third digit in the decimal for the quotient. Are the digits the same?

G Look at example D in D5. This is the fourth step in dividing 7 by 11. Is the fourth step similar to the second step? Explain your answer.

H How do you know that, if you wish, you can continue the division process beyond the fourth step when you are finding the quotient of 7 and 11?

I Imagine that you have carried out ten steps of the division process in finding the quotient of 7 and 11. For each step, decide what digit in the decimal for the quotient you will obtain. For each step, decide what remainder you will obtain.

J In finding the quotient of 7 and 11, will you ever obtain a remainder of 0 in the division process? Will there be a last digit in the decimal for $\frac{7}{11}$? Using a decimal, express the quotient of 7 and 11 to the nearer ten-thousandth.

As you carry out the division process, it is not necessary to rewrite your work for each successive step. Just be sure to plan your work in such a way that you leave enough room to insert any necessary digits.

Notice that the digits in a decimal for $\frac{7}{11}$ form the pattern 63. This pattern keeps repeating without end in the decimal for $\frac{7}{11}$. Such a decimal is a *repeating decimal.* The pattern of digits, 63, is the *repetend* (rep′ə tend′) of the decimal for $\frac{7}{11}$.

K How do you know that $\frac{7}{11} = .6363$ is not a true statement?

Study D6, which shows how to write and read a decimal for $\frac{7}{11}$. The bar above the digits 6 and 3 tells you that the pattern of digits, 63, is the repetend of the decimal and that the pattern 63 repeats without end in the decimal for $\frac{7}{11}$.

L Is $\frac{7}{11} = .\overline{63}$ a true statement?

M The work in D7 shows how to obtain a decimal for $\frac{1}{6}$. Is $\frac{1}{6}$ the same number as $1 \div 6$?

N What decimal in D7 expresses the dividend?

O What digit in the decimal for $\frac{1}{6}$ is obtained in each of the two steps shown in D7?

A

$$\begin{array}{r|l} 11\overline{)7.0} & 6 \\ 66 & \\ \hline 4 & \end{array}$$

B

$$\begin{array}{r|l} 11\overline{)7.0} & 6 \\ 66 & \\ \hline 40 & 3 \\ 33 & \\ \hline 7 & \end{array}$$

D4

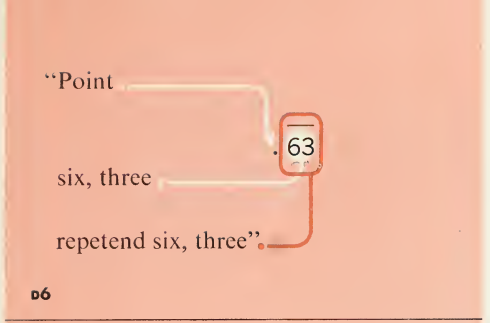
C

$$\begin{array}{r|l} 11\overline{)7.00} & 60 \\ 660 & \\ \hline 40 & 3 \\ 33 & \\ \hline 70 & 6 \\ 66 & \\ \hline 4 & \end{array}$$

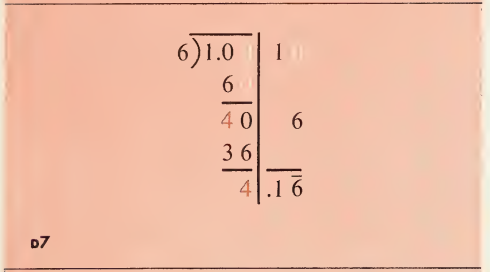
D

$$\begin{array}{r|l} 11\overline{)7.000} & 600 \\ 6600 & \\ \hline 400 & 30 \\ 330 & \\ \hline 70 & 6 \\ 66 & \\ \hline 40 & 3 \\ 33 & \\ \hline 7 & .6363 \end{array}$$

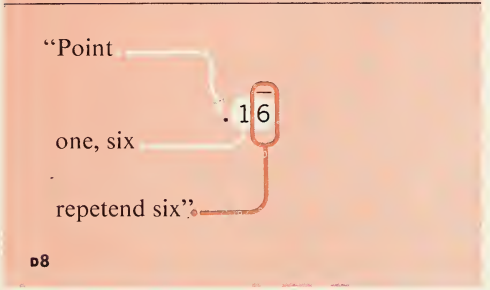
D5



D6



D7



D8

P If you carry out five steps of the division process, what digits will you obtain in the decimal for $\frac{1}{6}$?

Q In the division process for finding the decimal for $\frac{1}{6}$, is the third step similar to the second step? Explain your answer.

R When you divide 1 by 6, you will never obtain a remainder of 0. Explain how you know that this is true.

Study D8, which shows how to read the decimal for $\frac{1}{6}$.

S What is the repetend in the decimal for the quotient of 1 and 6? Why is the bar placed above the digit 6, but not above the digit 1?

T Is $\frac{1}{6} = .16$?

Use the method shown in D4, D5, and D7 to find a repeating decimal for each of the rational numbers named below.

U $\frac{1}{3}$ **V** $\frac{5}{9}$ **W** $\frac{3}{7}$ **X** $\frac{11}{6}$ **Y** $\frac{5}{6}$ **Z** $\frac{2}{3}$

Earlier in this lesson you learned that, if a remainder of 0 is obtained at some step in the division process, then the decimal for the quotient is a terminating decimal. You also learned that certain rational numbers cannot be named by terminating decimals. Now you will decide whether or not a terminating decimal is also a repeating decimal.

A Is $\frac{1}{4} = .25$? Is $.25 = .250$? Does the decimal .250 name the same rational number as the decimal .25?

B Write a decimal that names $\frac{1}{4}$ and that has ten digits to the right of the decimal point.

C Suppose that you inserted 50 digits for zero after the digit 5 in the decimal .25. Would you obtain a decimal for $\frac{1}{4}$?

D Does the decimal $.25\overline{0}$ express $\frac{1}{4}$?

You can insert the digit 0 as many times as you wish to the right of the digits in the decimal .25 without changing the number expressed. Therefore, the terminating decimal .25 and all other terminating decimals are repeating decimals. However, the repetend 0 is not usually indicated when we write such a decimal. For example, we usually express $\frac{1}{4}$ by the decimal .25, and not the decimal $.25\overline{0}$.

If a terminating decimal can be used to name a given rational number, then the given rational number can be indicated by a fraction whose denominator is 1 or a power of 10. For example, you know that $\frac{1}{4} = \frac{25}{100}$; you also know that the denominator of the fraction $\frac{25}{100}$ is a power of 10.

E Now explain a method that will help you decide whether or not a given rational number can be expressed by a terminating decimal.

A $.4 < .5$.
B $.8\overline{3} < .875$.
C $.55 < .\overline{5}$.

D9

Tell whether or not each number named below can be expressed by a terminating decimal.

F $\frac{7}{4}$ G $\frac{10}{3}$ H $\frac{2}{7}$ I $\frac{6}{5}$ J $\frac{2}{6}$ K $\frac{3}{20}$

Now that you know how to convert common fraction numerals to decimals, you can use decimals when you order rational numbers.

A Read sentence A in D9. Is $.4 = \frac{4}{10}$? Is $.5 = \frac{5}{10}$? Is $\frac{4}{10} < \frac{5}{10}$? Explain your answers.

REMINDER

The universe for a and c is N.
The universe for b and d is C.
For each replacement of the variables,
if $ad < cb$, then $\frac{a}{b} < \frac{c}{d}$.

See lesson 73, page 321.

B Does sentence A express a true statement?

C Read sentence B in D9. Is the decimal $.8\overline{3}$ a repeating decimal? A terminating decimal?

D Does sentence B express a true statement?

E Read sentence C. Is the decimal .55 a repeating decimal? A terminating decimal?

F Is the decimal $.\overline{5}$ a repeating decimal? A terminating decimal?

G How do you know that sentence C expresses a true statement?

H Explain how you can order the numbers expressed by two given terminating decimals by simply inspecting the decimals.

Tell whether each of the following sentences expresses a true statement or a false statement.

I $.25 > .23$. K $1.8 < 1.8\overline{3}$. M $.8 > .75$.
J $.\overline{3} < .3$. L $.1875 < .875$. N $.1\overline{6} < .06$.

Now you know how to convert common fraction numerals to decimal numerals and decimal numerals to common fraction numerals. You have learned what repeating decimals and terminating decimals are, and you have also learned how to order rational numbers by using decimals.

On your own

Find the basic fraction that indicates the rational number named in each of exercises 1 through 12.

1 .48	4 2.03	7 15.42	10 .0132
2 .007	5 .09	8 .0088	11 8.06
3 1.34	6 7.8	9 .904	12 .00001

Use a decimal numeral to express the rational number named in each of exercises 13 through 24.

13 $\frac{1}{2}$	15 $\frac{4}{7}$	17 $\frac{5}{8}$	19 $\frac{12}{11}$	21 $\frac{7}{15}$	23 $\frac{5}{3}$
14 $\frac{3}{5}$	16 $\frac{4}{9}$	18 $\frac{7}{6}$	20 $\frac{1}{13}$	22 $\frac{1}{25}$	24 $\frac{3}{4}$

For each of exercises 25 through 28, arrange the rational numbers named in order, according to the idea of "less than." Begin with the name of the least number.

25 .375 .037 .073 .307 .357 .370
26 .4086 .406 .4 .468 .4068 .46
27 $.\overline{6}$ .6 .63 .601 $.1\overline{6}$ .61
28 .12 $.\overline{12}$ $.1\overline{2}$ .122 .21 .012

The sentences in exercises 29 through 40 express statements about two rational numbers. For each exercise, first replace each common fraction numeral by a decimal that expresses the same number. Then write another sentence that expresses the same statement as the original sentence. Finally, tell whether the sentence that you wrote expresses a true statement or a false statement.

29 $\frac{4}{5} < \frac{3}{4}$.	33 $\frac{17}{25} > \frac{11}{16}$.	37 $\frac{3}{5} > \frac{5}{8}$.
30 $\frac{7}{8} > \frac{5}{6}$.	34 $\frac{5}{9} < \frac{4}{7}$.	38 $\frac{1}{3} < \frac{3}{8}$.
31 $\frac{9}{25} > \frac{7}{20}$.	35 $\frac{7}{13} < \frac{6}{11}$.	39 $\frac{2}{3} > \frac{3}{5}$.
32 $\frac{2}{12} < \frac{3}{11}$.	36 $\frac{5}{4} > \frac{21}{20}$.	40 $\frac{2}{25} < \frac{3}{20}$.

SPECIAL CHALLENGE

In lesson 69 you learned that two prime numbers are twin primes if there is only one natural number between them.

A Are 3 and 5 twin primes? Name six other twin primes.

B Name the only natural number that is between 3 and 5. Is 7 a prime? How many natural numbers are between 5 and 7?

If three primes can be ordered so that there is only one natural number between the first and second primes and only one natural number between the second and third primes, then these prime numbers are *triple primes*. Thus, 3, 5, and 7 are triple primes.

C Are 1, 3, and 5 triple primes? Explain your answer.

The work that follows will help you decide whether or not there is more than one set of triple primes.

D Is any even number greater than 2 a prime number? Explain your answer.

E What two successive natural numbers are primes? Is there more than one such pair? Explain your answer.

Since every even number greater than 2 is a composite number, you know that each member of a set of triple primes must be an odd number. Every odd number can be thought of in one of three ways. It is a multiple of 3, or it is the successor of a multiple of 3, or it is the *predecessor* of a multiple of 3. The predecessor of a given natural number is the natural number that comes immediately before the given number.

Name the successor and the predecessor of each of the numbers named below.

F 1 **G** 27 **H** 656 **I** 1975

For each odd number named in exercises J through Q, tell whether it is a multiple of 3, the successor of a multiple of 3, or the predecessor of a multiple of 3.

J 19 **L** 23 **N** 111 **P** 7901

K 3 **M** 59 **O** 113 **Q** 9157

Now think about three consecutive odd numbers, such as 15, 17, and 19. 15 is a multiple of 3; 17 is the predecessor of a multiple of 3; and 19 is the successor of a multiple of 3.

In each group of numbers named below, which number is a multiple of 3? The predecessor of a multiple of 3? The successor of a multiple of 3?

R 13, 15, 17 **U** 61, 63, 65

S 23, 25, 27 **V** 75, 77, 79

T 43, 45, 47 **W** 101, 103, 105

X For any three consecutive odd numbers, show that one of these numbers will always be a multiple of 3.

Y Explain why {3, 5, 7} is the only set of triple primes.

KEEPING SKILFUL

For each exercise below, use a decimal to express the sum, difference, product, or quotient.

1 $14.62 + 9.43$

2 $.862 - .372$

3 7.62×8

4 $1.003 - .2145$

5 $.11 + .7 + .67$

6 $.92 - .086$

7 $7.6 + .019 + 3.8$

8 $3.62 - .835$

9 4.12×9.6

10 $.25 \times .13$

11 $6.5 + 18 + .792$

12 $.368 \times .75$

13 $9 - .04956$

14 $2.1075 + .934 + .7$

15 $(.186 \times 9.4)7$

16 $.6(4.9 \times .38)$

17 $21.5 \div 25$ (Hundredths)

18 $33.18 \div 4.2$ (Tenths)

19 $12 \div .16$ (Ones)

20 $.214 \div .3$ (Thousandths)

21 $2 \div 14$ (Ten-thousandths)

22 $64 \div .13$ (Hundredths)

23 $.9086 \div .72$ (Thousandths)

CHECKING UP

If you have trouble with these tests, you should refer to lessons 86, 87, and 88.

Test 88

From the list given below each exercise, choose the expression that correctly completes the exercise.

1 The expanded form of the numeral $2\frac{304}{10000}$ is the numeral ~~~.

a $2(1) + 3(\frac{1}{10^1}) + 0(\frac{1}{10^2}) + 4(\frac{1}{10^3})$

b $2(1) + 3(\frac{1}{10^2}) + 4(\frac{1}{10^3})$

c $2(1) + 0(\frac{1}{10^1}) + 3(\frac{1}{10^2}) + 0(\frac{1}{10^3}) + 4(\frac{1}{10^4})$

d $2(1) + 304(\frac{1}{10^3})$

2 The product of 47.832 and 10^3 is ~~~.

a .047,832 **c** 47,832

b 478.32 **d** 47,832,000

3 The number .052 is less than ~~~.

a .0052 **b** 0.52 **c** .0092 **d** .0500

4 The decimal for ~~~ is a terminating decimal.

a $\frac{2}{6}$ **b** $\frac{1}{7}$ **c** $\frac{5}{9}$ **d** $\frac{3}{8}$

5 In the decimal 328.1657, the digit 5 is associated with ~~~.

a 10^3 **b** $\frac{1}{10^3}$ **c** $\frac{1}{10^2}$ **d** 10^2

6 The decimal ~~~ is a name for $\frac{1}{11}$.

a $.0\overline{9}$ **b** $.\overline{09}$ **c** .09 **d** $.09\overline{0}$

7 The common fraction numeral ~~~ is a name for .00008.

a $\frac{8}{10000}$ **b** $\frac{4}{500000}$ **c** $\frac{1}{12500}$ **d** $\frac{2}{2500}$

8 The number .0536 divided by 100 is ~~~.

a 5.36 **b** .00536 **c** .000536 **d** 53.6

Test 89

9 Find the sum of 2.81 and .5009.

10 Find the sum of .09, 37, and 1.765.

11 Subtract 1.05 from 1.7.

12 Subtract 62.809 from 147.2.

13 Find the product of 236.5 and 1.79.

14 Find the product of .088 and .06.

15 Divide .09 by .23. (Tenths)

16 Divide 4 by 7. (Hundredths)

The density property

In unit 6 you learned that each natural number has a successor. You also learned that you can determine how many natural numbers there are between any two given natural numbers. In this lesson you will find out if the rational numbers have these properties.

A A picture of a natural-number line is shown in D1. Which natural numbers are associated with the points that are represented by dots? What do the three small dots to the right of the picture of the line tell you?

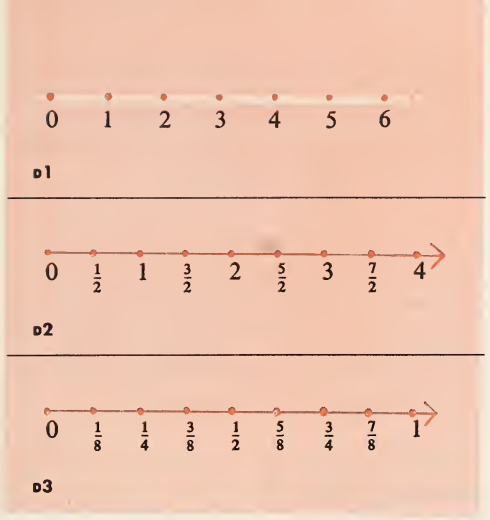
B Tabulate the set of natural numbers associated with points between point 0 and point 1 in the natural-number line. Is this set a finite set? Is this set the solution set of $0 < x \wedge x < 1$? U = N.

C Tabulate the set of natural numbers associated with points between point 0 and point 5. Is this set a finite set? Is this set the solution set of $0 < x \wedge x < 5$? U = N.

D In the set that you tabulated for exercise C, is $1 < 2$? Is $2 < 3$? Is $3 < 4$?

E You know that the successor of a given natural number is greater than the given number and that there are no natural numbers between a given number and its successor. Is 2 the successor of 1? Is 3 the successor of 2? Name the successor of 25. The successor of 87.

F Now look at the picture of a rational-number line shown in D2. Notice that the numerals 0, 1, 2, 3, and 4 are used as names of rational numbers. What does the arrow to the right of the picture of the line tell you?



G In D2, a dot is used to represent a point between point 0 and point 1. What number is associated with this point? Is $\frac{1}{2}$ a member of the solution set of $0 < x \wedge x < 1$? $U = R_a$.

H Is point $\frac{1}{4}$ represented by a dot in D2? How do you know that point $\frac{1}{4}$ is between point 0 and point $\frac{1}{2}$? Is $\frac{1}{4}$ a member of $\{x|0 < x \wedge x < \frac{1}{2}\}$? $U = R_a$.

I Is point $\frac{1}{8}$ between point 0 and point $\frac{1}{4}$? Is point $\frac{1}{16}$ between point 0 and point $\frac{1}{8}$? Name a point that is between point 0 and point $\frac{1}{16}$.

J You know that $\frac{1}{5} < \frac{4}{15}$ because $1 \cdot 15 < 4 \cdot 5$. How do you know that $\frac{4}{15} < \frac{1}{3}$? Since $\frac{1}{5} < \frac{4}{15}$ and $\frac{4}{15} < \frac{1}{3}$, we say that $\frac{4}{15}$ is between $\frac{1}{5}$ and $\frac{1}{3}$.

K Is $\frac{3}{4}$ between $\frac{5}{8}$ and 1? Explain your answer.

Now you will find a rational number that is between two given rational numbers.

L Think of the rational numbers 5 and 9. Is $5 < 9$? Is $(5 + 9) \div 2$ between 5 and 9? Explain your answer.

M Think of the rational numbers $\frac{13}{2}$ and $\frac{19}{2}$. Is $\frac{13}{2} < \frac{19}{2}$?

N Now you will find a rational number that is between $\frac{13}{2}$ and $\frac{19}{2}$. Is $(\frac{13}{2} + \frac{19}{2}) \div 2 = \frac{32}{4}$? Is $\frac{32}{4}$ between $\frac{13}{2}$ and $\frac{19}{2}$? How do you know?

O How do you know that $(\frac{13}{2} + \frac{32}{4}) \div 2$ is between $\frac{13}{2}$ and $\frac{32}{4}$?

P How do you know that $(\frac{32}{4} + \frac{58}{8}) \div 2$ is between $\frac{32}{4}$ and $\frac{58}{8}$?

Q If $x < y$, do you think that, for each replacement of the variables, $(x + y) \div 2$ is between x and y? The universe for x and y is R_a.

R Explain how you can find a number that is between $\frac{4}{5}$ and $\frac{9}{7}$. What is this number?

You have learned that it is always possible to find at least one rational number that is between any two given rational numbers. Therefore, we say that the set of rational numbers is *dense*. This property is the *density property of the set of rational numbers*. The property is expressed below.

The universe for x, y, *and* r *is* R_a. *For each replacement of* x *and* y, *if* $x < y$, *then there is a rational number* r *such that* $x < r \wedge r < y$.

S Suppose that a is any natural number. Is there a natural number between a and $a + 1$? Does the set of natural numbers have a density property?

T You know that, in a rational-number line, point $\frac{1}{2}$ is between point 0 and point 1 because $0 < \frac{1}{2} \wedge \frac{1}{2} < 1$. Is point $\frac{3}{4}$ between point 0 and point 1? Explain your answer.

U Look at D3. Which points between point 0 and point 1 are represented by dots? Are the rational numbers that are associated with these points members of $\{x|0 < x \wedge x < 1\}$? $U = R_a$. Name three other rational numbers that are members of $\{x|0 < x \wedge x < 1\}$.

V Suppose that x and y are any two given rational numbers. Use the density property of the set of rational numbers to explain why the set of rational numbers between x and y is an infinite set.

W Is the set of points associated with rational numbers between point x and point y an infinite set?

You know now that the set of rational numbers between any two given rational numbers is infinite. Next you will decide whether or not there is a rational number that immediately follows a given rational number.

A Tabulate $\{x|0 < x \wedge x < 1\}$. $U = N$. In the ordering of the natural numbers, does 1 immediately follow 0? Is 1 the successor of 0?

B Now suppose that the universe is R_a. Is $\{x|0 < x \wedge x < 1\} = \{\ \}$?

C Is $\{x|0 < x \wedge x < 1\}$ an infinite set? Is 0 the least member of R_a? Explain your answers.

D Now try to name the number in R_a that immediately follows 0. Suppose that you choose $\frac{1}{2}$. Is $\frac{1}{4}$ between 0 and $\frac{1}{2}$? In the ordering of the rational numbers, does $\frac{1}{2}$ immediately follow 0?

E Is $\frac{1}{8} < \frac{1}{4}$? Explain why $\frac{1}{4}$ does not immediately follow 0 in R_a. Explain why $\frac{1}{8}$ does not immediately follow 0.

F Use the density property of the set of rational numbers to explain why there is no rational number that immediately follows a given rational number.

As you know, you tabulate an infinite set of natural numbers by naming the least member of the set and then naming the next few greater members. To tabulate N, for example, you write $\{0, 1, 2, \ldots\}$.

G Now think about an infinite set of rational numbers. Is $\{x|x < \frac{1}{3}\}$ an infinite set? Is 0 the least member of this set? $U = R_a$.

H Can you name the next two greater members of $\{x|x < \frac{1}{3}\}$? Can you tabulate this set of rational numbers in the same way that you tabulate an infinite set of natural numbers?

I Name the greatest member of $\{x|x < 21\}$. $U = N$. Tabulate $\{x|x < 21\}$.

J Can you name the greatest member of $\{x|x < \frac{1}{3}\}$? $U = R_a$. Explain your answer.

Since you cannot tabulate a set like $\{x|x < \frac{1}{3}\}$ in the way that you learned to tabulate sets of natural numbers, you should use a standard description instead of a tabulation.

In this lesson you studied the density property of the set of rational numbers. You also learned that there is an infinite set of rational numbers between any two given rational numbers and that there is no rational number that immediately follows a given rational number.

On your own

1 Which members of $\{2\frac{1}{2}, 5, \frac{7}{4}, \frac{23}{52}, 3, \frac{9}{2}\}$ are between 0 and 3?

2 Which members of $\{5, 3\frac{4}{5}, \frac{8}{12}, \frac{7}{9}, \frac{20}{3}\}$ are between $\frac{2}{3}$ and 7?

3 Which members of $\{\frac{7}{4}, 2, \frac{7}{2}, \frac{20}{8}, 1\frac{2}{3}\}$ are between $\frac{3}{4}$ and $3\frac{1}{2}$?

4 Which members of $\{\frac{7}{48}, 1, \frac{0}{8}, \frac{10}{24}, \frac{16}{10}\}$ are between $\frac{1}{8}$ and $\frac{15}{12}$?

5 Which members of $\{\frac{2}{2}, \frac{5}{8}, \frac{3}{1}, \frac{12}{16}, \frac{7}{8}\}$ satisfy $\frac{3}{4} < x$? $U = R_a$.

6 Which members of $\{\frac{2}{5}, 1, \frac{1}{2}, \frac{12}{21}, 500\}$ are also members of $\{d|d < \frac{3}{5}\}$? $U = R_a$.

7 Which members of $\{6, 6\frac{1}{2}, 7, 7\frac{1}{2}, 8\}$ are also members of $\{y|6 < y \wedge y < 7\frac{3}{4}\}$? $U = R_a$.

8 Which members of $\{1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4, 4\frac{1}{2}\}$ satisfy $5 > z \wedge z > 3\frac{1}{3}$? $U = R_a$.

9 Which members of $\{\frac{3}{7}, \frac{8}{14}, \frac{7}{5}, \frac{9}{16}, \frac{5}{9}\}$ satisfy $\frac{1}{2} < m \wedge m < \frac{5}{7}$? $U = R_a$.

For each of exercises 10 through 13, list in order the rational numbers named. Begin with the name of the least number.

10 $0, \frac{1}{8}, \frac{1}{15}, \frac{1}{9}, \frac{1}{20}$

11 $14\frac{1}{2}, \frac{67}{10}, \frac{0}{8}, \frac{12}{12}, 16$

12 $\frac{4}{5}, \frac{1}{3}, \frac{7}{8}, \frac{1}{6}, \frac{19}{20}$

13 $6, \frac{17}{3}, 7, \frac{22}{3}, \frac{13}{2}$

For each of exercises 14 through 21, decide whether the set described is finite or infinite.

14 $\{x|2000 < x \wedge x < 3{,}759{,}000\}$ $U = N$.

15 $\{x|125\frac{1}{2} < x \wedge x < 126\}$ $U = R_a$.

16 $\{x|17 > x \wedge x > 16\}$ $U = C$.

17 $\{x|\frac{3}{2} < x \wedge x < \frac{6}{4}\}$ U = R_a.
18 $\{x|x < 5\}$ U = N.
19 $\{x|\frac{1}{4} > x \wedge x > \frac{1}{8}\}$ U = R_a.
20 $\{x|1 < x \wedge x < 100\}$ U = C.
21 $\{x|0 < x \wedge x < 3\}$ U = R_a.

For each set described in exercises 22 through 30, first tell if there is a least member. If there is, name this member. Then tell if there is a greatest member. If there is, name this member. Remember that each member of the solution set must also be a member of the universe.

22 $\{x|x < 3\}$ U = N.
23 $\{x|x < 3\}$ U = R_a.
24 $\{x|x > 15\}$ U = C.
25 $\{x|x < \frac{7}{8}\}$ U = R_a.
26 $\{x|6 < x \wedge x < 7\frac{1}{2}\}$ U = R_a.
27 $\{x|\frac{1}{8} < x \wedge x < 5\}$ U = R_a.
28 $\{x|2 < x \wedge x < 5\}$ U = N.
29 $\{x|5 < x \wedge x < 7\}$ U = R_a.
30 $\{x|3 < x \wedge x < 9\}$ U = C.

KEEPING SKILFUL

For each exercise below, find a mixed numeral that expresses the rational number named.

1 $\frac{29}{3}$ 2 $\frac{148}{11}$ 3 $\frac{76}{9}$ 4 $\frac{103}{3}$ 5 $\frac{129}{4}$ 6 $\frac{89}{13}$ 7 $\frac{212}{8}$ 8 $\frac{96}{7}$

For each exercise below, use a decimal to express the rational number named.

9 $\frac{5}{3}$ 10 $\frac{25}{8}$ 11 $\frac{1}{6}$ 12 $\frac{11}{5}$ 13 $\frac{7}{9}$ 14 $\frac{13}{25}$ 15 $\frac{47}{4}$ 16 $\frac{1}{11}$ 17 $\frac{5}{8}$ 18 $\frac{12}{15}$ 19 $\frac{3}{7}$ 20 $\frac{36}{144}$

For each exercise below, use a decimal to express the sum, difference, or product named in the exercise.

21 .1672 − .0938
22 .067 + .9 + .008
23 8.43 + 27 + .59
24 13 × .87
25 4.9 × .65
26 1 − .213
27 6 + .6 + 6.67
28 .71 × .42
29 4.21 − .3658
30 .75 + .88 + .32
31 .83 × .692
32 .5 − .1972
33 .003 + .8 + 1.56
34 .9(3 × .382)

Solution sets of conditions in one variable

You know how to find the solution sets of such conditions as $x + 2 = 9$ and $x - 7 > 11$ when the universe for the variable is N or C. In this lesson you will learn how to find the solution sets of such conditions as $x + \frac{5}{8} = 7$ and $x - \frac{2}{5} > \frac{2}{3}$. The universe for the variable will be R_a.

A When you find the solution set of a condition that involves addition or subtraction, you can use the relationship between addition and subtraction. Sentences A and B in D1 show how subtraction is related to addition. Does each sentence express a true statement?

B If you know that the difference of $\frac{3}{4}$ and $\frac{1}{2}$ is $\frac{1}{4}$, do you also know that $\frac{3}{4}$ is the sum of $\frac{1}{2}$ and $\frac{1}{4}$?

C What is the universe for x in the conditions expressed by sentences C and D?

D If you know that the difference of x and $\frac{1}{2}$ is $\frac{1}{4}$, do you also know that x is the sum of $\frac{1}{2}$ and $\frac{1}{4}$?

E Tabulate the solution set of $x - \frac{1}{2} = \frac{1}{4}$. What computation did you use to obtain the member of the solution set?

U = R_a.

A $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$.
B $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$.
C $x - \frac{1}{2} = \frac{1}{4}$.
D $x = \frac{1}{2} + \frac{1}{4}$.

D1

F The sentences in D2 show how addition is related to subtraction. Study sentences E and F. Does each sentence express a true statement?

G If you know that the sum of $\frac{2}{3}$ and $\frac{1}{6}$ is $\frac{5}{6}$, do you also know that $\frac{1}{6}$ is the difference of $\frac{5}{6}$ and $\frac{2}{3}$?

H What is the universe for x in the conditions expressed by sentences G and H?

I If you know that the sum of $\frac{2}{3}$ and x is $\frac{5}{6}$, do you also know that x is the difference of $\frac{5}{6}$ and $\frac{2}{3}$?

J Tabulate the solution set of $\frac{2}{3} + x = \frac{5}{6}$. What ? computation did you use to obtain the member of the solution set?

K Study sentences I and J in D3. Does each sentence express a true statement?

L You know that the sum of .31 and .18 is .49. How do you know, without computing, that .31 is the difference of .49 and .18?

M What is the universe for x in the conditions expressed by sentences K and L in D3?

N You know that the sum of x and .18 is .49. How do you know that x is the difference of .49 and .18?

O Tabulate the solution set of $x + .18 = .49$. ? What computation did you use to obtain the member of the solution set?

P Now study sentences M, N, and O in D4. Does each sentence express a true statement?

Q You know that the difference of $\frac{3}{5}$ and $\frac{1}{2}$ is $\frac{1}{10}$. How do you know, without computing, that $\frac{3}{5}$ is the sum of $\frac{1}{2}$ and $\frac{1}{10}$?

R You know that the sum of $\frac{1}{2}$ and $\frac{1}{10}$ is $\frac{3}{5}$. How do you know, without computing, that $\frac{1}{2}$ is the difference of $\frac{3}{5}$ and $\frac{1}{10}$?

S What is the universe for x in the conditions expressed by sentences P, Q, and R in D4?

T You know that the difference of $\frac{3}{5}$ and x is $\frac{1}{10}$. How do you know that $\frac{3}{5}$ is the sum of x and $\frac{1}{10}$?

$U = R_a$.

E $\frac{2}{3} + \frac{1}{6} = \frac{5}{6}$.

F $\frac{1}{6} = \frac{5}{6} - \frac{2}{3}$.

G $\frac{2}{3} + x = \frac{5}{6}$.

H $x = \frac{5}{6} - \frac{2}{3}$.

D2

$U = R_a$.

I $.31 + .18 = .49$.

J $.31 = .49 - .18$.

K $x + .18 = .49$.

L $x = .49 - .18$.

D3

$U = R_a$.

M $\frac{3}{5} - \frac{1}{2} = \frac{1}{10}$.

N $\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$.

O $\frac{1}{2} = \frac{3}{5} - \frac{1}{10}$.

P $\frac{3}{5} - x = \frac{1}{10}$.

Q $\frac{3}{5} = x + \frac{1}{10}$.

R $x = \frac{3}{5} - \frac{1}{10}$.

D4

U You know that the sum of x and $\frac{1}{10}$ is $\frac{3}{5}$. How do you know that x is the difference of $\frac{3}{5}$ and $\frac{1}{10}$?

V Tabulate the solution set of $\frac{3}{5} - x = \frac{1}{10}$. What ? computation did you use to obtain the member of the solution set?

Tabulate the solution set of each condition expressed below. Then tell what computation you used to find the member of the solution set. $U = R_a$.

W $\frac{3}{5} + x = \frac{7}{8}$. X $y + 1\frac{3}{8} = 4$. Y $5 - n = 2.6$.

In your work with natural numbers, you used conditions for equality to help you find the solution sets of conditions for inequality. When you are working with rational numbers and conditions for inequality, it is also helpful to use conditions for equality.

A Look at sentence S in D5. How do you know that $\frac{1}{4}$ satisfies $x + \frac{3}{8} = \frac{5}{8}$?

B You can use $\frac{1}{4}$ to help you find the solution set of the condition expressed by sentence T in D5. If you replace x by $\frac{1}{4}$ in $x+\frac{3}{8}>\frac{5}{8}$, do you obtain a true statement? Is $\frac{1}{4}$ a member of the solution set of $x+\frac{3}{8}>\frac{5}{8}$?

C Now you can decide if any rational number less than $\frac{1}{4}$ is a member of the solution set. Is $\frac{1}{8}<\frac{1}{4}$? Is $\frac{1}{16}<\frac{1}{4}$? Is $\frac{1}{8}$ a solution of $x+\frac{3}{8}>\frac{5}{8}$? Is $\frac{1}{16}$ a solution?

D Is any rational number less than $\frac{1}{4}$ a solution of $x+\frac{3}{8}>\frac{5}{8}$?

E Next you can decide if any rational number greater than $\frac{1}{4}$ is a solution of $x+\frac{3}{8}>\frac{5}{8}$. Is $\frac{5}{16}$ a solution? Is 2 a solution? Is $5\frac{1}{2}$ a solution?

F Is every rational number that is greater than $\frac{1}{4}$ a solution of $x+\frac{3}{8}>\frac{5}{8}$?

G You can write a standard description of the solution set of $x+\frac{3}{8}>\frac{5}{8}$ in the way shown at the right. Is $\{x|x>\frac{1}{4}\}=\{x|x+\frac{3}{8}>\frac{5}{8}\}$? Explain your answer.

$\{x|x>\frac{1}{4}\}$

The symbols $\{x|x+\frac{3}{8}>\frac{5}{8}\}$ and $\{x|x>\frac{1}{4}\}$ are both standard descriptions of the solution set of $x+\frac{3}{8}>\frac{5}{8}$. However, the symbol $\{x|x>\frac{1}{4}\}$ tells simply and clearly what numbers satisfy $x+\frac{3}{8}>\frac{5}{8}$. So from now on, when you are asked to give a standard description of a solution set, give the simpler description.

H Is $\{x|x>\frac{1}{4}\}$ an infinite set?

I Can you tabulate $\{x|x>\frac{1}{4}\}$ in the way that you have been tabulating infinite sets?

J Look again at D5. Think about the condition expressed by sentence U. You can also use $\frac{1}{4}$ to help you find the solution set of $x+\frac{3}{8}<\frac{5}{8}$. Is $\frac{1}{4}$ a member of the solution set of $x+\frac{3}{8}<\frac{5}{8}$?

K Is $\frac{1}{2}$ a solution of $x+\frac{3}{8}<\frac{5}{8}$? Is $\frac{9}{10}$ a solution? Is $5\frac{1}{3}$ a solution? Is any rational number that is greater than $\frac{1}{4}$ a solution of $x+\frac{3}{8}<\frac{5}{8}$?

L Is $\frac{3}{16}$ a solution of $x+\frac{3}{8}<\frac{5}{8}$? Is $\frac{2}{9}$ a solution? Is $\frac{1}{1000}$ a solution? Is every rational number that is less than $\frac{1}{4}$ a solution?

$U = R_a$.

S $x+\frac{3}{8}=\frac{5}{8}$.

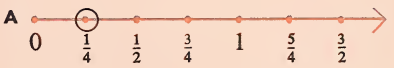
T $x+\frac{3}{8}>\frac{5}{8}$.

U $x+\frac{3}{8}<\frac{5}{8}$.

D5

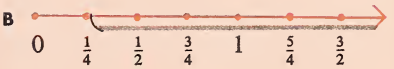
$U = R_a$.

A



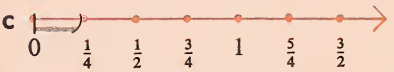
$\{x|x=\frac{1}{4}\}$

B



$\{x|x>\frac{1}{4}\}$

C



$\{x|x<\frac{1}{4}\}$

D6

M Is $\{x|x<\frac{1}{4}\}$ the solution set of $x+\frac{3}{8}<\frac{5}{8}$?

N Is $\{x|x<\frac{1}{4}\}$ an infinite set?

O Why can you not tabulate $\{x|x<\frac{1}{4}\}$?

Now you will learn how to use pictures of rational-number lines to make graphs of solution sets of conditions that involve rational numbers.

A D6 shows three pictures of a rational-number line. The sets described in D6 are the solution sets of the conditions expressed in D5. What is the universe for x?

B Look at graph A in D6. What point is represented by the encircled dot? Is the encircled dot the graph of $\{x|x=\frac{1}{4}\}$? Is the graph of $\{x|x=\frac{1}{4}\}$ a subset of the graph of R_a?

C Look at graph B. Is each point that is associated with a rational number greater than $\frac{1}{4}$ located to the right of point $\frac{1}{4}$? The bar below some of the dots in the graph of R_a indicates the dots in the graph of $\{x|x>\frac{1}{4}\}$. The curved stroke indicates that this graph does not contain the dot for point $\frac{1}{4}$. The bar extends to the arrowhead to indicate that the graph of $\{x|x>\frac{1}{4}\}$ contains the dots for all the points to the right of point $\frac{1}{4}$.

D Do the dots indicated by the bar form the graph of $\{x|x>\frac{1}{4}\}$?

E In graph C the vertical stroke through the dot for point 0 indicates that the graph of $\{x|x<\frac{1}{4}\}$ contains the dot for point 0. Does this graph contain the dot for point $\frac{1}{4}$?

F Does the graph of $\{x|x<\frac{1}{4}\}$ contain all the dots that represent points associated with members of R_a that are less than $\frac{1}{4}$?

G Look at D7. What is the universe for x? Is $\{x|x<4\}$ the solution set of $x+2<6$?

H Tabulate the set of numbers associated with the points whose dots are encircled in D7. Is the set a finite set? Do the encircled dots form the graph of $\{x|x<4\}$? Is the graph of $\{x|x<4\}$ a subset of the graph of N?

I D8 also shows a graph of $\{x|x<4\}$. Is it also a graph of $\{x|x+2<6\}$? What is the universe for the variable in $x+2<6$?

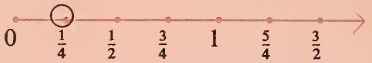
J In D8, how are the dots in the graph of $\{x|x<4\}$ indicated?

K Is $\{x|x<4\}$ a finite set when $U=R_a$? Can you tabulate $\{x|x<4\}$ when $U=R_a$?

Now you will find solution sets of compound conditions in one variable when the universe is a set of rational numbers.

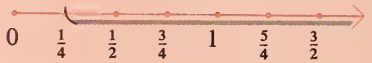
$U = R_a$.

A



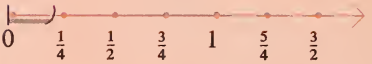
$\{x|x=\frac{1}{4}\}$

B



$\{x|x>\frac{1}{4}\}$

C

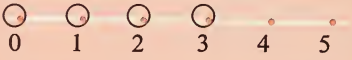


$\{x|x<\frac{1}{4}\}$

D6

$U = N$.
$x+2<6$.

D



$\{x|x<4\}$

D7

$U = R_a$.
$x+2<6$.

E

$\{x|x<4\}$

D8

A What compound condition is expressed in D9? What is the universe for x?

B Is $\{x|x>\frac{1}{3}\}$ the solution set of $\frac{2}{3}+x>1$?

C What is the solution set of $\frac{1}{2}+x<2$?

D Is $\{x|x>\frac{1}{3}\wedge x<\frac{3}{2}\}$ the solution set of the compound condition expressed in D9?

E Is $\{x|x>\frac{1}{3}\}\cap\{x|x<\frac{3}{2}\}$ the same set as $\{x|x>\frac{1}{3}\wedge x<\frac{3}{2}\}$? Explain your answer.

> REMINDER
> The intersection of two sets is the set that contains those objects, and only those objects, that belong to both sets.
> *See lesson 22, page 83.*

F Can you tabulate $\{x|x>\frac{1}{3}\wedge x<\frac{3}{2}\}$? Explain your answer.

Next you will learn to make graphs of solution sets of compound conditions. D10 shows graphs of the solution set of each simple condition and the solution set of the compound condition expressed in D9.

G Look at graph F in D10. Do the dots indicated by the bar form the graph of $\{x|x>\frac{1}{3}\}$?

H Look at graph G. How are the dots in the graph of $\{x|x<\frac{3}{2}\}$ indicated?

I Look at graph H. The dots for the points between point $\frac{1}{3}$ and point $\frac{3}{2}$ form the graph of $\{x|x>\frac{1}{3}\wedge x<\frac{3}{2}\}$. How are these dots indicated? Is $\{x|x>\frac{1}{3}\wedge x<\frac{3}{2}\}$ an infinite set?

J Look at D11. What is the universe for x?

K Is $\{x|x>2\}$ the solution set of $2+x>4$? What is the solution set of $x-1>3$?

L Is $\{x|x>2\}\cap\{x|x>4\}$ the solution set of $2+x>4\wedge x-1>3$?

M Is $\{x|x>4\}$ the solution set of $2+x>4\wedge x-1>3$? Explain your answer.

N Look again at D11. Which dots form the graph of the solution set of the compound condition? How is this graph indicated?

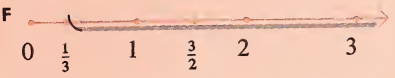
$U=R_a$.

$\frac{2}{3}+x>1\wedge\frac{1}{2}+x<2$.

D9

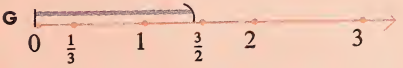
$U=R_a$.

F



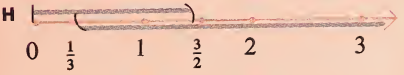
$\{x|x>\frac{1}{3}\}$

G



$\{x|x<\frac{3}{2}\}$

H



$\{x|x>\frac{1}{3}\wedge x<\frac{3}{2}\}$

D10

$U=R_a$.

$2+x>4\wedge x-1>3$.

I

$\{x|x>2\wedge x>4\}$

D11

O Is the graph of $\{x|x>4\}$ the same as the graph of $\{x|2+x>4\wedge x-1>3\}$? Explain your answer.

You have often used N and C as universes for variables. As you know, the only difference

$U = Z.$

$3 - x > 2 \wedge .5 + x > 1.5.$

$\{x | x < 1 \wedge x > 1\}$

D12

between N and C is that zero is not contained in C. Now you will use as a universe a set of rational numbers that does not contain zero.

In lesson 85 you learned about the non-zero rational numbers of arithmetic. The set of non-zero rational numbers of arithmetic is the set of all the rational numbers of arithmetic except zero. We will use the letter Z as a name of this set.

P Look at D12. What compound condition is expressed in the display? What is the universe for x?

Q Is $\{x | x < 1\}$ the solution set of $3 - x > 2$? Is 0 a member of $\{x | x < 1\}$?

R What is the solution set of $.5 + x > 1.5$? Are there any members of $\{x | x > 1\}$ that are also members of $\{x | x < 1\}$?

S Is $\{x | x < 1 \wedge x > 1\}$ the solution set of the compound condition expressed in D12?

T Look again at D12. Is any dot that is in the graph of the solution set of $3 - x > 2$ also in the graph of the solution set of $.5 + x > 1.5$?

U Tabulate $\{x | 3 - x > 2 \wedge .5 + x > 1.5\}$.

Now you know how to find solution sets of simple and compound conditions in one variable when the universe is R_a or Z. You also know how to use a picture of a rational-number line to make a graph of a set of rational numbers.

On your own

For each of exercises 1 through 13, tell whether the sentence expresses a true statement or a false statement. $U = R_a$.

1 $\{x | \frac{2}{3} + x = \frac{9}{5}\} = \{\frac{37}{15}\}$.

2 $\{x | x + 3 > 5\} = \{x | x > 2\}$.

3 $\{x | .5 - x > .25\} = \{x | x < .25\}$.

4 $\{x | x - 2 > \frac{2}{3}\} = \{x | x > 2\frac{2}{3}\}$.

5 $\{x | 3 + x < 7\} = \{0, 1, 2, 3\}$.

6 $\{x | x - \frac{1}{2} > \frac{1}{3} \wedge x + \frac{1}{2} < \frac{4}{3}\} = \{\frac{5}{6}\}$.

7 $\{x | 1 + x = 4 \wedge x + \frac{1}{4} > 4\frac{1}{3}\} = \{\ \}$.

8 $\{x | x > \frac{1}{3} \wedge x < \frac{7}{3}\} = \{\frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3}\}$.

9 $\{x | x + \frac{1}{4} > \frac{7}{4} \wedge x - 2 > 5\} = \{x | x > 7\}$.

10 $\{x | x + .2 > 3 \wedge x - .8 > 2.2\} = \{x | x > 3\}$.

11 $\{x | 5 - x > 4\frac{1}{5} \wedge x + 3 > \frac{17}{5}\} = \{\frac{3}{5}\}$.

12 $\{x | x + \frac{4}{9} > 1 \wedge x + 2 < \frac{7}{2}\} = \{x | x > \frac{3}{2}\}$.

13 $\{x | \frac{1}{3} - x > \frac{2}{15} \wedge \frac{1}{15} < x + \frac{1}{30}\} = \{x | x < \frac{1}{15} \wedge x > \frac{1}{30}\}$.

For each of exercises 14 through 23, first make a picture of a rational-number line. Then use the picture to make a graph of the solution set described in the exercise. For exercises 14, 20, and 21, $U = Z$. For each of the other exercises, $U = R_a$.

14 $\{x | x > 2\}$

15 $\{x | x < 2.3\}$

16 $\{x | x = \frac{4}{5}\}$

17 $\{x | x < \frac{7}{2}\}$

18 $\{x | x > .4\}$

19 $\{x | x > \frac{1}{2} \wedge x < 1\}$

20 $\{x | x > \frac{1}{6} \wedge x < \frac{5}{4}\}$

21 $\{x | x > \frac{13}{5} \wedge x < 3\frac{1}{2}\}$

22 $\{x | x > 5 \wedge x < 3\}$

23 $\{x | x > .6 \wedge x < 1.2\}$

For each of the conditions expressed in exercises 24 through 35, tabulate the solution set, if possible; if not, give a standard description of the solution set. Then make a picture of a rational-number line and use it to make a graph of the solution set. $U = R_a$.

24 $x - \frac{2}{3} > \frac{5}{9}$.

25 $\frac{4}{5} \div x > \frac{1}{2}$.

26 $.3 + x < 1.8$.

27 $4.8 - x > 2.2$.

28 $x - 2\frac{1}{5} = 1\frac{7}{10}$.

29 $x + .7 > 3$.

30 $x + \frac{2}{7} < \frac{1}{2} \wedge 2\frac{1}{2} + x > 3$.

31 $x - \frac{1}{6} > \frac{2}{5} \wedge \frac{1}{2} + x > 1$.

32 $x + \frac{1}{5} > \frac{3}{2} \wedge 2 - x > 0$.

33 $\frac{5}{3} - x > 1 \wedge x + \frac{6}{3} > 2$.

34 $\frac{7}{8} - x > \frac{1}{8} \wedge x - \frac{1}{4} > \frac{1}{4}$.

35 $x - .2 > 1 \wedge 3 + x < 6$.

Conditions in one variable for problems

In this lesson you will solve problems that involve simple and compound conditions. The problems are much like problems you have solved earlier in this book, but they involve rational numbers instead of natural numbers.

Read the problem in D1. $U = R_a$.

A From the problem, you know that Mrs. Jones had $2\frac{1}{4}$ yd. of oilcloth after she had bought $\frac{1}{2}$ yd. Does the sentence at the right express the condition for the problem? For what is y a variable?

$y + \frac{1}{2} = 2\frac{1}{4}$.

B Why is $y + \frac{1}{2}$, and not $y - \frac{1}{2}$, used in the condition?

C Why is the idea of equality used in the condition?

D What computation should you use to find the number that satisfies $y + \frac{1}{2} = 2\frac{1}{4}$? What is the number?

E Tabulate $\{y \mid y + \frac{1}{2} = 2\frac{1}{4}\}$.

Now that you know what number satisfies $y + \frac{1}{2} = 2\frac{1}{4}$, you can use the number to get the answer to the problem.

Mrs. Jones had $1\frac{3}{4}$ yd. of oilcloth before she bought the $\frac{1}{2}$ yd.

Read the problem in D2. Also read the sentence that expresses the condition for the problem. $U = R_a$.

F Why is $13.50 - n$, and not $13.50 + n$, used in the condition? Why is the idea of equality used in the condition? For what is n a variable?

After Mrs. Jones had bought $\frac{1}{2}$ yd. of oilcloth, she had $2\frac{1}{4}$ yd. of oilcloth in all. How much oilcloth did she have before she bought the $\frac{1}{2}$ yd.?

D1

Jerry had \$13.50. He bought a present for his father and then had \$10.91 left. How much money did he spend for the present?

$$13.50 - n = 10.91.$$

D2

Sue had $1\frac{1}{4}$ yd. of lace. After she had bought some more, she still had less than $2\frac{3}{4}$ yd. of lace. How much lace had she bought?

D3

G What computation should you use to find the number that satisfies $13.50 - n = 10.91$?

H Tabulate $\{n \mid 13.50 - n = 10.91\}$ and give the answer to the problem.

Read the problem in D3. $U = Z$. Remember that Z is the set of all the rational numbers of arithmetic except 0.

I Explain why $1\frac{1}{4} + x < 2\frac{3}{4}$ is the condition for the problem.

J Why is $1\frac{1}{4} + x$, and not $1\frac{1}{4} - x$, used in the condition?

K Why is the idea of "less than" used in the condition?

You can use a condition for equality to help you find the solutions of $1\frac{1}{4} + x < 2\frac{3}{4}$.

L Imagine that Sue had $2\frac{3}{4}$ yd. of lace after she had bought some more. What number satisfies $1\frac{1}{4} + x = 2\frac{3}{4}$?

M Is $1\frac{1}{2}$ a member of $\{x \mid 1\frac{1}{4} + x < 2\frac{3}{4}\}$?

N Each member of $\{x \mid 1\frac{1}{4} + x < 2\frac{3}{4}\}$ must be less than what rational number?

O You can use the symbol $\{x \mid x < 1\frac{1}{2}\}$ to describe the solution set of $1\frac{1}{4} + x < 2\frac{3}{4}$. How do you know that $\{x \mid x < 1\frac{1}{2}\}$ is the same set as $\{x \mid 1\frac{1}{4} + x < 2\frac{3}{4}\}$?

P Can you tabulate $\{x \mid x < 1\frac{1}{2}\}$? Explain your answer.

Q Is 0 a member of $\{x \mid x < 1\frac{1}{2}\}$? Explain your answer.

You can use $\{x \mid x < 1\frac{1}{2}\}$ to get the answer to the problem.

Sue had bought less than $1\frac{1}{2}$ yd. of lace.

Sue had $1\frac{1}{4}$ yd. of lace. After she had bought some more, she still had less than $2\frac{3}{4}$ yd. of lace. How much lace had she bought?

D3

Ken had more than \$9.10. He then earned \$2.75, but he still had less than \$14.00. How much money did Ken have before he earned the \$2.75?

D4

So far in this lesson you have solved problems involving simple conditions in one variable. In the work that follows, you will learn to solve problems involving compound conditions in one variable when $U = R_a$.

Read the problem in D4. $U = R_a$.

A Explain why the sentence below at the right expresses one of the simple conditions for the problem.

$s > 9.10.$

B How do you know that the sentence below at the right expresses another simple condition for the problem?

$s + 2.75 < 14.00.$

C Does the sentence in red below express the compound condition for the problem in D4?

$$s > 9.10 \wedge s + 2.75 < 14.00.$$

Now you can find the solution set of the compound condition. Remember that each member of the solution set must satisfy both simple conditions.

D What condition for equality can you use to help you find the solution set of $s + 2.75 < 14.00$?

E How do you know that $\{s \mid s < 11.25\}$ is the solution set of $s + 2.75 < 14.00$?

F Does each member of $\{s \mid s < 11.25\}$ satisfy the compound condition for the problem? Explain your answer.

G Is the intersection of $\{s \mid s > 9.10\}$ and $\{s \mid s < 11.25\}$ the same set as the solution set of $s > 9.10 \wedge s + 2.75 < 14.00$?

H Which description below is the more useful description of the solution set of the compound condition? Explain your answer.

$\{s \mid s > 9.10 \wedge s + 2.75 < 14.00\}$

$\{s \mid s > 9.10 \wedge s < 11.25\}$

You can use $\{s \mid s > 9.10 \wedge s < 11.25\}$ to get the answer to the problem. You can give the answer in either of the following ways:

Ken had more than \$9.10 and less than \$11.25.

Ken had between \$9.10 and \$11.25.

I You know that any member of R_a between 9.10 and 11.25 is a solution of the compound condition for the problem in D4. How do you know that 10.256 is a member of the solution set? That 11.0012 is a member? However, you should not use a solution like 10.256 or 11.0012 to get a sensible answer.

J Is 9.11 the least number that you can use to get a sensible answer to the problem? What is the greatest number?

K The set tabulated in D5 contains the solutions of $s > 9.10 \wedge s < 11.25$ that give sensible answers to the problem in D4. Is 9.76 a member of this set? Is 10.85 a member?

L Explain why 10.013 and 11.115 are not ? members of the set tabulated in D5.

Read the problem in D6. $U = R_a$.

M Is $3\frac{1}{4} - m < 1$ one of the simple conditions for the problem? For what is m a variable?

N Did Ralph use all of the paint? Why is $m < 3\frac{1}{4}$ another simple condition for the problem?

O What is the compound condition for the problem in D6? Give a standard description of the solution set of $3\frac{1}{4} - m < 1$.

P Is $\{m | m > 2\frac{1}{4}\}$ the solution set of the compound condition? Explain your answer.

Q How do you know that each number that satisfies the compound condition must be less than $3\frac{1}{4}$?

R Give a standard description of the solution ? set of $3\frac{1}{4} - m < 1 \wedge m < 3\frac{1}{4}$. Then give the answer to the problem.

For the preceding problems you used the solutions of conditions to get answers to problems about persons and things. Now you will use solutions of conditions as answers to problems that concern numbers only. $U = R_a$.

A Read the problem in D7. Use x as a variable for the number you are to find. Is $x > 1\frac{1}{2}$ one of the simple conditions for the problem? Is $2\frac{1}{3} + x < 5\frac{2}{3}$ another simple condition?

B What is the compound condition for the problem in D7?

C Is $\{x | x > 1\frac{1}{2} \wedge x < 3\frac{1}{3}\}$ the solution set of ? $x > 1\frac{1}{2} \wedge 2\frac{1}{3} + x < 5\frac{2}{3}$? How do you know?

D Is each member of $\{x | x > 1\frac{1}{2} \wedge x < 3\frac{1}{3}\}$ an answer to the problem?

E Read the problem in D8. Use s as a variable for the number you are to find. Does the problem involve a compound condition? What is the condition for the problem?

F Give a standard description of the solution set of $\frac{6}{5} + s > \frac{23}{10}$. Is each member of the solution set an answer to the problem?

G Read the problem in D9. Use n as a variable for the number you are to find. What is the condition for the problem?

H Give a standard description of the solution set of $n < .8 \wedge 1.5 + n > 2.3$.

I Is there any rational number that is both less than .8 and greater than .8?

J Tabulate the solution set of $n < .8 \wedge 1.5 + n > 2.3$.

$\{9.11, 9.12, 9.13, \ldots, 11.24\}$

D5

Ralph had $3\frac{1}{4}$ gal. of paint. After he had used some, but not all, of this paint, he had less than 1 gal. left. How much paint had he used?

D6

What numbers greater than $1\frac{1}{2}$ can be added to $2\frac{1}{3}$ so that each sum is less than $5\frac{2}{3}$?

D7

What numbers can be added to $\frac{6}{5}$ so that each sum is greater than $\frac{23}{10}$?

D8

What numbers less than .8 can you add to 1.5 so that each sum is greater than 2.3?

D9

In this lesson you have used simple and compound conditions in one variable to solve problems involving rational numbers.

On your own

For each of problems 1 through 16, write a sentence that expresses a condition. If the condition is a condition for equality, tabulate the solution set. If the condition is a condition for inequality, give a standard description of the solution set. Then give the answer to the problem. The universe is given for each problem.

1 Jill bought 2 cans of salted nuts. One can contained $\frac{7}{8}$ lb. of nuts. The other can contained $\frac{5}{8}$ lb. How many pounds of salted nuts did Jill buy? $U = R_a$.

2 Tom lost $9\frac{1}{2}$ lb. during the baseball season. At the end of the season he weighed $145\frac{3}{4}$ lb. How much did Tom weigh before he lost the $9\frac{1}{2}$ lb.? $U = R_a$.

3 Mrs. Harris bought some mint ice cream and also $\frac{1}{4}$ gal. of vanilla ice cream. She bought less than $1\frac{3}{8}$ gal. of ice cream in all. How much mint ice cream did Mrs. Harris buy? $U = Z$.

4 Henry had a board that was $5\frac{1}{2}$ ft. long. He used $2\frac{3}{4}$ ft. of this board to build a shelf. How much of the board did he have left after he had built the shelf? $U = R_a$.

5 Before Mr. Larson drove from his home to Charlottetown, the odometer on his car read 13,648.8 mi. When he reached Charlottetown, the odometer read 13,824.5 mi. What is the distance from his home to Charlottetown? $U = R_a$.

6 Mrs. Allen used $\frac{2}{3}$ cup of carrots, $\frac{2}{3}$ cup of potatoes, and $\frac{1}{4}$ cup of onion for a meat and vegetable pie. How many cups of vegetables did she use for the pie? $U = R_a$.

7 On Monday the school lunch supervisor ordered $48\frac{1}{2}$ pt. of milk. She ordered less than $97\frac{1}{2}$ pt. of milk in all on Monday and Tuesday. How much milk did she order on Tuesday? $U = Z$.

8 Precipitation is moisture in the form of rain, snow, or dew. The normal yearly precipitation in Halifax is 54.26 in. In Calgary it is 17.47 in. Halifax receives how much more precipitation than Calgary? $U = R_a$.

9 Kurt worked in his father's store $1\frac{1}{2}$ hr. on Tuesday, $1\frac{3}{4}$ hr. on Thursday, and $4\frac{3}{4}$ hr. on Saturday. How many hours did Kurt work in the store on these three days? $U = R_a$.

10 Ruth had more than \$1.75. After she had spent \$1.59, she had less than \$.43 left. How much money did she have before she spent the \$1.59? $U = R_a$.

11 Last year Mrs. Murphy spent \$134.95 for a washing machine, \$199.50 for a clothes dryer, and \$239.75 for a stove. How much in all did she spend for the three appliances? $U = R_a$.

12 A passenger ship, the "France," is $45\frac{1}{6}$ ft. longer than the passenger ship the "United States." The "France" is $1035\frac{1}{6}$ ft. long. What is the length of the "United States"? $U = R_a$.

13 The elevation of Lake Erie above sea level is 326.52 feet greater than the elevation of Lake Ontario. The elevation of Lake Ontario is 245.88 feet. What is the elevation of Lake Erie? $U = R_a$.

14 A plumber had a piece of copper pipe $18\frac{1}{3}$ ft. long. He used some, but not all, of this pipe and then had less than $6\frac{1}{4}$ ft. left. How much of the copper pipe did he use? $U = Z$.

15 The Drama Club members made more than $5\frac{1}{2}$ gal. of punch for a party. They served $4\frac{3}{4}$ gal. of punch and had less than $1\frac{1}{2}$ gal. left at the end of the party. How much punch had they made? $U = R_a$.

16 Mr. Roberts drove less than 759.3 mi. last week. If he had driven 49.9 mi. farther, he would have driven more than 800.0 mi. How far did Mr. Roberts drive last week? $U = R_a$.

For each of problems 17 through 22, first write a sentence that expresses a condition. Tabulate the solution set of each condition for equality. Give a standard description of the solution set of each condition for inequality. $U = R_a$.

17 What number is the quotient of $6\frac{3}{5}$ and $2\frac{1}{2}$?

18 What number can be added to 3.024 so that the sum is 5.112?

19 What numbers greater than $\frac{1}{6}$ can be added to $5\frac{1}{6}$ so that each sum is less than $5\frac{1}{3}$?

20 What numbers are less than the product of $1\frac{1}{3}$ and $2\frac{1}{7}$?

21 What numbers greater than 7.136 can be added to 1.084 so that each sum is greater than 8.429?

22 What numbers less than $3\frac{1}{8}$ can be subtracted from $9\frac{1}{4}$ so that each difference is greater than 4?

CHECKING UP

If you have trouble with these tests, you can find help in lessons 89, 90, and 91.

Test 90

Tell whether or not each of the numbers named in exercises 1 through 5 is between $\frac{3}{7}$ and $\frac{8}{15}$.

1 $\frac{1}{2}$ **2** $\frac{3}{4}$ **3** $\frac{1}{6}$ **4** $\frac{4}{9}$ **5** $\frac{7}{11}$

Tell whether or not each of the numbers named in exercises 6 through 10 is a solution of $m < \frac{5}{6}$. $U = R_a$.

6 $\frac{11}{12}$ **7** $\frac{10}{13}$ **8** $\frac{3}{4}$ **9** $\frac{13}{15}$ **10** $\frac{0}{2}$

Tell whether or not each of the numbers named in exercises 11 through 15 is a member of $\{x | x - \frac{5}{6} > \frac{1}{3} \wedge 5 - x > 3\}$. $U = R_a$.

11 $\frac{7}{6}$ **12** $1\frac{5}{8}$ **13** 2 **14** $\frac{15}{12}$ **15** $\frac{15}{16}$

Tabulate the solution set of each condition expressed in exercises 16 through 19. $U = R_a$.

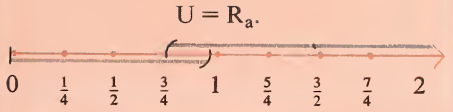
16 $7\frac{1}{2} = x - \frac{5}{11}$.

17 $5 - x = \frac{9}{5}$.

18 $11 = 4\frac{2}{9} + x$.

19 $x + \frac{4}{3} = 2\frac{1}{7}$.

$U = R_a$.



For each set described in exercises 20 through 25, tell whether or not a graph of the set is indicated in the display above.

20 $\{x | x - \frac{3}{8} > \frac{3}{8}\}$

21 $\{x | \frac{5}{4} - x > \frac{1}{2}\}$

22 $\{x | x + \frac{1}{4} < \frac{1}{2}\}$

23 $\{x | 2\frac{1}{2} + x < 3\frac{1}{2}\}$

24 $\{x | 2\frac{1}{4} - x < \frac{10}{8} \wedge \frac{5}{12} + \frac{1}{3} < x\}$

25 $\{x | x < 1 \wedge x + \frac{1}{4} > 1\}$

Test 91

For each of the following problems, write a sentence that expresses a condition. Tabulate the solution set of each condition for equality. Give a standard description of the solution set of each condition for inequality. Then give the answer to the problem. The universe is given for each problem.

26 On Wednesday, Russell spent $1\frac{2}{3}$ hr. on his homework. On Wednesday and Thursday, he spent $3\frac{1}{4}$ hr. in all on his homework. How many hours did he spend on his homework on Thursday? $U = R_a$.

27 Carol had \$.75 before she bought lunch in the school cafeteria. After she had paid for her lunch, she had more than \$.40 left. How much could Carol have paid for her lunch? $U = Z$.

28 Edmonton had more than .3 in. of rain on Sunday. After 1.1 in. of rain fell on Monday, the total rainfall for the two days was less than 2 in. How many inches of rain could Edmonton have had on Sunday? $U = R_a$.

29 What numbers can be subtracted from $7\frac{1}{5}$ so that each difference is greater than $3\frac{1}{7}$? $U = R_a$.

30 A given number is less than $9\frac{1}{2}$. The difference of the given number and $2\frac{3}{4}$ is greater than 5. What is the given number? $U = R_a$.

End-of-block tests on simple and compound conditions in one variable and on problem solving

Rational-number components of rate pairs

In unit 4 you learned to use rate pairs to represent rates and comparisons. The components of these rate pairs were natural numbers. In this lesson you will learn about rate pairs whose components are rational numbers.

A Read the problem in D1. You can use the rate pair expressed at the right to represent the comparison of the amount of peanuts with the amount of cashew nuts. Notice that each component of the rate pair is a rational number. To what does the first component refer? To what does the second component refer?

$1\frac{1}{4}/\frac{1}{2}$

B You know one rate pair that represents the comparison. You want to find a rate pair that is equivalent to $1\frac{1}{4}/\frac{1}{2}$. How do you know that $7\frac{1}{2}$ is the first component of the rate pair whose second component you are to find?

C Does the rate pair expressed below at the right also represent the comparison of the amount of peanuts with the amount of cashew nuts? For what is y a variable?

$7\frac{1}{2}/y$

D Does the sentence in D2 express a condition for equivalence for the problem in D1? What is the universe for y? Remember that Z is the set of non-zero rational numbers.

You have often used the definition of equivalent ordered pairs to find a natural number that satisfies a condition for equivalence. You know that, for each replacement of the variables, if $a/b \sim c/d$, then $ad = cb$; and if $ad = cb$, then $a/b \sim c/d$. The universe for the variables is C.

You can use the same definition to find a rational number that satisfies a condition for equivalence like the one expressed in D2. Now, however, the universe for the variables is Z.

E Look at D3. What do the arrows indicate? How was $1\frac{1}{4}y = \frac{15}{4}$ obtained from $1\frac{1}{4}/\frac{1}{2} \sim 7\frac{1}{2}/y$?

F Does the replacement for y that satisfies $1\frac{1}{4}y = \frac{15}{4}$ also satisfy $1\frac{1}{4}/\frac{1}{2} \sim 7\frac{1}{2}/y$?

G Now look at sentence A in D4. Is the condition expressed by sentence A the same condition as $1\frac{1}{4}y = \frac{15}{4}$? What is the universe for y?

In the work that follows, you will use some of the properties of rational numbers to find

D1

A mixture of nuts contains $1\frac{1}{4}$ lb. of peanuts for each $\frac{1}{2}$ lb. of cashew nuts. A certain amount of the mixture contains $7\frac{1}{2}$ lb. of peanuts. How many pounds of cashew nuts does this amount of the mixture contain?

D2

U = Z.

$1\frac{1}{4}/\frac{1}{2} \sim 7\frac{1}{2}/y.$

D3

U = Z.

$1\frac{1}{4} \times y$, or $1\frac{1}{4}y$

$1\frac{1}{4}/\frac{1}{2} \sim 7\frac{1}{2}/y.$

$7\frac{1}{2} \times \frac{1}{2}$, or $\frac{15}{4}$

$1\frac{1}{4}y = \frac{15}{4}.$

the solution of $\frac{5}{4}y = \frac{15}{4}$. In other words, you will find the rational number that can be multiplied by $\frac{5}{4}$ to obtain the product $\frac{15}{4}$.

H Look at sentence B in D4. You are going to obtain a condition that involves $1y$ instead of $\frac{5}{4}y$. By what number is $\frac{5}{4}y$ multiplied?

You can obtain the condition expressed by sentence B by using the well-defined property of multiplication of rational numbers. You have obtained $\frac{4}{5}(\frac{5}{4}y)$. You know, from sentence A, that $\frac{15}{4}$ is equal to $\frac{5}{4}y$. Therefore, in the product $\frac{4}{5}(\frac{5}{4}y)$, you can replace $(\frac{5}{4}y)$ by $\frac{15}{4}$ without affecting the product. The product $\frac{4}{5}(\frac{15}{4})$ is the same as the product $\frac{4}{5}(\frac{5}{4}y)$.

I Study sentence B again. For each replacement of y, is $\frac{4}{5}(\frac{5}{4}y) = (\frac{4}{5} \cdot \frac{5}{4})y$? Upon what property of rational numbers does your answer depend? Now we can write sentence C.

J Study sentence C again. For each replacement of y, is $(\frac{4}{5} \cdot \frac{5}{4})y = 1y$? Explain your answer. Now we can write sentence D.

K Compare sentences D and E in D4. For each replacement of y, is $1y = y$? Is $\frac{4}{5}(\frac{15}{4}) = 3$? Explain your answers.

L Does 3 satisfy $\frac{5}{4}y = \frac{15}{4}$? Does 3 also satisfy $1\frac{1}{4}/\frac{1}{2} \sim 7\frac{1}{2}/y$?

M Tabulate $\{y \mid 1\frac{1}{4}/\frac{1}{2} \sim 7\frac{1}{2}/y\}$.

Now you can give the answer to the problem in D1.

In an amount of the nut mixture that contains $7\frac{1}{2}$ lb. of peanuts, there are 3 lb. of cashew nuts.

N Now look at D5. Is the condition expressed by sentence F a condition for equivalence? What is the universe for x?

O Study sentences F and G. How can you obtain the condition $\frac{209}{4} = 6\frac{1}{2}x$ from the condition $3\frac{2}{3}/6\frac{1}{2} \sim x/14\frac{1}{4}$? Will the same number satisfy both conditions?

P Compare sentences G and H. Is $\frac{209}{4} = \frac{13}{2}x$ the same condition as $\frac{209}{4} = 6\frac{1}{2}x$?

U = Z.

A $\frac{5}{4}y = \frac{15}{4}$.

B $\frac{4}{5}(\frac{5}{4}y) = \frac{4}{5}(\frac{15}{4})$.

C $(\frac{4}{5} \cdot \frac{5}{4})y = \frac{4}{5}(\frac{15}{4})$.

D $1y = \frac{4}{5}(\frac{15}{4})$.

E $y = 3$.

D4

U = Z.

F $3\frac{2}{3}/6\frac{1}{2} \sim x/14\frac{1}{4}$.

G $\frac{209}{4} = 6\frac{1}{2}x$.

H $\frac{209}{4} = \frac{13}{2}x$.

D5

U = Z.

$.2/10 \sim x/75$.

D6

Now think about the condition $\frac{209}{4} = \frac{13}{2}x$. The exercises that follow show how to find a replacement for x that satisfies this condition.

Q What property do you use to obtain $\frac{2}{13}(\frac{209}{4}) = \frac{2}{13}(\frac{13}{2}x)$?

R Explain how to obtain $\frac{2}{13}(\frac{209}{4}) = (\frac{2}{13} \cdot \frac{13}{2})x$ from $\frac{2}{13}(\frac{209}{4}) = \frac{2}{13}(\frac{13}{2}x)$.

S Explain how to obtain $\frac{2}{13}(\frac{209}{4}) = 1x$ from $\frac{2}{13}(\frac{209}{4}) = (\frac{2}{13} \cdot \frac{13}{2})x$.

T Explain how to obtain $\frac{2}{13}(\frac{209}{4}) = x$ from $\frac{2}{13}(\frac{209}{4}) = 1x$. What is the solution of $\frac{2}{13}(\frac{209}{4}) = x$?

U Look again at D5. What is the solution of $\frac{209}{4} = \frac{13}{2}x$? What is the solution of $3\frac{2}{3}/6\frac{1}{2} \sim x/14\frac{1}{4}$?

V What condition is expressed in D6? What is the universe for x?

W Explain how you can obtain $15 = 10x$ from the condition expressed in D6.

X What number is the reciprocal of 10? Explain how you can use this number to find the solution of $15 = 10x$.

Y Tabulate $\{x \mid .2/10 \sim x/75\}$.

Z Think about the condition $\frac{a}{b}(x) = \frac{c}{d}$. In this condition, $\frac{a}{b}$ and $\frac{c}{d}$ are any two non-zero rational numbers. The universe for x is Z. Is $\frac{b}{a}$ the reciprocal of $\frac{a}{b}$? Explain how to find the solution of $\frac{a}{b}(x) = \frac{c}{d}$.

Now you know how to find solution sets of conditions for equivalence that involve rational numbers.

On your own

For each of exercises 1 through 6, use the definition of equivalent ordered pairs to decide whether or not the rate pair named is equivalent to $\frac{1}{5}/6$.

1 $\frac{3}{5}/18$
2 $\frac{1}{10}/3$
3 $1/60$
4 $12/\frac{2}{5}$
5 $\frac{7}{20}/10\frac{1}{2}$
6 $1/30$

For each of exercises 7 through 12, use the definition of equivalent ordered pairs to decide whether or not the rate pair named is equivalent to 3.5/5.25.

7 $14/21$
8 $7/3$
9 $7/10\frac{1}{2}$
10 $17.5/26.25$
11 $5.25/3.5$
12 $\frac{2}{3}/1$

Decide whether each sentence in exercises 13 through 20 expresses a true statement or a false statement.

13 $\frac{1}{3}/6 \sim \frac{5}{3}/18$.
14 $4\frac{1}{2}/1 \sim 2/\frac{11}{42}$.
15 $10/.8 \sim 50/4$.
16 $\frac{3}{4}/\frac{2}{3} \sim 3/2$.
17 $6/8 \sim .3/.8$
18 $1\frac{1}{2}/2 \sim 2\frac{1}{2}/1$.
19 $11/15\frac{1}{2} \sim 1/\frac{31}{32}$.
20 $7\frac{1}{3}/6 \sim 22/18$.

For each of exercises 21 through 40, tabulate the set described. U = Z.

21 $\{x|\frac{2}{3}/1\frac{1}{2} \sim x/5\}$
22 $\{y|3\frac{3}{4}/7\frac{1}{2} \sim 2/y\}$
23 $\{y|\frac{5}{8}/y \sim 15/25\}$
24 $\{x|6\frac{1}{2}/3\frac{2}{3} \sim x/\frac{1}{4}\}$
25 $\{x|x/.7 \sim 3.5/6\}$
26 $\{y|5\frac{3}{8}/5 \sim 2\frac{1}{4}/y\}$
27 $\{x|\frac{1}{3}/\frac{3}{4} \sim x/4\}$
28 $\{y|\frac{7}{2}/y \sim 7/24\}$
29 $\{y|7\frac{7}{8}/6\frac{1}{2} \sim 3/y\}$
30 $\{x|x/7 \sim 8/11\}$
31 $\{x|5/10 \sim x/.25\}$
32 $\{y|3/5 \sim y/6\}$
33 $\{x|\frac{1}{8}/4 \sim x/9\frac{1}{3}\}$
34 $\{x|x/1.6 \sim 2.5/5\}$
35 $\{x|1\frac{1}{7}/x \sim 4/\frac{2}{3}\}$
36 $\{x|.8/80 \sim 800/x\}$
37 $\{x|x/21 \sim \frac{1}{3}/\frac{2}{7}\}$
38 $\{x|9/11 \sim 11/x\}$
39 $\{x|.05/.5 \sim x/.007\}$
40 $\{x|\frac{1}{4}/\frac{1}{3} \sim 4/x\}$

Problems involving rate pairs and rational numbers

In the last lesson you learned that the components of rate pairs can be rational numbers. You also learned how to find solution sets of conditions for equivalence. In this lesson you will use this knowledge to solve problems that involve rate pairs and rational numbers.

Read the problem in D1. U = Z.

A You know that Kay sold $\frac{3}{4}$ lb. of candy for each $\frac{1}{2}$ lb. that Agnes sold. Does the rate pair named at the right represent this comparison? What does the first component of the rate pair refer to? What does the second component refer to?

$\frac{3}{4}/\frac{1}{2}$

B What are you asked to find in the problem? How do you know that 6 is the first component of the rate pair whose second component you are to find? For what is n a variable?

$6/n$

C Does the sentence below at the right express a condition for the problem? What does the condition tell you about $6/n$?

$\frac{3}{4}/\frac{1}{2} \sim 6/n$.

D Now you can find the solution set of $\frac{3}{4}/\frac{1}{2} \sim 6/n$. Explain how to obtain $\frac{3}{4}n = 6 \cdot \frac{1}{2}$ from $\frac{3}{4}/\frac{1}{2} \sim 6/n$.

E What number satisfies $\frac{3}{4}n = 6 \cdot \frac{1}{2}$? Remember that U = Z.

F Does the number that satisfies $\frac{3}{4}n = 6 \cdot \frac{1}{2}$ also satisfy $\frac{3}{4}/\frac{1}{2} \sim 6/n$? Explain your answer.

G Tabulate $\{n|\frac{3}{4}/\frac{1}{2} \sim 6/n\}$.

Now you can give the answer to the problem: Agnes sold 4 lb. of candy.

At a Girl Scout bazaar, Kay sold $\frac{3}{4}$ lb. of candy for each $\frac{1}{2}$ lb. that Agnes sold. Kay sold 6 lb. of candy. How much candy did Agnes sell?

D1

A Four books of different prices cost a total of \$6.36. Suppose that all the books were the same price. How much would each book cost?

B Four books of different prices cost a total of \$6.36. What is the average cost per book?

D2

Lorraine worked in a bakery 4 days last week. She worked an average of $1\frac{1}{2}$ hr. per day. How many hours in all did she work during the 4 days?

D3

Read problem A in D2. U = Z.

H You know that \$6.36 is the total cost of the 4 books. You can use the rate pair 6.36/4 to represent this rate. What does the first component of 6.36/4 refer to? What does the second component refer to? 6.36/4

I You are asked to find how much each book would cost if all the books were the same price. Use m as a variable for the amount of money each book would cost. Why do you use $m/1$, and not $1/m$? $m/1$

J Does the sentence below at the right express a condition for the problem? What does it tell you about $m/1$? $6.36/4 \sim m/1.$

K Tabulate $\{m|6.36/4 \sim m/1\}$.

You can use the number that satisfies $6.36/4 \sim m/1$ to get the answer to the problem.

If all 4 books were the same price, each book would cost \$1.59.

For problem A in D2, the number that satisfies $6.36/4 \sim m/1$ is the *arithmetical mean* (ar′ith met′ə kəl mēn). The arithmetical mean is the first component of a rate pair that has a second component of 1. In this case, the rate pair is equivalent to a second rate pair whose first component is the total cost of the books and whose second component is the number of books bought.

In this book we will use the word "average" for the arithmetical mean. For problem A in D2, the average is the amount of money that one book would cost if you assume that all the books are the same price and that the total cost of 4 books is \$6.36.

L Now read problem B in D2. You can use the condition $6.36/4 \sim m/1$ to solve this problem, also. Give the answer to the problem.

Read the problem in D3. U = Z.

M Could Lorraine have worked more than $1\frac{1}{2}$ hr. on one day? Less than $1\frac{1}{2}$ hr. on one day? Is it possible that she did not work exactly $1\frac{1}{2}$ hr. on any one of the 4 days?

N You know that Lorraine worked an average of $1\frac{1}{2}$ hr. per day. Does the rate pair $1\frac{1}{2}/1$ represent the average number of hours she worked per day?

O What does the first component of $1\frac{1}{2}/1$ refer to? What does the second component refer to?

P Is $1\frac{1}{2}/1 \sim n/4$ a condition for the problem? For what is n a variable? To what does 4 refer?

Q Why is $n/4$, and not $4/n$, used in the condition? Tabulate $\{n|1\frac{1}{2}/1 \sim n/4\}$.

Now you can give the answer to the problem.

Lorraine worked 6 hours during the 4 days.

You have often used the Cartesian set $C \times C$ as the universe for the variables in conditions involving two variables. Now you will use the Cartesian set $Z \times Z$, which is formed from the set of non-zero rational numbers.

> REMINDER
> A Cartesian set is the set of all ordered pairs that can be formed by matching each member of one set in turn with each member of a second set.
> *See lesson 31, page 127.*

A The first components of the members of $Z \times Z$ are members of what set? The second components of the members of $Z \times Z$ are members of what set?

B Is the Cartesian set $Z \times Z$ an infinite set? How many members of $Z \times Z$ have $3\frac{1}{4}$ as the first component? How many members of $Z \times Z$ have 11 as the second component?

C How many members of $Z \times Z$ have a first component of $\frac{3}{5}$ and a second component of 7?

In some problems that involve the idea of average, you must find the total amount before you can find the average. The problem in D4 is such a problem. Read this problem. $U = Z \times Z$.

D You are asked to find the average weight per package. In the rate pair expressed at the right, for what is x a variable? $x/1$

E Do you know the total weight of the three packages? Use y as a variable for the total weight. Why do you use $y/3$, and not $3/y$? $y/3$

F Does the sentence below at the right express one of the simple conditions for the problem? $x/1 \sim y/3.$

G Does the sentence below at the right express another simple condition for the problem? Why is y, and not x, used in this condition? $y = 1\frac{3}{4} + 1\frac{1}{2} + 1\frac{1}{4}.$

The sentence below in red expresses a compound condition for the problem.

$$x/1 \sim y/3 \wedge y = 1\tfrac{3}{4} + 1\tfrac{1}{2} + 1\tfrac{1}{4}.$$

Now you can find the solution set of the compound condition.

H What number satisfies $y = 1\frac{3}{4} + 1\frac{1}{2} + 1\frac{1}{4}$?

I If you replace y by $4\frac{1}{2}$ in $x/1 \sim y/3$, do you obtain $x/1 \sim 4\frac{1}{2}/3$?

J What number satisfies $x/1 \sim 4\frac{1}{2}/3$?

K Tabulate the solution set of the compound condition for the problem in D4.

L Why do you use the first component of $(1\frac{1}{2}, 4\frac{1}{2})$ to get the answer to the problem?

Now you can give the answer to the problem.

The average weight per package was $1\frac{1}{2}$ lb.

Read the problem in D5. A sentence that expresses a condition for the problem is also given. $U = Z$.

D4

Margaret bought 3 packages of hamburger. The weights of the packages were $1\frac{3}{4}$ lb., $1\frac{1}{2}$ lb., and $1\frac{1}{4}$ lb. What was the average weight per package?

D5

The pencils that Mark wants to buy cost 8¢ each. He has 25¢ to spend for these pencils. How many pencils can he buy?

$$8/1 \sim 25/x.$$

A In $8/1 \sim 25/x$, what does the first component of $8/1$ refer to? What does the second component of $8/1$ refer to?

B For what is x a variable? Why is $25/x$, and not $x/25$, used in the condition?

C Tabulate $\{x \mid 8/1 \sim 25/x\}$.

D Would it be sensible to say that Mark can buy $3\frac{1}{8}$ pencils? Explain your answer.

E Would it be sensible to say that Mark can buy 4 pencils? Explain your answer.

You know from your work above that $3\frac{1}{8}$ satisfies $8/1 \sim 25/x$. But, since Mark could not buy part of a pencil, $3\frac{1}{8}$ pencils is not a sensible answer to the problem. You can, however, use $3\frac{1}{8}$ to get a sensible answer. Because Mark does not have enough money to buy 4 pencils, you conclude that he can buy only 3 pencils with his 25¢. He will, of course, have 1¢ left over.

F Suppose that the problem in D5 concerned weight or length. Why is it sensible to talk about part of a pound or part of a foot?

Read the problem in D6 and the sentence that expresses a condition for the problem. U = Z.

G In $3/34 \sim 1/b$, to what does the first component of $3/34$ refer? To what does the second component of $3/34$ refer?

H For what is b a variable? Why is $1/b$, and not $b/1$, used in the condition?

I Tabulate $\{b \mid 3/34 \sim 1/b\}$.

J Can you buy one can of lemonade for $11\frac{1}{3}$¢? Explain your answer.

You have found that the price of one can of lemonade is $11\frac{1}{3}$¢. Stores, however, would charge 12¢ for one can. Therefore, to give a sensible answer to the problem in D6, you would use 12, and not 11 or $11\frac{1}{3}$.

K Give the answer to the problem in D6.

Read the problem in D7 and the sentence that expresses a condition for the problem. U = Z.

Three cans of a certain brand of frozen lemonade cost 34¢. How much would you have to pay if you bought 1 can of this lemonade?

$$3/34 \sim 1/b.$$

D6

A total of 1715 persons attended a play. The play was given on 6 different nights. What was the average attendance per night? (Hundredths)

$$1715/6 \sim s/1.$$

D7

L In $1715/6 \sim s/1$, what does the first component of $1715/6$ refer to? What does the second component of $1715/6$ refer to?

M For what is s a variable? Why is $s/1$, and not $1/s$, in the condition?

N Tabulate $\{s \mid 1715/6 \sim s/1\}$.

Look again at D7. The word "hundredths" in parentheses after the problem tells you to use a decimal to express your answer and to round off your answer to the nearer hundredth.

O Explain why you should use 285.83 when you give the answer to the problem in D7.

P Give the answer to the problem.

Remember that the average is a number used to give some idea of how many persons would have attended the play each night if the attendance had been the same for all 6 nights. For this reason it is permissible to use a number like 285.83 or 6.9 or $3\frac{1}{3}$, even when the number refers to a number of persons.

In this lesson you have learned to use conditions for equivalence to solve problems that involve rational numbers. You have also learned how to find averages.

On your own

For each problem, write a sentence that expresses a condition. Tabulate the solution set of the condition and give the answer to the problem. The universe is given for each problem.

1 Martha lives $2\frac{1}{4}$ times as far from school as Sandy does. Sandy lives $6\frac{1}{2}$ blocks from school. Martha lives how far from school? U = Z.

2 A professional football team scored 311 points in 12 games. What was the average number of points scored per game? (Tenths) U = Z.

3 The cost of transporting oil from Edmonton to Winnipeg was $4.50 per 100 barrels. At this rate, what was the cost of transporting 1350 barrels of oil from Edmonton to Winnipeg? U = Z.

4 Mrs. Owen bought 3 bars of soap that was priced at 2 bars for 29¢. How much did she pay for the 3 bars? U = Z.

5 The area of Lake Nipissing in Ontario is 350 sq. mi. The area of Lake Dauphin in Manitoba is 200 sq. mi. Lake Nipissing is how many times larger than Lake Dauphin? U = Z.

6 Mr. Russell drove 327 mi. on Monday. He averaged 34.5 miles per hour. How long did it take him to drive the 327 miles? U = Z.

7 On a series of tests, Marge received scores of 94, 98, 93, 85, and 91. What was her average score per test? (Tenths) U = Z × Z.

8 Mary had $5\frac{1}{2}$ yd. of printed cotton. She made as many aprons as possible out of this cloth. Each apron took $1\frac{1}{2}$ yd. of cloth. How many aprons did she make? U = Z.

9 Harold earns $1.40 an hour. Last week he worked $20\frac{1}{5}$ hr. How much did he earn last week? U = Z.

10 Ken and his father hiked for 3 hours. They hiked $3\frac{1}{4}$ mi. the first hour, $2\frac{1}{2}$ mi. the second hour, and 3 mi. the third hour. What was their average rate in miles per hour? U = Z × Z.

7 94 Exploring ideas

Per cents involving rational numbers

You know that per cents are rate pairs that represent comparisons and that have second components of 100. So far the first components of these rate pairs have been counting numbers. Now you will study per cents that have nonzero rational numbers as first components.

A The chart in D1 lists the yearly incomes of five families and also the amounts of money put into a savings account during the year. For each family, what was its income and how much money was put into a savings account?

B You can use a rate pair to represent the comparison of the amount each family put into a savings account with the income of that family. The rate pairs named in D2 represent these comparisons. To what does the first component of each rate pair refer? To what does the second component refer?

	Dollars put into savings account	Income in dollars for one year
Wilsons	325	5000
Powells	75	10,000
Hansons	700	6000
Benders	375	6000
Abrams	850	8500

D1

Wilsons	325/5000
Powells	75/10000
Hansons	700/6000
Benders	375/6000
Abrams	850/8500

D2

	U = Z.
Wilsons	$325/5000 \sim x/100.$
Powells	$75/10000 \sim x/100.$
Hansons	$700/6000 \sim x/100.$
Benders	$375/6000 \sim x/100.$
Abrams	$850/8500 \sim x/100.$

D3

Wilsons	$325/5000 \sim 6\frac{1}{2}/100.$
Powells	$75/10000 \sim \frac{3}{4}/100.$
Hansons	$700/6000 \sim 11\frac{2}{3}/100.$
Benders	$375/6000 \sim 6\frac{1}{4}/100.$
Abrams	$850/8500 \sim 10/100.$

D4

	A	B	C
Wilsons	325/5000	$6\frac{1}{2}/100$	$6\frac{1}{2}\%$
Powells	75/10000	$\frac{3}{4}/100$	$\frac{3}{4}\%$
Hansons	700/6000	$11\frac{2}{3}/100$	$11\frac{2}{3}\%$
Benders	375/6000	$6\frac{1}{4}/100$	$6\frac{1}{4}\%$
Abrams	850/8500	10/100	10%

D5

C Suppose that you want to find what per cent of its income each family put into a savings account. Can you use the conditions expressed in D3 to do this? For what is x a variable in each condition?

D How can you find the replacement for x that satisfies each condition?

E Now look at D4. Does each sentence express a true statement? How do you know?

F What comparison is represented by each rate pair whose second component is 100?

G Is each of these rate pairs a per cent?

> REMINDER
> A per cent is a rate pair that represents a comparison and that has a second component of 100.
> *See lesson 45, page 187.*

H Study D5. For each family, is the per cent named in column B equivalent to the rate pair named in column A?

I Now read the ratios in column C. For each family, is the per cent named in column C the same as the per cent named in column B?

Now that you know what per cent of its income each family put into a savings account, you can easily compare the savings of the different families by comparing the first components of the rate pairs.

J Look again at column C in D5. Which family put the greatest per cent of its income into a savings account? Which family put the next greatest per cent into a savings account?

K Is $6\frac{1}{2}\%$ greater than $6\frac{1}{4}\%$? Which family, the Wilsons or the Benders, put a greater per cent of its income into a savings account?

L Sometimes we say that a per cent is "less than 1%." This means that the first component of the rate pair is less than 1 and the second component is 100. Which family put less than 1% of its income into a savings account?

M Starting with the name of the family that put the greatest per cent of its income into a savings account, list the names in order. Then tell how much money out of each $100 of income each family put into a savings account.

	A	B	C
Wilsons	325/5000	$6\frac{1}{2}/100$	$6\frac{1}{2}\%$
Powells	75/10000	$\frac{3}{4}/100$	$\frac{3}{4}\%$
Hansons	700/6000	$11\frac{2}{3}/100$	$11\frac{2}{3}\%$
Benders	375/6000	$6\frac{1}{4}/100$	$6\frac{1}{4}\%$
Abrams	850/8500	10/100	10%

D5

	Average annual precipitation	
Precipitation in 1960		
Gander	43.18 in.	40 in.
Montreal	37.06 in.	42 in.
Toronto	31.82 in.	31 in.
Winnipeg	15.59 in.	20 in.
Edmonton	19.64 in.	18 in.
Regina	13.70 in.	15 in.

D6

N Did the family that put the greatest per cent of its income into a savings account put the greatest amount of money into the account?

O The chart in D6 lists the amount of precipitation recorded in six centres across Canada in 1960 and the average annual precipitation for each centre. In each centre, what was the precipitation during 1960? What is the average annual precipitation for each centre?

P For each centre listed in D6, write a ratio that expresses a rate pair. The first component of the rate pair should refer to the number of inches of precipitation in 1960. The second component should refer to the number of inches of average annual precipitation.

Q For each of the rate pairs you named for exercise P, find an equivalent rate pair that has a second component of 100. Use a decimal to express the first component of this rate pair in hundredths.

R Express each rate pair that you obtained for exercise Q by using the per cent sign.

S List in order the per cents you just named. Begin with the name of the least per cent.

T In which centres named in D6 was the precipitation for 1960 greater than 100% of the average annual precipitation?

Now you know how to use non-zero rational numbers as first components of per cents.

On your own

Express the rate pairs named in exercises 1 through 6 by using the per cent sign.

1 $25\frac{1}{2}/100$ **3** $\frac{1}{10}/100$ **5** .06/100

2 1.72/100 **4** 100.6/100 **6** $133\frac{1}{3}/100$

7 Find the per cent equivalent to 1/10. Is this per cent the same as $\frac{1}{10}\%$?

8 Which per cent is greater, .75/100 or 75/100?

9 Is .5/100 the same per cent as 50%?

10 Which per cent is greater, $\frac{1}{5}/100$ or 20%?

For each rate pair named in exercises 11 through 16, find an equivalent rate pair that has a second component of 100. Use a fraction numeral or a mixed numeral to express the first component of each rate pair. Then express each rate pair by using the per cent sign.

11 1/6 **13** 2/3 **15** $15\frac{1}{2}/6$

12 3/8 **14** 1/30 **16** $12\frac{1}{2}/14\frac{2}{7}$

For each rate pair named in exercises 17 through 28, find an equivalent rate pair that has a second component of 100. Use a decimal to express to the nearer hundredth the first component of each rate pair. Then express each rate pair by using the per cent sign.

17 3.8/25 **21** 22.4/16 **25** 5.174/7.35

18 18/40 **22** 51.66/63 **26** 28.57/6

19 6/13.2 **23** 70/80 **27** 1.7/26

20 11.4/9.8 **24** 10.5/5.2 **28** 4.2/.63

Problems involving per cents and rational numbers

In lesson 94 you learned that rational numbers can be the first components of per cents. In this lesson you will solve problems that involve per cents and rational numbers. U = Z.

Read the problem in D1. The exercises that follow will help you solve the problem.

A The rate pair named at the right below represents the comparison of the number of members who paid their dues with the total number of members. What does the first component of the rate pair refer to? What does the second component refer to? 5/15

B The question in the problem asks you to find what per cent 5 is of 15. To do this, you must find the first component of a rate pair that has a second component of 100 and that is equivalent to 5/15. For what is n a variable? $n/100$

C The sentence at the right below expresses a condition for the problem. What does the condition tell you about the rate pair whose first component you are to find? $5/15 \sim n/100.$

D Explain how to obtain $500 = 15n$ from $5/15 \sim n/100$.

E Does the number that satisfies $500 = 15n$ also satisfy $5/15 \sim n/100$? Explain your answer.

F Tabulate $\{n \mid 5/15 \sim n/100\}$.

Now you can give the answer to the problem in D1.

There are 15 members in the Western Camera Club. At the first meeting of the year, 5 members paid their dues. What per cent of the members paid their dues at that time?

D1

Mr. Brown receives a 6.5% commission on his total monthly sales. His total sales for May amounted to $6000. How much was Mr. Brown's commission for May?

D2

$33\frac{1}{3}\%$ of the camera club members paid their dues at the first meeting.

G Read the problem in D2. What are you asked to find?

Since Mr. Brown receives a 6.5% commission on his total monthly sales, he receives 6.5 dollars for each 100 dollars' worth of merchandise that he sells.

H Is the rate pair named at the right the same rate pair as 6.5%? $6.5/100$

I Mr. Brown's total sales for May amounted to $6000. Why is 6000 the second component of the rate pair whose first component you are to find? For what is n a variable? $n/6000$

J Does the sentence at the right express a condition for the problem? What does the condition tell you about the rate pair whose first component you are to find? $6.5/100 \sim n/6000.$

K Tabulate $\{n \mid 6.5/100 \sim n/6000\}$.

You can use the number that satisfies $6.5/100 \sim n/6000$ to get the answer to the problem.

Mr. Brown's commission for May was $390.

L Read the problem in D3. What are you asked to find?

M Is the rate pair named at the right the same rate pair as $\frac{1}{2}$%? $\frac{1}{2}/100$

N How do you know that 4 is the first component of the rate pair whose second component you are to find? For what is x a variable? $4/x$

O What does the condition expressed at the right tell you about the rate pair whose second component you are to find? $\frac{1}{2}/100 \sim 4/x.$

P Tabulate $\{x|\frac{1}{2}/100 \sim 4/x\}$.

Q Is the number that satisfies $\frac{1}{2}/100 \sim 4/x$ the answer to the problem?

Now you will find answers to other problems that involve per cents and rational numbers. U = Z.

A Read the problem in D4. What are you asked to find?

B What was the price of a ticket last season? What was the increase?

C What comparison is represented by the rate pair 25/40?

D What does the first component of 25/40 refer to? What does the second component refer to?

E You are to find what per cent 25 is of 40. How do you know that 100 is the second component of the rate pair you are to find?

F Is $25/40 \sim x/100$ a condition for the problem? For what is x a variable?

G Tabulate $\{x|25/40 \sim x/100\}$.

Now you can give the answer to the problem.

The increase is $62\frac{1}{2}$% of last season's price.

For the problem in D4, you found what per cent the increase was of the original price. This per cent of the original price is the *per cent of increase*.

4 is $\frac{1}{2}$% of what number?

D3

Last season a ticket to a Milbank High School football game cost 40¢. This season the price was increased by 25¢. The increase is what per cent of last season's price?

D4

The original value of Mr. Hill's car was \$2250. Now the value has decreased by \$750. The decrease is what per cent of the original value?

D5

Last year 240 persons were enrolled in evening classes at Adams High School. This year 500 persons are enrolled in evening classes. What is the per cent of increase in enrollment?

D6

H Read the problem in D5. What are you asked to find?

I What was the original value of the car? By how much has the value decreased?

J What comparison is represented by the rate pair 750/2250? What does the first component of 750/2250 refer to? What does the second component refer to?

K How do you know that 100 is the second component of the rate pair whose first component you are to find?

L Is $750/2250 \sim m/100$ a condition for the problem? For what is m a variable?

M Tabulate $\{m|750/2250 \sim m/100\}$.

You can use the number that satisfies $750/2250 \sim m/100$ to get the answer to the problem.

The decrease in value is $33\frac{1}{3}\%$ of the original value.

For the problem in D5, you found what per cent the decrease was of the original value. This per cent of the original value is the *per cent of decrease*.

Now you will use compound conditions in two variables to solve problems that involve per cents and rational numbers. $U = Z \times Z$.

Read the problem in D6. The exercises that follow will help you solve the problem.

A The problem asks you to find the per cent of increase. To do this, you must find the increase for each 100 persons enrolled last year. How do you know that 100 is the second component of the rate pair whose first component you are to find? This rate pair is expressed at the right. For what is b a variable?

$b/100$

B Do you know the increase in enrollment? Use c as a variable for the increase in enrollment.

C Why do you use $c/240$, and not $240/c$, for the second rate pair? Why do you use $c/240$, and not $c/500$?

$c/240$

D Does the sentence at the right express one of the simple conditions for the problem?

$b/100 \sim c/240.$

E Does the sentence at the right below express the other simple condition for the problem? Why is c, and not b, used in this simple condition?

$500 - 240 = c.$

The sentence in red below expresses the compound condition for the problem.

$b/100 \sim c/240 \wedge 500 - 240 = c.$

Now you can find the solution set of the compound condition.

F What number satisfies $500 - 240 = c$?

G If you replace c by 260 in $b/100 \sim c/240$, do you obtain $b/100 \sim 260/240$?

H What number satisfies $b/100 \sim 260/240$?

I How do you know that the solution set of $b/100 \sim c/240 \wedge 500 - 240 = c$ is a set of ordered pairs?

J Tabulate the solution set of the compound condition.

K What does the first component of $(108\frac{1}{3}, 260)$ refer to? What does the second component refer to?

L Why should you use the first component of $(108\frac{1}{3}, 260)$ to get the answer to the problem?

Now you can give the answer to the problem.

The per cent of increase in enrollment is $108\frac{1}{3}\%$.

You have learned how to solve problems that involve per cents whose first components are rational numbers. You also have learned how to find per cent of increase and of decrease.

On your own

For each of problems 1 through 14, write a sentence that expresses a condition. Then tabulate the solution set of the condition and give the answer to the problem. The universe is given for each problem.

1 The sales tax in a certain state is $3\frac{1}{4}\%$ of the selling price. What is the sales tax on a car priced at \$2200? $U = Z$.

2 Last week Frank worked $12\frac{1}{2}$ hr. This week he worked $2\frac{1}{2}$ hr. less than he did last week. What is the per cent of decrease in the number of hours he worked? $U = Z$.

3 Mr. Hardy's house, which cost \$2976, is now worth 325% of the original cost. How much is the house worth now? $U = Z$.

4 Last year Bill had a job that paid \$1.00 per hour. This year the same job pays \$.30 more per hour. What is the per cent of increase in his hourly wages? U = Z.

5 During a sale, a \$21 jacket was marked down \$7. What was the per cent of decrease in the price? U = Z.

6 Mr. Keys paid 6% interest on his loan from a local bank. At the end of one year, the interest on his loan was \$46.50. How much money had Mr. Keys borrowed? U = Z.

7 According to the 1921 census, the population of the Yukon was about 4,000. At the time of the 1951 census, the population of the Yukon was about 9,000. What was the per cent of increase in population? U = Z × Z.

8 Judy paid a sales tax of \$.63 on a skirt priced at \$14.00. The sales tax was what per cent of the selling price? U = Z.

9 There are about 124,800,000 telephones in the world. About 1.4% of these telephones are in Africa. About how many telephones are there in Africa? U = Z.

10 The sales tax on a certain book is \$.07. If the tax is 2% of the price, what is the selling price of this book? U = Z.

11 Last year Mr. Ward raised 96 hogs. This year he raised only 84 hogs. What was the per cent of decrease in the number of hogs that he raised? U = Z × Z.

12 One year, farmers received an average price of \$.50 per pound for maple sugar. The following year, they received an average price of \$.56 per pound. What was the per cent of increase in the price per pound? U = Z × Z.

13 Carl saved \$26.55 in May and \$17.70 in June. What was the per cent of decrease in the amount he saved? U = Z × Z.

14 The amount of fresh water that fills a certain container weighs 62.4 lb. The same amount of salt water weighs 64.0 lb. The weight of the fresh water is what per cent of the weight of the salt water? U = Z.

For each problem that follows, write a sentence that expresses a condition and tabulate the solution set of the condition. U = Z.

15 34 is what per cent of 51?

16 1.5 is 50% of what number?

17 What number is 75% of $5\frac{1}{3}$?

18 What number is .4% of 20?

19 5.2 is what per cent of 16?

20 What number is 125% of $\frac{3}{4}$?

21 72 is $\frac{1}{4}$% of what number?

22 40 is what per cent of 8000?

23 $\frac{3}{5}$ is 45% of what number?

24 What number is $22\frac{1}{2}$% of 80?

25 42 is what per cent of 560?

26 15 is $1\frac{1}{2}$% of what number?

27 5000 is what per cent of 10?

KEEPING SKILFUL

Tabulate the solution set of each condition expressed below. Use a fraction numeral or a mixed numeral to express each solution. U = Z.

1 $\frac{1}{3}/1 \sim n/100$.

2 $n/100 \sim \frac{7}{8}/1$.

3 $83\frac{1}{3}/100 \sim k/1$.

4 $\frac{1}{6}/1 \sim r/100$.

5 $37\frac{1}{2}/100 \sim c/1$.

6 $y/1 \sim 66\frac{2}{3}/100$.

7 $\frac{1}{7}/1 \sim g/100$.

8 $12\frac{1}{2}/100 \sim x/1$.

9 $\frac{5}{8}/1 \sim m/100$.

10 $x/1 \sim 8\frac{1}{3}/100$.

11 $33\frac{1}{3}/100 \sim t/1$.

12 $d/100 \sim \frac{1}{16}/1$.

13 $\frac{1}{8}/1 \sim k/100$.

14 $16\frac{2}{3}/100 \sim x/1$.

Use a decimal to express each sum, difference, or product expressed below.

15 13(.429)

16 29 − 1.682

17 .7329 + .9465

18 .088(1.27)

19 .9701 − .097

20 4.5 + .76 + .82

21 59(.12 × 3.7)

22 .11 + 7.1 + 13.6

23 1.5 − .6623

24 126(4.1 × 6.5)

25 .9 − .8962

26 .723 + .067 + .9

27 (2.7 × .08).5

28 .9 − .099

APPLYING MATHEMATICS

For each problem in this lesson, write a sentence that expresses a condition. If possible, tabulate the solution set of the condition. Otherwise, give a standard description of the solution set. Then give the answer to the problem. The universe is given for each problem.

1 Joan bought $3\frac{1}{2}$ lb. of apples, 1 lb. of bananas, and $\frac{3}{4}$ lb. of grapes. How much fruit in all did she buy? $U = R_a$.

2 Mr. Smith needs 9 pickets for each 4 ft. of fence. If he makes the fence 196 ft. long, how many pickets will he need? $U = C$.

3 Ed bought some white popcorn and $\frac{3}{4}$ lb. of yellow popcorn. He bought less than $1\frac{1}{8}$ lb. of popcorn in all. How much white popcorn did Ed buy? $U = Z$.

4 Marie paid more than $3.50 for school supplies. She gave the clerk $5.00 and received more than $1.25 in change. How much did Marie pay for the school supplies? $U = R_a$.

5 The atomic weight of chemical elements is given in atomic mass units (a.m.u.). The atomic weight of carbon is 12.011 a.m.u. The atomic weight of silver is 95.869 a.m.u. more than the atomic weight of carbon. What is the atomic weight of silver? $U = R_a$.

6 The atomic weight of iodine is 126.91 a.m.u. The atomic weight of nickel is 68.20 a.m.u. less than the atomic weight of iodine. What is the atomic weight of nickel? $U = R_a$.

7 In June, Marilyn saved $5.20. In July, she saved $1.30 less than she saved in June. What was the per cent of decrease in her savings? $U = Z$.

8 During a ski-jumping contest, a contestant made jumps of 290 ft. and 272 ft. What was the average distance per jump? $U = C \times C$.

9 Mrs. Moore received a 2% discount on her telephone bill for April. The discount amounted to $.17. How much was her telephone bill for April? $U = Z$.

10 One winter, Toronto had .5 in. of snow in November and 8.0 in. of snow in December. What was the per cent of increase in the amount of snowfall? $U = Z \times Z$.

11 Last week Mr. Lyons drove $1\frac{1}{2}$ times as far as Mr. Jones. Mr. Jones drove 135.4 mi. How far did Mr. Lyons drive last week? $U = Z$.

12 The daily high temperatures in Vancouver one week were 75°, 83°, 75°, 78°, 73°, 73°, and 72°. What was the average daily high temperature that week? (Hundredths) $U = Z \times Z$.

13 In Sue's history class, 25% of the students wear glasses. Eight students in the class wear glasses. How many students in the class do not wear glasses? $U = C \times C$.

14 In the year 1951, the population of Quebec province was about 4,055,000. About .7% of the residents of Quebec lived in Lachine. About how many persons lived in Lachine at that time? $U = Z$.

15 Mr. Wilson sold $24\frac{1}{2}$ gal. of ice cream on Monday and $30\frac{5}{8}$ gal. of ice cream on Tuesday. What was the per cent of increase in the amount of ice cream he sold? $U = Z \times Z$.

16 Last year, 700 students were enrolled in Westport High School. This year, 686 students are enrolled. What is the per cent of decrease in enrollment? $U = Z \times Z$.

17 What number is the difference of $37\frac{4}{5}$ and $19\frac{3}{7}$? $U = R_a$.

18 What number is 175% of 72? $U = C$.

19 What numbers can be added to 4.62 so that each sum is less than 5.01? $U = R_a$.

20 The sum of two numbers is less than 9. The second number is 3 more than the first number. What can the numbers be? $U = N \times N$.

21 What numbers greater than $5\frac{3}{4}$ can be subtracted from 12 so that each difference is greater than $5\frac{1}{2}$? $U = R_a$.

CHECKING UP

If you have trouble with these two tests, you can find help in lessons 92 through 95.

Test 92

For each of exercises 1 through 4, tell whether the sentence expresses a true statement or a false statement.

1 $8/60 \sim \frac{1}{10}/\frac{3}{4}$.
2 $\frac{7}{3}/\frac{6}{5} \sim \frac{35}{2}/\frac{9}{2}$.
3 $.4/.001 \sim 3.2/.8$.
4 $\frac{3}{2}/\frac{1}{8} \sim 4/\frac{1}{3}$.

Tabulate the solution set of each condition expressed in exercises 5 through 8. U = Z.

5 $.7/x \sim 1.4/.28$.
6 $\frac{1}{3}/\frac{7}{5} \sim x/\frac{7}{30}$.
7 $4\frac{1}{5}/1\frac{1}{2} \sim 2\frac{1}{3}/x$.
8 $x/.1 \sim 1/.001$.

For each rate pair expressed below, name a per cent that is equivalent.

9 .1/4 10 .075/5 11 7/1000 12 2/500

13 List in order the per cents you named for exercises 9 through 12. Begin with the name of the greatest per cent.

Test 93

For each problem, write a sentence that expresses a condition, tabulate the solution set, and give the answer to the problem. The universe is given for each problem.

14 Each volume of a set of books costs $4.98. At this rate, what does a set of 8 volumes cost? U = Z.

15 It took Lynn $1\frac{1}{2}$ times as long as it took Mary to read a history assignment. If it took Mary 30 minutes to read the assignment, how long did it take Lynn? U = Z.

16 Jack had scores of 133, 115, and 140 for three games of bowling. What was his average score per game? (Ones) U = Z × Z.

17 In two months, John's weight increased from 150 lb. to $157\frac{1}{2}$ lb. What was the per cent of increase in John's weight? U = Z × Z.

18 What number is 5% of 250? U = Z.

19 1.25 is what per cent of 250? U = Z.

20 .1250 is .05% of what number? U = Z.

7 96 Exploring problems

Problems involving comparisons with one

Now you will learn more about the use of rational numbers when you compare quantities.

Read problem A in D1. U = Z.

A Does the rate pair named at the right below represent the comparison of the life expectancy of a dog with the life expectancy of a horse? What does each component refer to?

11/33

The question in problem A asks you to find what fraction 11 is of 33. This means that you are to find what part of a whole year a dog may be expected to live for each year that a horse may be expected to live. To do this, you will find the first component of a rate pair that has a second component of 1 and that is equivalent to 11/33. This rate pair is expressed at the right above.

$x/1$

B For what is x a variable?

C Does the sentence at the right express a condition for problem A in D1? Tabulate the solution set of this condition.

$11/33 \sim x/1$.

You can give the answer to problem A in the following way:

The life expectancy of a dog is $\frac{1}{3}$ of the life expectancy of a horse.

D Read problem B in D1. This problem really asks you to find the number that has the same comparison with 1 as 11 has with 33. What do you know about the rate pair whose first component you are to find?

A The normal life expectancy of a dog is 11 years. The normal life expectancy of a horse is 33 years. The life expectancy of a dog is what fraction of the life expectancy of a horse?

B 11 is what fraction of 33?

D1

C There are .5 as many calories in a glass of whole milk as there are in a piece of apple pie. There are 330 calories in a piece of apple pie. How many calories are there in a glass of milk?

D What number is .5 of 330?

D2

E Saturn has 9 satellites. Saturn has $\frac{3}{4}$ as many satellites as Jupiter has. How many satellites does Jupiter have?

F 9 is $\frac{3}{4}$ of what number?

D3

E Write a sentence that expresses a condition for problem B. Then tabulate the solution set of the condition.

The answer to problem B is $\frac{1}{3}$. You can also express the answer in the following way:

11 is $\frac{1}{3}$ of 33.

F Read problem C in D2. It tells you that there is .5 calorie in a glass of whole milk for each calorie in a piece of apple pie. What rate pair that has a second component of 1 expresses this comparison? U = Z.

G Read problem D. This problem really asks you to find the number that has the same comparison with 330 as .5 has with 1. Can you use the rate pair .5/1? U = Z.

H Does the sentence at the right express a condition for problem C? For problem D?

$.5/1 \sim m/330.$

I To what does m refer in problem C? In problem D?

J Tabulate $\{m|.5/1 \sim m/330\}$.

K Explain why the sentence $\frac{1}{2}/1 \sim m/330$ also expresses a condition for both problems in D2.

L How many calories are there in a glass of whole milk? What number is .5 of 330?

M Read problem E in D3. It really tells you that Saturn has $\frac{3}{4}$ of a satellite for each satellite that Jupiter has. What rate pair expresses this comparison? U = Z.

N Read problem F. This problem really asks you to find the number with which 9 has the same comparison as $\frac{3}{4}$ has with 1. Can you use the rate pair $\frac{3}{4}/1$? U = Z.

O Does the sentence at the right express a condition for problem E? For problem F?

$\frac{3}{4}/1 \sim 9/y.$

P To what does y refer in problem E? To what does y refer in problem F?

Q Tabulate $\{y|\frac{3}{4}/1 \sim 9/y\}$.

R How many satellites does Jupiter have? 9 is $\frac{3}{4}$ of what number?

In this lesson you have learned how to develop conditions for equivalence for problems that involve comparisons with one. You have also learned the meaning of such sentences as "What number is $\frac{2}{3}$ of 16?"

On your own

For each problem, first write a sentence that expresses a condition. Then tabulate the solution set of the condition and give the answer to the problem. U = Z.

1 One year Mr. Nichols sold $\frac{3}{5}$ of his corn crop and kept the rest for cattle feed. His corn

crop that year was 10,000 bushels. How many bushels of corn did Mr. Nichols sell?

2 Ed spent $\frac{1}{2}$ as much time watching television on Friday as he spent on Saturday. He spent $1\frac{1}{4}$ hr. watching television on Friday. How many hours did he spend watching television on Saturday?

3 Sally has a weekly allowance of \$.75. One week she spent \$.60 of her allowance for a movie ticket. What fraction of her weekly allowance did she spend for the ticket? (Tenths)

4 There are 30 seventh-grade students at Henderson School. .9 of them went to a class picnic. How many seventh-grade students went to the picnic?

5 The Wilson High School basketball team won 14 of the 21 games it played this year. The number of games won was what fraction of the number of games played?

6 Mrs. Taylor took to a bake sale $\frac{3}{4}$ of the cupcakes she had made. She took 36 cupcakes to the sale. How many cupcakes had Mrs. Taylor made?

7 Mrs. Thomas had some drapery material. She used $\frac{7}{8}$ of it to make a pair of drapes. She used $4\frac{2}{3}$ yd. How much drapery material did she have to begin with?

8 The speed of sound in steel is 15,000 ft. per second. The speed of sound in a certain kind of wood is approximately $\frac{2}{3}$ of the speed of sound in steel. What is the approximate speed of sound in this kind of wood?

9 When a rocket reached a height of 20 mi. above the earth, it was traveling at a speed of 3500 miles per hour. When the rocket reached a height of 250 mi., it was traveling 5000 miles per hour. The speed of the rocket at 20 mi. above the earth was what fraction of its speed at 250 mi. above the earth?

10 A cake recipe that Jean has calls for $2\frac{1}{2}$ cups of flour. Jean made a cake using .5 of the ingredients called for in the recipe. How many cups of flour should she have used?

11 The Miller family uses $\frac{3}{4}$ gal. of milk a day. The King family uses $1\frac{1}{4}$ gal. of milk a day. The amount of milk that the Miller family uses in a day is what fraction of the amount of milk that the King family uses in a day?

12 Tony answered .85 of the questions on a test correctly. He answered 102 questions correctly. How many questions were there on the test?

13 6.01 is what fraction of 30.05?

14 12 is $\frac{2}{3}$ of what number?

15 What number is .35 of 60?

16 What number is $\frac{11}{12}$ of $\frac{1}{3}$?

17 30 is what fraction of 48?

18 3.6 is .4 of what number?

19 $10\frac{1}{5}$ is $\frac{3}{8}$ of what number?

20 750 is what fraction of 1000?

21 What number is .67 of 125?

KEEPING SKILFUL

For each exercise below, find the sum, difference, product, or quotient.

1 $4\frac{1}{2} \times 3\frac{1}{3}$

2 $17\frac{1}{4} + 39\frac{1}{2} + 45\frac{3}{4}$

3 $.6 + .5 + .8 + .9$

4 $11.07 - 2.196$

5 $5\frac{3}{4} \times 1\frac{7}{9}$

6 $.33 - .1265$

7 $\frac{3}{5} \div \frac{6}{7}$

8 $2\frac{4}{5} + 3\frac{7}{8}$

9 $10\frac{1}{7} - 5\frac{5}{7}$

10 $39 \times .175$

11 $7\frac{3}{8} - 2\frac{1}{6}$

12 $6\frac{1}{9} - 4\frac{2}{3}$

13 $1\frac{1}{4} \div \frac{7}{8}$

14 1.96×3.42

15 $.7695 + .834 + 2$

16 $4\frac{1}{2} + 6\frac{2}{3} + 7\frac{5}{6}$

17 $.027 \times .684$

18 $9\frac{2}{5} - 8\frac{3}{4}$

19 $2\frac{4}{5} \div 3\frac{2}{3}$

20 $\frac{5}{8} \times 6\frac{1}{3}$

21 $.324 + 1.09 + .93$

22 $\frac{9}{11} \div 4\frac{3}{8}$

23 $4.56 \div .23$ (Tenths)

24 $15 \div 9$ (Hundredths)

25 $70 \div .56$ (Ones)

26 $.386 \div 1.4$ (Thousandths)

27 $.9628 \div .032$ (Hundredths)

Per cents related to rate pairs with second components of one

You have used conditions for equivalence to solve problems that involve per cents. In these conditions the second component of one of the rate pairs was 100. In this lesson you will learn how to use other conditions for equivalence to solve such problems.

Read the problem in D1. The following exercises will help you understand why you can develop more than one condition for the problem.

A Look at D2. Sentence A expresses a condition for the problem in D1. What does each component of $37\frac{1}{2}/100$ refer to? For what is n a variable? What is the universe for n in $37\frac{1}{2}/100 \sim n/14$?

B Look at sentence B. How do you know that .375/1 is equivalent to $37\frac{1}{2}/100$? How can you obtain the rate pair .375/1 from the rate pair $37\frac{1}{2}/100$?

Since .375/1 is equivalent to $37\frac{1}{2}/100$, the rate pairs .375/1 and $37\frac{1}{2}/100$ represent the same comparison. The condition expressed by sentence B in D2, therefore, is also a condition for the problem in D1.

C What basic fraction is named by the decimal fraction numeral .375? How do you know that sentence C in D2 expresses the same condition as sentence B? How do you know that sentence C also expresses a condition for the problem in D1?

Now you will find the solution set of each of the conditions expressed in D2.

Joe received a discount of $37\frac{1}{2}\%$ on the price of a pair of ice skates. The regular price of the skates was \$14. How much discount did Joe receive? (Hundredths)

D1

$U = Z$.

A $37\frac{1}{2}/100 \sim n/14$.

B $.375/1 \sim n/14$.

C $\frac{3}{8}/1 \sim n/14$.

D2

D Look again at sentence A. How can you obtain $37\frac{1}{2} \times 14 = 100n$ from the condition expressed by sentence A? How can you obtain $525 = 100n$ from $37\frac{1}{2} \times 14 = 100n$?

E Does the number that satisfies the condition $525 = 100n$ also satisfy $37\frac{1}{2}/100 \sim n/14$? What computation do you use to find the number that satisfies $525 = 100n$?

F Tabulate $\{n \mid 37\frac{1}{2}/100 \sim n/14\}$.

G Next look at sentence B. How can you obtain $.375 \times 14 = n$ from the condition expressed by sentence B?

H What computation do you use to find the number that satisfies $.375 \times 14 = n$?

I Tabulate $\{n \mid .375/1 \sim n/14\}$.

J Now look at sentence C in D2. How can you obtain $\frac{3}{8} \times 14 = n$ from the condition expressed by sentence C?

K To find the number that satisfies $\frac{3}{8} \times 14 = n$, you must find the product of what two numbers? Use a decimal to express the product in hundredths.

L Tabulate $\{n \mid \frac{3}{8}/1 \sim n/14\}$.

M Does each of the conditions expressed in D2 have the same solution set?

N Give the answer to the problem in D1.

O Read the problem in D3 and sentence D in D4. Sentence D expresses a condition for the problem. What does each component of $33\frac{1}{3}/100$ refer to? For what is y a variable?

P Think about the rate pair $33\frac{1}{3}/100$. What rate pair that has a second component of 1 is equivalent to $33\frac{1}{3}/100$? How can you obtain $\frac{1}{3}/1$ from $33\frac{1}{3}/100$?

Q Look at sentence E in D4. Because $\frac{1}{3}/1$ is equivalent to $33\frac{1}{3}/100$, the rate pairs $\frac{1}{3}/1$ and $33\frac{1}{3}/100$ represent the same comparison. Is the condition expressed by sentence E also a condition for the problem in D3?

Now you can find the solution set of each of the conditions expressed in D4.

R Tabulate $\{y|33\frac{1}{3}/100 \sim 132/y\}$.

S Look again at sentence E in D4. Why can you multiply 132 by 3 to find the number that satisfies $\frac{1}{3}y = 132$?

T Tabulate $\{y|\frac{1}{3}/1 \sim 132/y\}$.

U Does each of the conditions expressed in D4 have the same solution set? Can you use either of these conditions as the condition for the problem in D3?

It is easier to multiply or divide by $\frac{1}{3}$ than by $33\frac{1}{3}$. Therefore, when we have a choice of conditions like the two conditions expressed in D4, we prefer to use the one expressed by sentence E.

In lesson 88 you learned that the rational number $\frac{1}{3}$ can be expressed by the repeating decimal $.\overline{3}$. Therefore, the rate pairs $.\overline{3}/1$ and $\frac{1}{3}/1$ are the same. However, since you do not know how to compute with a numeral like $.\overline{3}$, it is not very helpful to write the sentence $.\overline{3}/1 \sim 132/y$ to express a condition for the problem in D3. Sometimes a decimal like $.\overline{3}$ or $.\overline{6}$ expresses a member of the solution set of a condition. In that case, $.\overline{3}$ or $.\overline{6}$ may be used to get an answer to a problem.

V Give the answer to the problem in D3.

$33\frac{1}{3}\%$ of the students enrolled at Lakeview Junior High School are in the ninth grade. There are 132 ninth graders at Lakeview. How many students in all are enrolled at Lakeview?

D3

$$U = Z.$$

D $33\frac{1}{3}/100 \sim 132/y.$

E $\frac{1}{3}/1 \sim 132/y.$

D4

Mr. Harris had saved \$2500. He used \$1000 of his savings to buy a car. What per cent of his savings did he use to buy the car?

$$1000/2500 \sim x/100.$$

D5

---

48 is $66\frac{2}{3}\%$ of what number?

D6

In the problems that follow, you will be asked to find what per cent one number is of another number. Now you will see how these problems are related to problems that ask you to find what fraction one number is of another number.

Read the problem in D5. A sentence that expresses a condition for the problem is also given. U = Z.

A What does each component of $1000/2500$ refer to? In the condition $1000/2500 \sim x/100$, for what is x a variable?

B Tabulate $\{x|1000/2500 \sim x/100\}$. Give the answer to the problem in D5.

C For the problem in D5, you found what per cent of his savings Mr. Harris used to buy the car. Suppose that you were asked to find what *fraction* of his savings he used. You can use $1000/2500 \sim y/1$ to find what fraction of his savings he used. For what is y a variable?

D Tabulate $\{y \mid 1000/2500 \sim y/1\}$. First use a decimal to name the member of the solution set. Then use a common fraction numeral.

E Did Mr. Harris use .4 of his savings to buy the car? Did he use $\frac{2}{5}$ of his savings?

F Is $.4/1$ equivalent to $40/100$, or 40%? Is $\frac{2}{5}/1$ equivalent to $40/100$, or 40%?

When you say that Mr. Harris used .4, or $\frac{2}{5}$, of his savings, you are using a fraction to tell what part of his savings he used. When you say that he used 40% of his savings, you are using a per cent to tell what part of his savings he used.

G Does $.4/1$, or $\frac{2}{5}/1$, represent the same comparison as $40/100$? Is 40% of \$2500 the same number of dollars as .4, or $\frac{2}{5}$, of \$2500?

Now you will solve a problem that concerns numbers only. U = Z.

H Read the problem in D6. $66\frac{2}{3}$ is which component of the rate pair that you are given?

I $66\frac{2}{3}/100 \sim 48/y$ is a condition for the problem. For what is y a variable?

J Write another sentence that expresses a condition for the problem in D6. Tabulate the solution set of this condition.

K 48 is $66\frac{2}{3}\%$ of what number?

In this lesson you learned that you can develop more than one condition for problems that involve rate pairs. You should use the condition that is most convenient for you.

On your own

For each of problems 1 through 12, write a sentence that expresses a condition. Then tabulate the solution set of the condition and give the answer to the problem. U = Z.

1 Ed saves 50% of the money he earns each week. He saves \$4.80 a week. How much does Ed earn each week?

2 Cal had 540 stamps in his collection. He then sold $16\frac{2}{3}\%$ of his collection. How many stamps did Cal sell?

THINK Which condition is easier to use, $16\frac{2}{3}/100 \sim x/540$ or $\frac{1}{6}/1 \sim x/540$?

3 The distance from Murdock to Drayton is 69.6 mi. The distance from Murdock to Clark City is 92.8 mi. The distance from Murdock to Drayton is what per cent of the distance from Murdock to Clark City?

4 Mrs. Thomas received a $37\frac{1}{2}\%$ discount on the price of a table that she bought. The regular price of the table was \$120. How much discount did Mrs. Thomas receive?

5 Margaret bought a book at a discount of 10%. The discount amounted to \$.65. What was the regular price of the book?

6 The Carters received a bill of \$35 for repairs on a television set. The bill included a charge of \$25 for a new picture tube. The charge for the picture tube was what fraction of the total bill? Use a common fraction numeral to express your answer.

THINK Should you use $x/1$ or $x/100$ in a condition for the problem?

7 There are 16 members on the staff of the Martin School newspaper. 10 of the members are boys. What per cent of the staff are boys? (Tenths)

8 The population of Redford is $66\frac{2}{3}\%$ of the population of Hill City. The population of Redford is 10,600. What is the population of Hill City?

9 Archie had 30 newspapers to sell one day. He sold 27 of them. What per cent of his papers did he sell that day?

10 Jim had correct answers for 40 of the questions on a history test. He had correct answers for 80% of the questions. How many questions did the test contain?

11 Mr. Andrews paid $6\frac{1}{4}\%$ interest on a loan of \$1600 for one year. How much interest did he pay?

12 Mrs. Brown used $2\frac{1}{4}$ cups of tomato juice from a can that contained $3\frac{3}{4}$ cups of juice in all. The amount of tomato juice that she used was what fraction of the total amount in the can? Use a common fraction numeral to express your answer.

For each of problems 13 through 18, write a sentence that expresses a condition. Then tabulate the solution set of the condition. $U = Z$.

13 25 is what per cent of 75? Use a mixed numeral to express your answer.

14 What number is $12\frac{1}{2}\%$ of 40?

15 3.5 is 25% of what number?

16 $5\frac{1}{4}$ is what per cent of $10\frac{1}{2}$?

17 11 is 55% of what number?

18 What number is 90% of 85.6? (Hundredths)

KEEPING SKILFUL

For each condition expressed below, tabulate the solution set if possible. Otherwise, give a standard description of the solution set. $U = R_a$.

1 $x + 4\frac{1}{2} = 6\frac{4}{5}$.

2 $9.8 < x + 1.75$.

3 $1.67 - m = .98$.

4 $x - 7\frac{7}{8} = 2\frac{1}{5}$.

5 $4\frac{1}{7} - 2\frac{6}{13} = b$.

6 $\frac{8}{3} - g > \frac{2}{5}$.

7 $5\frac{7}{9} + p = 11\frac{1}{3}$.

8 $9\frac{1}{5} - v = 3\frac{3}{4}$.

9 $d - 1\frac{1}{8} > 4\frac{1}{7}$.

10 $3.9 + 4.68 = f$.

11 $x - .768 = 1.307$.

12 $13\frac{1}{6} + 9\frac{6}{7} > k$.

13 $n + 3\frac{1}{11} = 7\frac{1}{3}$.

14 $7.04 + c < 9$.

15 $y - 1\frac{2}{3} = 6\frac{4}{5}$.

16 $11\frac{1}{4} + 3\frac{5}{9} = r$.

17 $\frac{2}{7} + \frac{3}{4} < x \wedge \frac{9}{4} - \frac{13}{14} > x$.

18 $x + \frac{1}{4} > \frac{7}{8} \wedge \frac{3}{2} - x > \frac{5}{8}$.

19 $g + \frac{1}{3} < \frac{4}{5} \wedge g > 7 - 5\frac{1}{5}$.

20 $c + \frac{3}{8} < \frac{6}{5} \wedge \frac{3}{4} + c > \frac{13}{10}$.

APPLYING MATHEMATICS

In this lesson you will solve problems similar to those you have studied so far in this book. For each problem, write a sentence that expresses a condition. If possible, tabulate the solution set of the condition. Otherwise, give a standard description of the solution set. Then give the answer to the problem. The universe is given for each problem.

1 In all, Marion bought less than 54 in. of red ribbon and green ribbon. She bought 27 in. of red ribbon. How many inches of green ribbon could she have bought? $U = C$.

2 Carl has lost $11\frac{1}{2}$ lb. since school started. He now weighs $125\frac{1}{4}$ lb. How much did Carl weigh when school started? $U = R_a$.

3 The Harvey Corporation employs $\frac{2}{3}$ as many women as men. They employ 118 women. How many men do they employ? $U = Z$.

4 The temperature at 6:00 A.M. was less than 24° above zero. By 4:00 P.M. the temperature had risen 17° and was more than 36° above zero. What could the temperature have been at 6:00 A.M.? $U = N$.

5 Mrs. Fenton spent \$42 for a chair and \$266 for a sofa. She spent how many times as much money for the sofa as for the chair? $U = Z$.

6 Sue bought 9 oranges priced at 6 for 49¢. How much did she pay for the 9 oranges? $U = Z$.

7 If Mrs. Hunter buys $2\frac{1}{2}$ qt. of ice cream, she will have more than 3 qt. of ice cream in the freezer. If she buys $\frac{1}{2}$ qt., she will have less than 2 qt. of ice cream in the freezer. How much ice cream does she have now? $U = R_a$.

8 Together, Lois and Evelyn spent 95¢ for 6 tropical fish. Evelyn spent 21¢ for 2 black mollies and 25¢ for 2 red platties. How much did Lois spend for the 2 fish she bought? $U = N \times N$.

9 Bill's weight is $\frac{7}{8}$ of John's. Bill weighs 112 lb. How much does John weigh? U = Z.

10 Janet has $2.50 to spend for socks. The socks she wants to buy are priced at 3 pairs for $1.60. How many pairs of these socks can she buy? U = Z.

11 Together, Tom and Harvey sold fewer than 24 tickets to the school talent show. Tom sold 3 times as many tickets as Harvey. How many tickets could each of the boys have sold? U = C × C.

12 Mr. Carey had 126 maple trees to sell. He sold $66\frac{2}{3}\%$ of these trees in one week. How many trees did he sell that week? U = Z.

13 Dave saved $5.00 in May, $11.56 in June, $9.70 in July, and $6.10 in August. Dave saved an average of how much per month for these four months? U = Z × Z.

14 In 1948, there were about 2,000,000 motor vehicles registered in Canada. By 1960, there were about 5,200,000 motor vehicles registered. What was the per cent of increase in the number of motor vehicles registered during this period? U = C × C.

15 Six hundred persons attended the Webster County Fair the first day. The second day, 542 persons attended the fair. What was the per cent of decrease in attendance? U = Z × Z.

16 What numbers greater than 716 can be subtracted from 989 so that each difference is greater than 173? U = N.

17 The sum of two numbers is 89. The second number is greater than the product of 12 and 7. What can the numbers be? U = N × N.

18 Certain rate pairs are equivalent to 5/15. The sum of the components of each of these rate pairs is less than 25. What are the rate pairs? U = C × C.

19 What number is $33\frac{1}{3}\%$ of 180? U = Z.

20 What numbers less than $1\frac{1}{5}$ can be added to $\frac{7}{8}$ so that each sum is greater than 1? $U = R_a$.

CHECKING UP

The small numeral within parentheses tells what page to turn to for help if you need it.

Test 94

The names of eight properties of R_a are listed below.

Closure property of multiplication
Closure property of addition
Density property
Commutative property of multiplication
Commutative property of addition
Associative property of multiplication
Associative property of addition
Distributive property of multiplication over addition

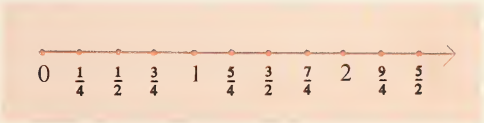
In each of exercises 1 through 7, a property of rational numbers is expressed. For each exercise, choose the name of the property from the list above. The universe for x, y, and z is R_a.

1 $(x + y) + z = x + (y + z)$. (353)
2 xy is a member of R_a. (326)
3 $x + y = y + x$. (350)
4 $(xy)z = x(yz)$. (352)
5 $x + y$ is a member of R_a. (329)
6 $x(y + z) = xy + xz$. (356)
7 $xy = yx$. (349)

Test 95

For each of the conditions expressed in exercises 8 through 11, first write a standard description of the solution set. Then make a copy of the graph of R_a shown in the display below and use it to make a graph of the solution set. $U = R_a$. (394, 395)

8 $x - \frac{1}{2} = \frac{4}{5}$.
9 $\frac{3}{4} + x = \frac{11}{12}$.
10 $x + \frac{1}{8} > \frac{7}{8} \wedge \frac{3}{2} > x - \frac{3}{4}$.
11 $\frac{5}{6} - x < \frac{2}{3} \wedge \frac{1}{2} - x > \frac{1}{4}$.



Test 96

For each of exercises 12 through 22, decide what words or symbols best complete the exercise.

12 The identity element for addition of rational numbers is the number ~~~~. (361)

13 The solution set of $.4/6 \sim x/33$ is ~~~~. U = Z. (403)

14 $1 \div \frac{7}{8}$ is the reciprocal of ~~~~. (369)

15 $4\frac{1}{4}/8\frac{1}{2}$ is equivalent to ~~~~ per cent. U = Z. (409)

16 The fraction ~~~~ is the basic fraction for 2.72. (382)

17 The decimal ~~~~ is a name for $\frac{7}{8}$. (383)

18 The digits ~~~~ form the repetend of the decimal $4.\overline{7926}$. (384)

19 The decimal ~~~~ is a name for $\frac{5}{9}$. (384)

20 The expanded form of the numeral $792\frac{34}{100}$ is the numeral ~~~~. (370)

21 The condition $\frac{a}{b} \sim \frac{ka}{kb}$ is the ~~~~ property of fractions. The universe for a is N. The universe for b and k is C. (314)

22 For each true statement obtained from $xy = 1$, the replacement for x is the ~~~~ of the replacement for y. The universe for x and y is R_a. (363)

Test 97

23 What is the product of $\frac{2}{5}$ and $\frac{4}{5}$? (325)

24 What is the sum of $\frac{1}{2}$ and $\frac{3}{5}$? (331)

25 What is the sum of 2.67 and 5.72? (375)

26 What is the difference of $\frac{11}{12}$ and $\frac{3}{8}$? (334)

27 What is the sum of 23.7, 4.592, and 169.82? (376)

28 What is the product of 3.0069 and 8.47? (376)

29 What is the difference of $12\frac{4}{9}$ and $6\frac{2}{3}$? (354)

30 What is the product of .0079 and 5.06? (376)

31 What is the difference of 46.22 and 13.57? (375)

32 What is the product of $2\frac{2}{3}$ and $4\frac{5}{7}$? (346)

33 What is the quotient of 5 and $\frac{4}{9}$? (367)

34 What is the sum of $7\frac{8}{11}$ and $3\frac{1}{2}$? (354)

35 What is the difference of 300.09 and 68.093? (376)

36 What is the quotient of 42.6 and 7.01? (Tenths) (379)

37 What is the quotient of 81.09 and 3.4? (Hundredths) (379)

38 What is the quotient of $2\frac{3}{4}$ and $7\frac{5}{6}$? (366)

Test 98

From the list given below each exercise, choose the expression that correctly completes the exercise.

39 ~~~~ is *not* a rational number. (309, 316)

a $\frac{1}{25}$ **b** $\frac{0}{6}$ **c** $\frac{6}{0}$ **d** $\frac{210}{7}$

40 The fraction ~~~~ is equivalent to the fraction $\frac{8}{34}$. (311)

a $\frac{4}{16}$ **b** $\frac{34}{8}$ **c** $\frac{24}{80}$ **d** $\frac{20}{85}$

41 For each replacement of the variables, if $\frac{a}{b} \sim \frac{c}{d}$, then ~~~~. The universe for a and c is N. The universe for b and d is C. (311)

a $ab = cd$ **b** $ad = cb$ **c** $ac = bd$ **d** $abc = d$

42 In a given rational-number line, point $\frac{1}{1}$ is to the right of point $\frac{0}{1}$. In the given line, point ~~~~ is to the left of point $\frac{5}{4}$. (320)

a $\frac{75}{72}$ **b** $\frac{6}{4}$ **c** $\frac{5}{3}$ **d** $\frac{21}{7}$

43 The greatest common divisor of 24 and 144 is ~~~~. (338)

a 6 **b** 12 **c** 24 **d** 72

44 The numeral ~~~~ is a name for $\frac{27}{6}$. (368)

a $2\frac{7}{6}$ **b** $3\frac{7}{6}$ **c** $4\frac{3}{2}$ **d** $4\frac{1}{2}$

45 ~~~~ is *not* a property. $U = R_a$. (361, 362)

a $x - x = 0$. **c** $x \cdot 1 = x$.

b $x + 1 = 1$. **d** $x - 0 = x$.

46 The reciprocal of $2\frac{1}{2}$ is ~~~~. (363)

a $2\frac{2}{1}$ **b** $\frac{5}{2}$ **c** $\frac{2}{5}$ **d** $\frac{4}{1}$

47 $\frac{5}{6} \div \frac{2}{5}$ is the same as ~~~~. (366)

a $\frac{6}{5} \times \frac{5}{2}$ **b** $\frac{5}{6} \div \frac{5}{2}$ **c** $\frac{6}{5} \times \frac{2}{5}$ **d** $\frac{5}{6} \times \frac{5}{2}$

48 ~~~~ is *not* a member of $\{x | x < \frac{7}{10}\}$. $U = R_a$. (321)

a $\frac{48}{70}$ **b** $\frac{10}{13}$ **c** $\frac{0}{4}$ **d** $\frac{3}{7}$

49 $\{x \mid x > 1 \wedge x < 2\}$ is ——. $U = R_a$. (390)

a the empty set
b a finite set
c an infinite set
d a subset of N

Test 99

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

50 The fraction (8, 3) represents 3 of 8 parts. (308)

51 The second component of the fraction (2, 7) is the denominator. (309)

52 The numbers 21 and 12 are relatively prime. (315)

53 Since $6 \times 8 < 7 \times 7$, then $\frac{7}{8} < \frac{6}{7}$. (321)

54 The least common multiple of 6 and 8 is 48. (340)

55 The repetend of a terminating decimal is the numeral zero. (385)

56 The product of 10^3 and 2.74 is 2740. (377)

57 The mate of the rational number $\frac{5}{5}$ is the natural number 5. (342)

58 Zero is the identity element for multiplication of rational numbers. (361)

59 Exactly one rational number is a reciprocal of $\frac{3}{4}$. (364)

60 The set of non-zero rational numbers is closed under division. (367)

61 $\{x \mid 3\frac{3}{4}/\frac{2}{3} \sim 5/x\} = \{\frac{8}{9}\}$. $U = Z$. (402)

62 $\frac{9}{5}/\frac{2}{7} \sim \frac{8}{3}/\frac{80}{189}$. (402)

Test 100

For exercises 63 through 71, $U = Z$.

63 8 is $\frac{1}{4}\%$ of what number? (412)

64 What number is 34% of $6\frac{1}{2}$? (412)

65 3.6 is what per cent of 60? (412)

66 What number is 4.05% of 3000? (412)

67 140 is .8% of what number? (412)

68 48 is what fraction of 88? (416)

69 14 is what per cent of 51? (Hundredths) (412)

70 $41\frac{1}{4}$ is $\frac{5}{8}$ of what number? (417)

71 $64\frac{2}{5}$ is 70% of what number? (412)

CHECKING UP

The small numeral within parentheses tells you what page to turn to if you need help with these tests.

Test 101

What words or symbols best complete exercises 1 through 10?

1 Every plane is a subset of ——. (31)

2 A line and a given point not in the line determine a ——. (35)

3 You must multiply 13 by —— to get a product of 221. (264)

4 If, for two sets A and B, $A \cup B$ is the empty set, then A and B are —— sets. (89)

5 The natural number 15 is a —— of the natural number 5. (339)

6 If ℓ_1 and ℓ_2 are not included in the same plane, then ℓ_1 and ℓ_2 are —— lines. (101)

7 The common endpoint of two rays that form an angle is the —— of the angle. (105)

8 Each member of a —— is a rate pair. (173)

9 $12 - 5 = 7$ also means $12 =$ ——. (259)

10 Two prime numbers that have just one natural number between them are ——. (297)

Test 102

11 Write a decimal for 1001_{two}. (229)

12 Write a decimal for 705_{eight}. (229)

13 Write a decimal for $4\mathcal{E}0_{twelve}$. (229)

14 Convert the numeral 47 to a base-five numeral. (231)

15 Convert the numeral 62 to a base-two numeral. (231)

16 Convert the numeral 3040 to a base-twelve numeral. (231)

17 Convert the numeral 443_{five} to a base-two numeral. (232)

18 Convert the numeral $5\mathcal{X}8_{eleven}$ to a base-eight numeral. (232).

19 Express the sum of 462_{twelve} and 998_{twelve} by a base-twelve numeral. (234)

Test 103

For each problem, write a sentence that expresses a condition, tabulate the solution set of the condition, and give the answer to the problem. The universe is given for each problem.

20 Richard put a new 12-exposure roll of film in his camera. He took some pictures one day, but had some exposures left. How many pictures could he have taken and how many exposures could have been left? $U = C \times C$. (151)

21 Mr. Frazer bought some stock in a corporation for $745. One month later he sold the stock for $810. His expenses in buying and selling the stock amounted to $22. How much money did Mr. Frazer make from this stock? $U = N \times N$. (156)

22 In a basketball game, Terry got 5 rebounds. In the same game, the other members of his team got 20 rebounds. The number of rebounds Terry got was what per cent of the total number of rebounds his team got? $U = C \times C$. (195)

23 Mr. Holmes worked in his garden for $4\frac{2}{3}$ hr. one Saturday morning. He worked a total of $6\frac{3}{4}$ hr. during the morning and the afternoon. How many hours did he work that afternoon? $U = R_a$. (397)

24 Bert spent $\frac{1}{4}$ hr. painting 9 pickets in a fence. At that rate, how long did it take him to paint the whole fence, which contained 183 pickets? $U = Z$. (404)

25 The Central School Auditorium has seats for 385 students. There are 1078 students in Central School. How many times must a movie be shown so that each student can see it? $U = Z$. (404)

26 Jim can throw a softball 180 feet. He can throw a baseball 195 feet. The distance Jim can throw a softball is what fraction of the distance he can throw a baseball? $U = Z$. (416)

CHECKING UP

The small numeral within parentheses tells what page to turn to for help if you need it.

Test 104

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

1 A circle is the boundary of two regions. (49)

2 If ℓ_1 and ℓ_2 are any two lines, and if $\ell_1 \cap \ell_2 \neq \{\ \}$, then ℓ_1 and ℓ_2 are coplanar lines. (102)

3 The set of natural numbers is closed under subtraction. (260)

4 Division of rational numbers has the commutative property. (368)

5 $\frac{1}{4}\%$ is the same rate pair as 25/100. (409)

6 If the universe for (a, b) is $N \times N$, then the solution set of $a-3=7$ is an infinite set. (135)

7 If plane ABC separates space into two half-spaces, and plane ABC $\cap$ $\overline{CD}$ $= \{C\}$, then points C and D are in the same half-space. (41, 83)

Test 105

Write each of the following numerals in expanded form. (228, 371)

8 934 **9** 7065 **10** 1011_{two} **11** $2\mathcal{E}\mathcal{X}_{twelve}$ **12** 85.01 **13** 601.317

Use scientific notation to express each number named below. (239)

14 70 **15** 4000 **16** 9,000,000 **17** 200,000

Test 106

From the list below each exercise, choose the word or symbol that correctly completes the exercise.

18 When you round 2155 to the next greater thousand, you obtain ~~~. (56)

a 2000 **b** 2160 **c** 2200 **d** 3000

19 When you round 978 to the next lesser ten, you obtain ~~~. (56)

a 70 **b** 970 **c** 980 **d** 900

20 For any two sets A and B that are not empty, if A is a subset of B, then $A \cup B =$ ~~~. (89)

a A **b** B **c** $A \cap B$ **d** $\{\ \}$

21 If $\overleftrightarrow{QR}$ is included in plane RST, then the intersection of plane RST and $\overleftrightarrow{QR}$ is ~~~. (101)

a $\overleftrightarrow{QR}$ **b** {R} **c** $\{\ \}$ **d** plane RST

22 It is impossible to find the ~~~ of the components of (12, 0). (292)

a sum **b** product **c** difference **d** quotient

23 If $A = \{0, 2, 4, 6, 8, 10\}$ and $B = \{5, 10\}$, then $n(A \cup B) =$ ~~~. (257)

a 2 **b** 7 **c** 9 **d** 18

24 $5 + 4 \cdot 2 - 4 \div 2 =$ ~~~. (280)

a 11 **b** 7 **c** $4\frac{1}{2}$ **d** 16

25 If the universe for x is R_a, and the universe for y is Z, then $x \div y = x \cdot 1/y$ is the ~~~ property of rational numbers. (369)

a product **c** quotient

b reciprocal **d** transitive

26 An arithmetical mean is ~~~. (405)

a an average **c** a sum

b a number of times as many **d** a product

27 If you use decimals to find the product of 47.165 and 8.03299, you can express that product by a decimal with ~~~ digits to the right of the decimal point. (377)

a 2 **b** 8 **c** 5 **d** 11

Test 107

For each problem, write a sentence that expresses a condition, tabulate the solution set of the condition, and give the answer to the problem. The universe is given for each problem.

28 In the last two seasons, the Centre High School football team played 20 games and lost fewer than 7 of the games they played. There were no ties. How many games did the team win in the last two seasons? $U = N \times N$. (153)

29 One week the number of logs that Mr. Graham burned was more than 10 but fewer than 25. How many logs did he burn that week? $U = N$. (94)

30 Patricia has fewer than 11 skirts and sweaters. She has the same number of skirts as sweaters. How many skirts and how many sweaters does she have? $U = C \times C$. (156)

31 Together, the two high school buildings in Springfield have 82 rooms. The newer building has 6 fewer rooms than the older building. How many rooms does each of the buildings have? $U = N \times N$. (156)

32 At the beginning of a trip, the odometer on Sam's bicycle read 2387.4 mi. At the end of the trip, the odometer read 2397.1 mi. How many miles did Sam ride his bicycle on this trip? $U = Z$. (397)

33 After Sally had gained 8 lb., she weighed 104 lb. What was the per cent of increase in her weight? $U = Z \times Z$. (411)

34 Certain rate pairs are equivalent to 6/10. The second component of each of these rate pairs is less than the sum of 9 and the first component. What are the rate pairs? $U = C \times C$. (199)

35 What number is $33\frac{1}{3}\%$ of 58? $U = Z$. (419)

36 18 is what per cent of 48? $U = Z$. (419)

37 7.08 is 40% of what number? $U = Z$. (419)

38 49 is what fraction of 84? $U = Z$. (416)

39 What number is $\frac{5}{8}$ of 200? $U = Z$. (416)

40 $39\frac{1}{6}$ is $\frac{5}{12}$ of what number? $U = Z$. (416)

Symbols and how to read them

These displays show you how to read the mathematical symbols used in this book. They are arranged in the order in which they were introduced. Whenever, in your work, you come to a symbol that you have forgotten how to read, look for one like it among those on this page and the next page. If you need more information about the symbol, turn to the page of the book whose numeral is given above the symbol. There you will find a complete explanation of the meaning of the symbol.

page 11

"The set whose members are

$\{ \text{Michigan, Huron, Superior, Erie, Ontario} \}$

Michigan, Huron, Superior, Erie, Ontario"

page 16

"x plus six equals

$x + 6 = 9.$

nine."

page 19

"K equals the set whose members are

$K = \{ 5, 6, 7, 8, 9, 10 \}.$

five, six, seven, eight, nine, ten."

page 20

"x plus two is greater than

$x + 2 > 6.$

six."

page 20

"Two times n is less than

$2n < 9.$

nine."

page 22

"The set whose members are

$\{ 0, 1, 2, \ldots, 1000 \}$

the natural numbers from 0 through 1000"

page 23

"The set whose members are

$\{ 0, 1, 2, \ldots \}$

the natural numbers"

page 27

"The set whose members are

$\{ x \mid x < 6 \}$

all x that satisfy the condition that x is less than six"

page 33

$\overleftrightarrow{RS}$ "Line RS"

page 38

$\overline{XZ}$ "Segment XZ"

page 39

"Segment AB is congruent to

$\overline{AB} \cong \overline{CD}.$

segment CD."

page 83

"The intersection of

$\{1, 2, 3, 4\} \cap \{2, 4, 6\}$

the set whose members are one, two, three, four and the set whose members are two, four, six"

page 83

"The intersection of C and D equals

$C \cap D = \{2, 4\}.$

the set whose members are two, four."

page 87

"The union of

$\{5, 10\} \cup \{6, 8, 10\}$

the set whose members are five, ten and the set whose members are six, eight, ten"

page 87

"The union of C and D equals

$C \cup D = \{5, 6, 8, 10\}.$

the set whose members are five, six, eight, ten."

page 91

"x plus three is greater than seven and

$x + 3 > 7 \wedge 6 + x < 14.$

six plus x is less than fourteen."

page 91

"The intersection of

$\{x|x+3>7\} \cap \{x|6+x<14\}$

the solution set of $x+3>7$ and

the solution set of $6+x<14$"

page 104

$\overrightarrow{AB}$

"Ray AB"

page 105

$\angle ABC$

"Angle ABC"

page 109

$\triangle ABC$

"Triangle ABC"

page 123

"The ordered pair

(Paris, Spain)

whose first component is Paris and

whose second component is Spain"

page 124

(1, 2)

"The ordered pair one, two"

page 127

$A \times B$

"A cross B"

page 128

(a, b)

"The ordered pair a, b"

page 138

$\{(x, y) \mid x > y\}$

"The set whose members are all ordered pairs x, y that satisfy the condition that x is greater than y"

page 139

point (1, 4)

"Point 1, 4"

page 169

$(2, 7) \sim (4, 14).$

"The ordered pair two, seven is equivalent to the ordered pair four, fourteen."

page 174

$1/15 \sim 2/30.$

"One to fifteen is equivalent to two to thirty."

page 174

$\{ 4/7, \ldots \}$

"The set whose members are the rate pair 4/7 and all rate pairs equivalent to 4/7"

page 175

$\{(x, y) \mid x/y \sim 4/7\}$

"The set whose members are the ordered pairs x, y that satisfy the condition that x to y is equivalent to four to seven"

page 187

88 %

"Eighty-eight per cent"

page 216

1033_{four}

"one zero three three, base four"

page 227

10^2

"The second power of ten"

page 238

256, 193, 462, 109

"256 billion 193 million 462 thousand 109"

page 249

"The natural number 0 equals the number of members in the empty set."

$0 = n\{\}$.

page 257

"The number of members in A"

$n(A)$

page 257

"The number of members in the union of A and B"

$n(A \cup B)$

page 262

"The number of members in D cross E"

$n(D \times E)$

page 311

$\frac{a}{b}$ "The fraction a over b"

page 311

"The fraction one third is equivalent to the fraction five fifteenths."

$\frac{1}{3} \sim \frac{5}{15}$.

page 314

"The set whose members are $\{\frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \ldots\}$ $\frac{1}{5}$ and all fractions equivalent to $\frac{1}{5}$"

page 371

"972 and 564 thousandths"

972.564

page 372

"57 hundredths"

$.57$

page 385

"Point six, three repetend six, three"

$.\overline{63}$

page 385

"Point one, six repetend six"

$.1\overline{6}$

Answers to ? exercises

Answers to ? exercises are keyed to your book by page number and exercise letter. For example, the answer for exercise O on page 6 is labelled 6 O. When an exercise includes two or more questions, the answers to these questions are separated by an asterisk (*).

6 O No * The sentence in D3 does not tell you who has a birthday in September. Sentence A in D4 tells you that Richard has a birthday in September.

7 F No. You obtain a different sentence that expresses the same false idea.

11 B The set of odd numbers from 1 through 9

13 E There are many sentences that express the same idea as each of the sentences in D1. Examples are: "Three is less than 4" and "III is less than four."

14 G The idea expressed by a closed sentence is either true or false. The idea expressed by an open sentence is a requirement that is neither true nor false.

17 H No * $3y = 12$ is a condition, not a statement, and therefore cannot be an equation. * Yes * $3(4) = 12$ is a true statement that includes the idea of equality and therefore is an equation.

18 P Yes * Since only members of the universe are used as replacements for the variable in a condition, each member of the solution set must also be a member of the universe.

19 E Yes. {7, 9, 11} and {9, 11, 7} have exactly the same members. * Yes. R and T are different names for the same set.

19 F Yes * S and T are different names for the same set. In the tabulation {13, 14, 15, 13}, a name for the number thirteen is listed twice but the members of {13, 14, 15, 13} are exactly the same as the members of {13, 14, 15}.

21 M Nine * No. None of the statements obtained includes the idea of equality. * No. The condition does not include the idea of equality.

23 I 2,000,000

23 L No. For any natural number that you name, you can name another natural number that is greater. * There is no end to the list of names of the members in the set of natural numbers.

23 P No * If you could, there would be a greatest natural number, but you know that there is no greatest natural number. There are more members in the set of natural numbers than any natural number you can name.

25 D Yes * Yes. When each member of one set is also a member of a second set, the first set is a subset of the second set.

25 E Yes * Yes * Yes When one set is a subset of a second set and the second set is also a subset of the first set, the two sets are the same set.

29 C If your pencil is sharp enough, you can place many dots between any two dots. There would be even more points between any two points represented by the dots you make.

29 D Think of the set of points between two given points. Picture E represents such a set. There is no end to the number of points in this set. Therefore picture E represents an infinite set of points.

30 N Space is an infinite set of points because you cannot use a natural number to tell how many points there are in space.

31 H No * Since a line extends endlessly in opposite directions you cannot say one line is longer than another line. * Line b contains different points from line c.

32 M Yes * You can picture as many lines as you want included in a plane.

32 N Yes. Even with a pencil having the sharpest tip possible, you cannot represent all the lines included in a plane.

33 D Yes * Think of making sketches of lines that contain C. You can make sketches of lines as close together as possible and there are still many lines containing C that you have not represented.

33 E No * There are many planes that contain C, so all the lines that contain C would not be included in one plane.

33 H Yes. D3 represents only 4 of the geometric figures, each of which contains E and F. It is impossible to make sketches of all the geometric figures that contain E and F.

34 P Yes * Since any two points determine a line, you know that two points are always contained in the same line.

34 D Yes. If the stick is straight, it will touch the board all along its length.

34 F Yes. Since each point in $\overleftrightarrow{AC}$ is also in the plane, $\overleftrightarrow{AC}$ is a subset of the plane. * Yes. $\overleftrightarrow{CB}$ is the same line as $\overleftrightarrow{AC}$.

35 B Yes * Points A and B are in the plane suggested by the surface of the book cover. Since $\overleftrightarrow{AB}$ contains two points of this plane, $\overleftrightarrow{AB}$ is included in the plane.

35 D No * The book could have more leaves to suggest other planes that include $\overleftrightarrow{AB}$.

35 E Yes * No matter how many leaves the book had, they could not suggest all the planes that include $\overleftrightarrow{AB}$.

35 K Yes * Points A, B, and D are noncollinear points. Two of these points determine a line and the third point is not in the line. There is exactly one plane that includes a given line and contains a point not in the line. * Yes * Points A, B, and E are also noncollinear and are contained in exactly one plane.

35 L Yes * Two of the points determine a line that does not contain the third point. Only one plane includes this line and contains the third point. * No

38 T The property that for any two points, there is at least one point between them. Suppose that the two given points are A and B. The property states that there is at least one point between A and B. You can name this point C. Now

you can locate at least two more points between A and B. One of these points is between A and C and the other point is between C and B. You can keep on locating points between A and B in this same way.

39 G Yes * Each member of $\overline{XW}$ is also a member of $\overleftrightarrow{XW}$.

39 C No * No * $\overline{AB}$ and $\overline{GH}$ do not contain the same points and therefore are not the same. Since equality means "the same as," $\overline{AB} = \overline{GH}$ is not a true statement.

41 F Yes. Since every plane extends endlessly, each plane separates space into two sets of points, or half-spaces.

42 M Yes. Since every line extends endlessly in opposite directions, each line included in a plane separates the plane into two half-planes.

42 S Yes * Yes * Yes * Yes. Each point in a line is the boundary of two half-lines.

42 T Yes * H is not between I and J.

43 J No * No * If a segment is less than another segment, the first segment cannot also be the same size or greater than the second segment.

46 J No * Any point that satisfies $\overline{AX} < \overline{AB}$ determines a segment that is less than $\overline{AB}$, while any point that satisfies $\overline{AX} \cong \overline{AB}$ determines a segment that is the same size as $\overline{AB}$. * No * Any point that satisfies $\overline{AX} > \overline{AB}$ determines a segment that is greater than $\overline{AB}$, and this point cannot also determine a segment that is the same size as $\overline{AB}$.

46 N Yes * Point B * Yes. B satisfies $\overline{AX} \cong \overline{AB}$.

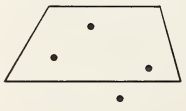
46 P Yes. Each point in a circle together with the centre of the circle determines a radius of the circle. * Since a circle contains infinitely many points and each of these points together with the centre determines a radius, a circle has infinitely many radii.

47 K Both endpoints of a chord are in the circle and only one endpoint of a radius is in the circle.

47 N A sphere. There would be infinitely many radii in infinitely many planes. The endpoints of all these radii in all the planes would determine a sphere.

47 A Yes * Three points must be either collinear or noncollinear. You already know that three noncollinear points determine a plane. You also know that if three points are collinear, they are contained in the same line, and that a plane that contains two of the points must contain the third point. Since the three points are in the same plane, they are coplanar points.

47 B In your sketch, one of the dots should not be in the picture of the plane. Your sketch should be similar to the one below.



48 I Circle R is the solution set of $\overline{RX} \cong \overline{RS}$ and the interior of circle R is the solution set of $\overline{RX} < \overline{RS}$.

48 M Circle R is the solution set of $\overline{RX} \cong \overline{RS}$ and the exterior of circle R is the solution set of $\overline{RX} > \overline{RS}$.

51 M No. Points to the right of point 26 are associated with natural numbers that are greater than 26. The condition $x < 26$ requires that each member of $\{x | x < 26\}$ be less than 26. * Yes. Points to the left of point 26 are associated with natural numbers that are less than 26.

53 Q The graph shown in D14 could be the graph of the solution set of infinitely many conditions. Some examples of these conditions are: $x > 3$, $x + 1 > 4$, and $x + 16 > 19$.

57 L No

57 Q No

62 B Because both $15 - n$ and 9 are equal to the number of quarters Sue had left.

62 D Because both $x + 5$ and 39 are equal to the number of coins George had after Joe gave him the 5 coins.

66 C Conditions for inequality * In lesson 17 you made conditions for equality for the problems.

69 F An example of a problem that can be solved by using the condition $15 + n = 27$ is: Bill had 15 model cars. After he made some more, he had 27 model cars. How many more model cars did Bill make?

70 K An example of a problem that can be solved by using the condition $x + 8 = 17$ is: After Mrs. Morgan bought 8 more hamburger buns, she had 17 hamburger buns in all. How many hamburger buns did she have before she bought the 8?

70 Q An example of a problem that can be solved by using the condition $n - 9 = 15$ is: Dick spent 9 cents of his money and then had 15 cents left. How much money did Dick have before he spent the 9 cents?

71 V An example of a problem that can be solved by using the condition $32 - n = 13$ is: Helen had 32 magazines. She gave some of these magazines to her sister and then had 13 magazines left. How many magazines did Helen give to her sister?

71 I An example of a problem that can be solved by using $18 + 24 = w$ is: Jane made 18 cupcakes in the morning and 24 cupcakes in the afternoon. How many cupcakes did she make altogether?

72 P An example of a problem that can be solved using $237 - 49 = n$ is: Mary is reading a book that has 237 pages. She has read 49 pages. How many pages does she have left to read before she finishes the book?

75 S The additional information given for problem C is different from the additional information given for problem D.

82 F Sets A and B were used to find members for the set {Peru, Chile}. The members of {Peru, Chile} are the only countries that belong to both set A and set B.

83 L Yes * The intersection of D and C is {2, 4} because 2 and 4 are the only objects that belong to both D and C. You already know that the intersection of C and D is {2, 4}. * Yes * Since $D \cap C$ and $C \cap D$ have the same members, they are equal sets, and $C \cap D = D \cap C$ is a true statement.

83 N No * No * 5 is a member of F but not E. 10 is a member of E but not F. The intersection of E and F contains those objects, and only those objects, that are members of both set E and set F.

85 I Their intersection is the empty set because disjoint sets have no common members.

85 O Their intersection is not the empty set because sets that meet have some common members.

86 V If one set is a subset of a second set, the intersection of the two sets is the first set.

86 W The intersection of a set with itself is the set itself. * The common members of a set and itself are all the members of the set.

86 X Yes * Since each member of the intersection of two sets is also a member of each of the two sets, you know that the intersection of two sets is a subset of each of the two sets. If the intersection of two sets is the empty set, however, the intersection is also a subset of both sets, since the empty set is a subset of every set.

86 Y The empty set * A set and the empty set have no common members and therefore have no members in their intersection.

86 E Sets S and M were used to furnish the members for this set. Each teacher who is a member of either S or M, or of both S and M, is a member of {Mr. Adams, Miss Evans, Mr. Wilson, Mr. Cox}.

87 N Yes * The union of D and C is {5, 6, 8, 10}. You already know that the union of C and D is {5, 6, 8, 10}. * Yes * Since $D \cup C$ and $C \cup D$ have the same members, they are equal sets. Therefore, $C \cup D = D \cup C$ is a true statement.

88 Q No * 6 does not belong to either E or F. * Yes * 0 belongs to E. * Yes * 11 belongs to F. The union of E and F contains those objects, and only those, that belong to either set E or set F or to both set E and set F.

88 Z $K \cap L = \{4, 5, 6, \ldots, 10\}$. * Yes * You know that $K \cup L = \{0, 1, 2, \ldots\}$. Therefore, each member of $K \cap L$ is also a member of $K \cup L$.

89 O Yes * Set A contains all the cities that are in either A or B, or in both A and B. A does not contain any other members, so set A is the union of sets A and B.

89 R Their union is the second set.

89 S Yes * Since each member of A is also a member of $A \cup B$, A is a subset of $A \cup B$. * Yes * Since each member of B is also a member of $A \cup B$, B is a subset of $A \cup B$. * Yes * Since each member of $A \cap B$ is also a member of $A \cup B$, $A \cap B$ is a subset of $A \cup B$.

89 T The given set * The given set contains all the members that are in the given set or the empty set, or both the given set and the empty set.

89 U The set itself * You know that the union of two sets contains only those members that are in either set, or in both sets. You also know that the union of two sets must contain all the members of each set. Therefore, the union of a set with itself must be the set itself.

89 V Each of the sets must be the empty set. If one of the sets had at least one member, the union of the two sets would have at least one member.

89 W The two sets are the same set. Both the union and the intersection of a set with itself are the set itself. So, if the union of two sets is the same as the intersection of these two sets, they must be the same set.

91 H Yes * $\{x|x+3>7\} = \{5, 6, 7, \ldots\}$. $\{x|6+x<14\} = \{0, 1, 2, \ldots, 7\}$. 5, 6, and 7 are the only numbers that belong to both $\{x|x+3>7\}$ and $\{x|6+x<14\}$. Therefore, {5, 6, 7} is the intersection of $\{x|x+3>7\}$ and $\{x|6+x<14\}$.

92 I Yes * They contain exactly the same members.

92 P The intersection of the solution sets of two simple conditions contains those members, and only those members, that satisfy both simple conditions. You know that the solution set of a compound condition that includes "and" also contains those members, and only those members, that satisfy both simple conditions. Therefore, the solution set of the compound condition and the intersection of the solution sets of the simple conditions are the same.

93 G In D11, the dots in the graph of $\{x|x<9\}$ are encircled. The dots in the graph $\{x|x<9 \wedge x+2<8\}$ are some of these encircled dots. In the Venn diagram you made for exercise E, the picture of a circle that represents $\{x|x<9 \wedge x+2<8\}$ should be in the interior of the picture of a circle that represents $\{x|x<9\}$. * In D11, all the dots that are enclosed by squares are also encircled. In the Venn diagram, the picture of a circle that represents $\{x|x+2<8\}$ also represents $\{x|x<9 \wedge x+2<8\}$.

95 F No * Each member of $\{x|x+6>12\}$ does not satisfy the other simple condition for the problem.

95 I The intersection, since it contains only those members that belong to both sets.

100 L One. There is exactly one plane that contains three noncollinear points. * If E were a member of both plane ABD and plane ABC, then two planes would contain the three noncollinear points A, B, and E.

100 M Since the two planes have common members, the planes must intersect. Two planes are parallel only if they have no common members. *

You know that the intersection of two planes that meet is a line. Since three points are in the intersection of the two planes, the points must be members of the same line.

100 N Since planes FGH and EGH have points G and H in common, points G and H are in the intersection of planes FGH and EGH. Therefore, the intersection of plane FGH and plane EGH is line GH.

101 C Points L and N are contained in ℓ_3. Since ℓ_3 contains two points of plane LMN it is included in the plane $* \ell_3$

101 G No * You do not know whether or not $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are included in the same plane. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel only if they are included in the same plane.

102 I ℓ_4 and ℓ_5 are not included in one plane and are skew lines. Their intersection is the empty set.

102 J ℓ_6 and ℓ_7 are included in the same plane and are parallel.

102 P One. If two lines intersect, the lines are included in the same plane.

104 C The set of points that contains A and all the points in the line to the left of A

105 I $\overrightarrow{QP}$ is the set of points consisting of Q and all points in ℓ_1 to the left of Q.

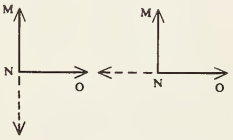
105 K No * They have different endpoints and extend in opposite directions.

105 E Yes. The vertex of an angle together with a point in each side of the angle are three noncollinear points. Any three noncollinear points are contained in the same plane. * Yes

105 F No * The sides of an angle are rays, and a ray extends endlessly in one direction, so you cannot say that one side of an angle is longer than the other side. No * For the same reason, it is not sensible to say that one angle has longer sides than another angle.

107 G $\angle GHI \cong \angle MNO$ is a true statement because congruent angles are the same size. If two angles are the same size as a third angle, then the two angles must also be the same size.

107 J The sketches below illustrate how the picture of either one of the sides of $\angle MNO$ may be extended to make a sketch of the supplement of $\angle MNO$. * Yes. A right angle is congruent to its supplement.



109 B Yes * The union of these three sets contains all the points that belong to each of the three sets.

109 E No * There is exactly one plane that contains three noncollinear points. Therefore, the vertices of a triangle are contained in one plane. Since a line that contains two points of a plane is included in that plane, the sides of a triangle are included in the plane determined by the vertices.

109 G No * The sides of $\angle ABC$ contain points that are not members of the sides of $\triangle ABC$.

110 G Yes * A triangle that has three congruent sides must have two congruent sides.

110 H Yes * Since each member of the set of equilateral triangles is also a member of the set of isosceles triangles, the set of all equilateral triangles is a subset of the set of all isosceles triangles.

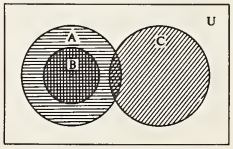
110 I No * Isosceles triangles have only two congruent sides and are not members of the set of all equilateral triangles.

110 N Any figure that has more than one right angle cannot be a triangle.

111 O A right triangle may be an isosceles triangle. * A right triangle cannot be an equilateral triangle.

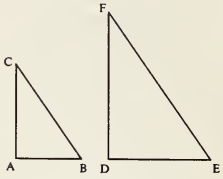
111 P See the diagram below that is to be made for both this exercise and for exercise Q. * $A \cap B$ is the set of all equilateral triangles and is a subset of set A.

111 Q See the diagram below. * $B \cap C$ is the empty set.



111 R $C \cap D$ is the set of all scalene triangles having one right angle. Sets C and D meet.

111 F The drawing below may be used to illustrate that the three sides of one triangle need not be congruent to the three sides of the other even though three angles of one are congruent to three angles of the other.



$\angle ABC \cong \angle DEF$.
$\angle ACB \cong \angle DFE$.
$\angle CAB \cong \angle FDE$.

111 G The three angles of one triangle would be congruent to the three angles of the other triangle if three sides of one are congruent to three sides of the other.

113 K By the portion of the sketch that has pictures of both horizontal and vertical lines.

114 M The intersection contains one point since there is a point in one of the sides of the angle that is between C and D.

114 R The intersection contains one point since there is a point in one of the sides of the triangle that is between M and N.

115 D Two. The simple closed curve is the boundary of each of these regions.

115 E It is not a simple closed curve or an angle, because each of these figures separates the plane into two regions.

115 F The intersection contains at least one point since there is at least one point in the simple closed curve that is between the two points.

122 C No * The sentence in D2 is an open sentence. An open sentence expresses

a condition, not a statement. A condition is neither true nor false.

123 R The symbol (Paris, Norway) is read "The ordered pair whose first component is Paris and whose second component is Norway." The symbol {Paris, Norway} is read "The set whose members are Paris, Norway." * (Paris, Norway) is an ordered pair while {Paris, Norway} is a set having two members.

124 H $x + x = 4$ has one variable. $x + y = 4$ has two variables. * $x + x = 4$ requires you to find a number which added to itself equals 4. $x + y = 4$ requires you to find any two numbers whose sum is 4.

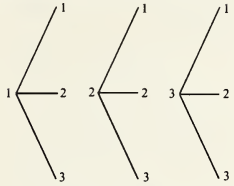
127 F By finding the product of the number of members in set M and the number of members in set N

127 G No * The first component of (June, Thanksgiving) is a member of B and the second component is a member of A. * Yes * (June, Thanksgiving) is a member of $B \times A$, not of $A \times B$. No * Since A and B are different sets, the first components of the members of $A \times B$ are different from the first components of the members of $B \times A$. The same is true for the second components.

128 L Yes * Each member of the solution set of "*a* is a holiday in *b*" is also a member of the universe for (*a*, *b*).

129 Q The members of set S. Since *y* is the first component of (*y*, *x*), you must use the first components of members of $S \times T$ as replacements for *y*. * The members of set T. Since *x* is the second component of (*y*, *x*), you must use the second components of members of $S \times T$ as replacements for *x*.

129 A One possible diagram is shown below.



129 E The universe is given for (*m*, *n*). *m* is the first component of (*m*, *n*) and must be replaced by the first component of (1, 3). *n* is the second component of (*m*, *n*) and must be replaced by the second component of (1, 3).

131 G Yes * Yes * (1, 0)

132 N Yes * Each dot in the graph of the solution set of $x + 2 < y$ is also in the graph of $A \times B$.

133 L Yes * The solution set of $y + 3 = x$ is {(3, 0)}. The encircled dot in the graph in D8 represents the point that is associated with (3, 0).

134 C The universe is a set of numbers, not a set of ordered pairs of numbers.

134 G The universe, $A \times A$, is given for (*x*, *y*), so the first components of the members of $A \times A$ may be used as replacements for *x*.

136 C No * Since no member of the set tabulated in exercise A satisfies $x < y$, the set cannot be the solution set of $x < y$. * Yes * The set tabulated in exercise A contains all the members of the universe that satisfy $x > y$, so this set is the solution set of $x > y$.

136 F There are many conditions that have {(0, 2), (1, 3), (2, 4), (3, 5)} as a solution set. Some examples of these conditions are $x + 2 = y$, $x + 3 = y + 1$, $y - 2 = x$.

138 C The condition $x > y$ * $\{(x, y) | x > y\}$ contains all the members of the universe, and only those members of the universe, that satisfy $x > y$.

139 C Yes * Both components of each member of $N \times N$ are natural numbers. Each ordered pair whose components are natural numbers is a member of $N \times N$.

139 D Yes * Since N is an infinite set, then the Cartesian set of N must also be infinite.

140 B Yes * Since the graph of the solution set is a subset of the universe, the graph of the universe includes a graph of the solution set.

141 P To make a graph of the solution set of $x + y < 4$, use all the dots below those in the solution set of $x + y = 4$. * To make a graph of the solution set of $x + y > 4$, use all the dots above or to the right of the dots in the graph of the solution set of $x + y = 4$.

144 L If you replaced *x* by 5, you would obtain the condition $5 + y = 4$. There is no member of B that satisfies this condition. If you replaced *x* by 6, you would obtain the condition $6 + y = 4$. There is no member of B that satisfies this condition.

Answers to ?page 148

144 C If a number greater than 3 is used as a replacement for *x* in $x + y < 4$, no member of B will satisfy the condition obtained.

145 F No * For each replacement of *r* by a natural number, there is a replacement for *s* that satisfies the condition obtained. Therefore, the solution set of $s = 2r + 1$ is an infinite set.

145 L $s > 2r + 1$. * One possible chart is shown below.

r	0	1	2	3	4
s	2	4	6	8	10
	3	5	7	9	
	4	6	8	10	
	5	7	9		
	6	8	10		
	7	9			
	8	10			
	9				
	10				

147 C The universe is a set of ordered pairs.

148 J Yes. $\{(x, y) | x + y = 6 \wedge y = 2x\}$ contains all the members of the universe that satisfy both $x + y = 6$ and $y = 2x$. The intersection of $\{(x, y) | x + y = 6\}$ and $\{(x, y) | y = 2x\}$ also contains all the members of the universe that satisfy both $x + y = 6$ and $y = 2x$.

148 Q $\{(x,y) | x + y < 7\} \cap \{(x,y) | y > x\}$ contains only those members of the universe that belong to both $\{(x,y) | x + y < 7\}$ and $\{(x,y) | y > x\}$. These are the same members of the universe that belong to $\{(x,y) | x + y < 7 \wedge y > x\}$.

149 C Yes. The condition $x + 1 = 4$ makes no requirement about the second component of an ordered pair. * Yes

149 J Yes. The condition $y + 1 < 7$ makes no requirement about the first component of an ordered pair.

159 K Each solution of the compound condition must satisfy both of the simple conditions. If a number less than 5 is used to replace a in $a - 5 = b$, the condition obtained cannot be satisfied by a member of N. For example, if a is replaced by 4, the condition obtained is $4 - 5 = b$. There is no member of N that satisfies $4 - 5 = b$.

168 J No * There are infinitely many ordered pairs that represent the rate at which the pencils are sold. Examples of other ordered pairs that represent this rate are (5, 20), (10, 40), (20, 80).

169 A No * To decide if (a,b) and (c,d) are equivalent ordered pairs, you must know what the replacements for the variables are.

170 Q Yes * Any ordered pair is equivalent, as well as equal, to itself. For example: you know that (2, 4) is equivalent to (2, 4) because $2 \times 4 = 2 \times 4$.

171 O Yes. The product of 6 and the second component of each ordered pair will be equal to the product of the first component of the ordered pair and 8.

174 G Each member of a proportional relation is equivalent to each of the other members. If you know one member, you can tell which rate pairs are members of the proportional relation and which are not members.

175 I Yes * 4/7 is a member of the proportional relation {8/14, . . .}, and each member of a proportional relation is equivalent to each of the other members. * Yes * Since 8/14 and 4/7 are equivalent rate pairs, {8/14, . . .} and {4/7, . . .} are the same proportional relation. * Yes * Since 100/175 and 4/7 are equivalent rate pairs, {100/175, . . .} and {4/7, . . .} are the same proportional relation.

175 M Yes. Since 8/14 and 4/7 are equivalent rate pairs, $\{(x,y) | x/y \sim 8/14\}$ and $\{(x,y) | x/y \sim 4/7\}$ are the same proportional relation.

176 B The first component of 16/8 refers to the number of plants. Since 4 is the number of plants for which you want to find the cost, 4 must be the first component of the rate pair you are to find.

178 O {2} * {2} * Yes. The two sets contain the same member.

178 G 10/32 * $10/32 \sim 15/48$.

182 L Both of the rate pairs 20/60 and 60/20 represent the rate at which cars can be washed. If you think of the rate pair 20/60, then $20/60 \sim w/15$ is the condition for the problem. If you think of the rate pair 60/20, then $60/20 \sim 15/w$ is the condition for the problem.

183 M Both of the rate pairs 35/2 and 2/35 represent the rate. If you think of the rate pair 35/2, then you will use either the condition $s/4 \sim 35/2$, or the condition $35/2 \sim s/4$. If you think of the rate pair 2/35, then you will use either the condition $2/35 \sim 4/s$, or the condition $4/s \sim 2/35$.

186 C Room 104 * Yes * Room 104 was the only one of the five rooms that raised more money than its goal.

187 J Yes * All you need to do is compare the first components of the rate pairs that have second components of 100.

188 D No * No * To receive a grade greater than 100%, the first component of the rate pair representing the comparison would have to be greater than the second component. This means that Susan would have had to answer more questions correctly than were on the test, which is impossible.

191 C You know that $x\%$ is the same as $x/100$. Therefore, both "6 is $x\%$ of 10" and $6/10 \sim x/100$ require you to find the first component of a rate pair that has a second component of 100 and that is equivalent to 6/10.

192 I Yes * Yes. "30% of 320 is s" is the same condition as $30/100 \sim s/320$ since both conditions require you to find the first component of a rate pair that has a second component of 320 and that is equivalent to 30/100.

192 R You want to compare the number of tickets Martin sold with the number of tickets June sold. 12/8, not 8/12, represents this comparison.

192 V The second component. The problem asks you to find the first component of a rate pair that has a second component of 32 and that is equivalent to 125/100.

207 H No * Since the same symbol cannot be used more than once in a code numeral, the code system does not have the property of repetition. * No * Since each numeral in a code system consists of just one symbol, addition cannot be used to find the number expressed. Therefore, the code system does not have the property of addition.

207 I Yes. In a code system, a different symbol is used to name each member of {1, 2, 3, . . .}. Since {1, 2, 3, . . .} is an infinite set, an infinite set of symbols must be used to name the members of {1, 2, 3, . . .}.

207 J Yes * A tally system is easier to learn because it has only one symbol which is used again and again. * Yes * In a code system, each number is expressed with just one symbol. A number like 5000 is easier to express in a code system than in a tally system.

208 H No. There are only 9 stars and strokes in all but there are 30 objects in all.

208 J Yes * 14. Since each numeral contains 3 stars and 2 strokes, they name the same number. The arrangement of the

stars and strokes in the numeral does not matter.
209 P Since you are using a base of four, you must group by 4's. The new group contains 4 groups of 4 groups of 4.
210 V You must know what number is expressed by each symbol.
212 E You must be able to tell which strokes express the number 1 and which strokes express the number 3.
212 L Yes * By distributing the six strokes in the four spaces in two different ways.
212 M You could leave an open space to separate the strokes in one position from those in another position. Numeral E in D3 could be written / // / //.
213 N No * The symbols in the numeral could be rearranged in any order, but the numeral would still express the same number.
213 D The numbers expressed by the symbols in the numeral are different. For example, a stroke in the second space from the right in a base-ten numeral expresses the number 10, while a stroke in the same position in a base-three numeral expresses the number 3.
213 H Three strokes in any space may be replaced by one stroke in the next space to the left.
213 I 9 * Ten strokes in any space may be replaced by one stroke in the next space to the left.
214 U The greatest number of strokes needed in a space is one less than the base.
215 J 1 is the fourth digit from the right, not the third digit from the right.
217 I Think of a base-ten numeration system that uses strokes. The most strokes you need in any space is 9. This means that you need 10 different digits to express the numbers 0 through 9. * In a base-four system, you need 4 different digits to express the numbers 0 through 3.
217 E No * Ordinarily, you use only the digits from 0 through 9. There is no single digit that expresses the number 10.
218 L Since there is exactly one digit in each position in numeral D, you can determine what position a digit is in without using a chart.
220 L The decimal numeration system requires fewer symbols to name most numbers than the Egyptian numeration system requires. For example, the decimal numeral 1607 has four symbols, while numeral I in D3 has 14 symbols. There are some exceptions, of course. The Egyptian numeral for 1,000,000, for example, has only one symbol, while the decimal numeral has seven.
221 N No. It is true that XC and CX express different numbers, so position makes some difference; but in most cases, the symbols can be in any order. * No. For example, in the numeral XXX, each of the X's expresses the number 10.
221 O If a symbol that expresses a lesser number is written to the left of a symbol that expresses a greater number, you subtract the lesser number from the greater. If a symbol that expresses a lesser number is written to the right of a symbol that expresses a greater number, you add the lesser number to the greater. To find the number expressed by XCIII, you subtract 10 from 100 and then add 3 to the difference. To find the number expressed by CXIII, you add 100, 10, and 3.
222 W The arrangement of the symbols is important in XCVII and XLV. The symbol X must remain on the left of C to show that 10 is subtracted from 100. Similarly, the symbol X must remain on the left of L to show that 10 is subtracted from 50.
222 X The decimal system is more convenient than the Roman system. * Fewer symbols are necessary to express most numbers in the decimal system than in the Roman system. The use of subtraction and a mixed base makes Roman numerals more difficult to understand than decimal numerals.
223 I Yes * In the Babylonian numeration system, the position of each symbol in a numeral determined the number expressed by the symbol.
223 O Yes * Numeral J, because fewer symbols are used than in numeral I.
223 P The decimal numeration system is more convenient because it has only one base and more code symbols. For example to express each of the numbers from 2 through 9 in the decimal system, you use only one symbol while in the Babylonian system you would use more than one symbol for each of the numbers from 2 through 9.
226 N Yes. You know that if you choose any member of {1, 2, 3, . . .} and multiply this number by 1, the product will be the number you chose.
226 O Yes * Yes. Since the product of 1 and the number you chose is the number, both 1 and the number are factors of the number.
227 N Yes * Yes.
227 T Yes * If you replace n by 1, a true statement is obtained because the first power of any number is the number. If you replace n by any other counting number, a true statement is obtained because, no matter how many times 1 is used as a factor, the product is always 1.
232 V $285 \div 144 = 1$, remainder 141.
$141 \div 12 = 11$, remainder 9.
$285 = 1\text{£}9_{\text{twelve}}$.
232 W Yes. You know that both the numerals 346_{nine} and $1\text{£}9_{\text{twelve}}$ express the number 285.
236 M 0 and 1 are the only digits used in the base-two numeration system.
240 G 3000 * 3000 * 3×10^3 * 3 is a number from 1 to 10, and 10^3 is a power of 10. 30×10^3 is not in scientific notation because 30 is not a number from 1 to 10.

246 I Yes * In fact, there are 120 different ways in which set X and set Y can be put in one-to-one correspondence. The diagram below represents a third way in which set X and set Y can be put in one-to-one correspondence.

$X = \{3, 6, 1, 9, 5\}.$
$\quad\quad\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
$Y = \{21, 14, 17, 0, 6\}.$

247 M A finite set cannot be put in one-to-one correspondence with an infinite set, since there are more members in an infinite set than in a finite set.

247 N Yes. If two sets are equal, then they are the same set. Every set can be put in one-to-one correspondence with itself. Therefore, every set is equivalent to itself. * No. From your answers to exercise K and exercise L, you know that two sets, like set P and set T, that are not equal, can be equivalent.

248 M The diagram below represents a way in which set E and set N can be put in one-to-one correspondence.

$N = \{0, 1, 2, 3, 4, \ldots\}.$
$\quad\quad\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
$E = \{0, 2, 4, 6, 8, \ldots\}.$

248 O The diagram below represents a way in which the set of natural numbers (N) and the set of odd numbers (O) can be put in one-to-one correspondence.

$N = \{0, 1, 2, 3, 4, \ldots\}.$
$\quad\quad\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
$O = \{1, 3, 5, 7, 9, \ldots\}.$

250 D $\{0\}$ is a set with one member, and $\{\ \}$ is the set with no members. * $\{0\}$ is a set which has one member—zero, and $n\{\ \}$ is the number of members in the empty set, or the natural number 0.

251 T $\{0, 1, 2, \ldots, m\text{-}1\}$ * Tabulate the set that contains the natural numbers from zero through the number that is one less than the given natural number.

251 U Yes. The number of members in any given standard set is the same as the natural number associated with the set; therefore, the standard set of any natural number is a finite set. * No. Since you cannot name the greatest natural number, the set of natural numbers is an infinite set.

252 D No * You know that a proper subset of a set does not contain all the members of the set. The standard set of 3 contains all the members of the standard set of 3.

253 L Yes. Each natural number is less than infinitely many other natural numbers. For example, if a is replaced by 5 in $a < b$, you obtain $5 < b$. The solution set of $5 < b$ is $\{6, 7, 8, \ldots\}$.

253 O Point a is to the left of point b.

254 S Point b is to the right of point a.

254 E Yes. Since the set contains 0, and 1 is the successor of 0, the set also contains 1. Since the set contains 1, and 2 is the successor of 1, the set also contains 2. Since 2 is a member of the set, so is the successor of 2, which is 3. You can continue in this way and show that any given natural number is contained in the set.

254 F No * Zero is the only natural number that is not the successor of a natural number because it does not immediately follow another natural number.

254 M Yes * The set of natural numbers between any two given natural numbers has a greatest member that is one less than the greater of the two given numbers. Any set of natural numbers that has a greatest member is a finite set.

257 J $S \cup T$

258 L No * Sets A and B are not disjoint sets. These two sets have two common members, 18 and 19.

259 K Yes. Natural numbers are associated only with finite sets, and each finite set has a natural number associated with it. The union of two finite sets is also a finite set. Therefore, the sum of two natural numbers is a number that is associated with a finite set, and the sum of the components of each pair of natural numbers is a natural number.

260 I No * There is no natural number that you can add to 5 to obtain a sum of 1.

260 N No * There is no natural number that is the difference of the components of an ordered pair in which the first component is less than the second component.

260 O The replacement for b. Since you know that the difference of a and b is a natural number, you also know that the sum of b and the natural number is a. Therefore, the replacement for a must be equal to or greater than the replacement for b.

262 H The Cartesian set $S \times T$

263 L The empty set * There are no members in $\{\ \}$ to match with each of the members of $\{2, 3, 4, 5\}$.

263 M Yes * The Cartesian set of $\{\ \}$ and $\{2, 3, 4, 5\}$ is the empty set. Therefore, the product of 0 and 4 is the number associated with the empty set, which is 0.

263 Q $a \times b$, or c

263 R Yes * An ordered pair of numbers is used to obtain a product.

265 K $A \cup B$ * $B \cup A$ * Yes, since $A \cup B = B \cup A$ * Yes

266 Q No * The sets do not have the same members. For example, (13, 0), which is a member of $R \times S$, is not the same as (0, 13), which is a member of $S \times R$.

266 V $A \times B$ * $B \times A$ * Yes. $A \times B$ and $B \times A$ are equivalent sets. * Yes. The same number is associated with equivalent sets.

267 B No * You know that $1 - 0 = 1$ because 1 is the sum of 0 and 1. You also know that $0 - 1 = 1$ is a false statement because 0 is not the sum of 1 and 1.

268 E One * If you obtain one false statement from $a - b = b - a$, then you know that, for each replacement of the variables in $a - b = b - a$, you do not obtain a true statement. This means that $a - b = b - a$ is not a property.

268 G No * You know that, for some replacements of the variables, you do not obtain a true statement from $a \div b = b \div a$.

270 J Because of the associative property of addition of natural numbers, you know that the way in which you group the numbers 4, 3, and 6 does not affect the sum. Therefore, you know that the symbol $4 + 3 + 6$ names the number 13.

271 I Because of the associative property of multiplication of natural numbers, you know that the way in which you group the numbers 2, 5, and 8 does not affect the product. Therefore, you know that the symbol $2 \cdot 5 \cdot 8$ names the number 80.

272 E No * If division of natural numbers had the associative property, then you would obtain a true statement from $(a \div b) \div c = a \div (b \div c)$ for each replacement of a, b, and c. You know that, for at least one set of replacements for a, b, and c, you do not obtain a true statement from $(a \div b) \div c = a \div (b \div c)$. Therefore, you know that division of natural numbers does not have the associative property.

272 G Yes * You know a first number, $(8 \cdot 73)59$, is the same as a second number, $59(8 \cdot 73)$. You also know that the second number, $59(8 \cdot 73)$, is the same as a third number, $(59 \cdot 8)73$. Therefore, you know that the first number is the same as the third number.

275 M One way is first to find the sum of b and c. Then multiply this sum by a. Another way is first to find the product of a and b and the product of a and c. Then find the sum of these products.

275 O In $a(b + c) = (ab) + (ac)$, a is the multiplier of $(b + c)$, of b, and of c. In the condition expressed in D4, a has been multiplied by $(b + c)$, by b, and by c.

275 U Yes * You know that $a(b + c) = (ab) + (ac)$ is a property of natural numbers. Since you obtained $(b + c)a = (ba) + (ca)$ from $a(b + c) = (ab) + (ac)$, you know that $(b + c)a = (ba) + (ca)$ is also a property of natural numbers.

276 A For any three natural numbers, the sum of the first number and the product of the second and third numbers is the same as the product of the sum of the first and second numbers and the sum of the first and third numbers.

276 E No. If addition did distribute over multiplication, then, for each replacement of a, b, and c, you would obtain a true statement from $a + (bc) = (a + b)(a + c)$. Since you know at least one set of replacements for which you do not obtain a true statement, you know that addition does not distribute over multiplication.

277 K Yes * Yes. When b is greater than or equal to c, you obtain a true statement from $a(b - c) = (ab) - (ac)$.

282 F Yes * Yes * The set of natural numbers that have 2 as a factor.

283 L Using the distributive property, you know that $2m + 2n = 2(m + n)$. You also know that $2(m + n)$ is an even number because it has a factor of 2.

283 M $(2m)(2n) = 2(m)(2n)$, because of the associative property of multiplication. $2(m)(2n) = 2(m \cdot 2n)$, because of the associative property of multiplication. $2(m \cdot 2n)$ is an even number because it has a factor of 2.

287 F Addition is commutative in set T because you always obtain a true statement from $a \oplus b = b \oplus a$ when the universe for a and b is set T. * Muldivision is commutative in set T because you always obtain a true statement from $a \otimes b = b \otimes a$ when the universe for a and b is set T.

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287 P Muldivision distributes over addition in set T. * For each replacement of the variables from set T, a true statement is obtained from $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$.

287 Q Yes * Set T, together with the operations of addivision and muldivision and their properties, behaves like the natural-number system.

289 I Yes * You know that $0 + a = a$ is a property. You can obtain $a + 0 = a$ from $0 + a = a$ by using the commutative property of addition. Since $0 + a = a$ is a property, $a + 0 = a$ is also a property.

290 L Yes * You know that $a + 0 = a$ is a property. $a + 0 = a$ is, of course, the same as $a = a + 0$. Since $a - a = 0$ also means $a = a + 0$, then $a - a = 0$ must be a property.

290 M Yes * Since $a = 0 + a$ is a property, $a - 0 = a$ must also be a property.

291 J Yes * You know that $1 \cdot a = a$ is a property. You also know that $1 \cdot a = a \cdot 1$ because of the commutative property of multiplication. Therefore, $a \cdot 1 = a$ is also a property.

291 L Yes * Since $a = 1 \cdot a$ is a property, you know that $a/a = 1$ must also be a property.

291 M Since you know that $a = a \cdot 1$ is a property, you

also know that $a/1 = a$ is a property.

292 G You know that $0 \cdot a = 0$ is a property. Since you can obtain $a \cdot 0 = 0$ from $0 \cdot a = 0$ by using the commutative property of multiplication, you know that $a \cdot 0 = 0$ is also a property.

292 H Yes * You know that you obtain a true statement from $0 = 0 \cdot a$ for each replacement of a from {1, 2, 3, . . .}. Since $0/a = 0$ also means $0 = 0 \cdot a$, you know that you obtain a true statement from $0/a = 0$ for each replacement of a from {1, 2, 3, . . .}.

292 L Yes * Yes. Since $0/0 = b$ also means $0 = b \times 0$, the solution set of $0/0 = b$ is the same as the solution set of $0 = b \times 0$. The solution set of $0 = b \times 0$ is {0, 1, 2, . . .}, so the solution set of $0/0 = b$ is also {0, 1, 2, . . .}.

296 Q You know that 1 is a factor of every member of C and that every member of C is a factor of itself.

296 D Yes * 2 is a factor of 4. * Yes * Each number that has 4 as a factor also has 2 as a factor. * You should write the numeral 4 after the names of the following numbers in your chart: 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, and 48.

297 F You know that each numeral that is crossed out names a number that has a factor other than itself and 1. * 25

297 G {2, 3, 5, 7} * No * The number 1 has only one factor.

297 L 49 * {1, 7, 49} * Three

301 S No * If 1 were considered a prime, every counting number greater than 1 could be factored into primes in infinitely many ways. For example, the number 6 could be expressed as 2×3, $1 \times 2 \times 3$, $1 \times 1 \times 2 \times 3$, and so on.

301 G Every prime has just two factors, itself and 1. Therefore, you know that 1 is the only common factor of any pair of prime numbers.

311 V Yes * For each replacement of b and d, $0 \times d = 0 \times b$; therefore, $\frac{0}{b}$ and $\frac{0}{d}$ are equivalent fractions.

312 E If you multiply both the numerator and denominator of $\frac{2}{3}$ by the same member of C, you will obtain a fraction that is equivalent to $\frac{2}{3}$.

313 C By using the definition of equivalent fractions * For each replacement of the variables, $a(kb) = (ka)b$.

313 D Associative property of multiplication of natural numbers.

314 E Commutative property of multiplication of natural numbers.

314 F After replacement of the variables, sentence A tells you that a first number, $a(kb)$, is equal to a second number, $(ak)b$. Sentence B tells you that the second number, $(ak)b$, is equal to a third number, $(ka)b$. Therefore, the first number must be equal to the third number. This is the condition expressed by sentence C. * Yes

315 O $\frac{4}{1}$. Since 3 is the only common factor of 12 and 3, other than 1, you obtain the basic fraction that is equivalent to $\frac{12}{3}$ by dividing both the numerator and denominator of $\frac{12}{3}$ by 3.

317 L Yes. By replacing n in $\frac{2n}{3n}$ by each member of C in turn, you obtain the set of all the fractions equivalent to the basic fraction $\frac{2}{3}$. * Yes. You know that, for each replacement of n, the fraction $\frac{2n}{3n}$ indicates the same rational number as the fraction $\frac{2}{3}$; therefore, the rational number $\frac{2n}{3n}$ is equal to the rational number $\frac{2}{3}$.

318 A Yes. Because of the reduction property of rational numbers, you know that the rational number $\frac{2}{2}$ is equal to the rational number $\frac{1}{1}$. * Yes. The fraction numeral $\frac{1}{1}$ names the rational number $\frac{2}{2}$.

318 E Yes. You know that there are infinitely many prime numbers. Therefore, there are infinitely many pairs of these primes that are basic fractions. There are also the basic fractions, one or both of whose components are not primes. * Yes. Since a rational number contains one, and only one, basic fraction, each basic fraction indicates a different rational number.

318 F Yes * Since each basic fraction indicates a different rational number and there are infinitely many basic fractions, the set of rational numbers of arithmetic is an infinite set.

319 R First separate the segment determined by points $\frac{0}{1}$ and $\frac{1}{1}$ into b congruent segments. Then continue to locate points in the line to the right of point $\frac{1}{1}$ so that they determine segments congruent to one of the b segments. You assign the rational number $\frac{a}{b}$ to the right endpoint of the a^{th} of b segments.

320 W Yes. There are infinitely many rational numbers. Each of these rational numbers is associated with a point in a rational-number line. Therefore, there are infinitely many points in a rational-number line. * Yes. The name of any fraction in a rational number may be used to name the point associated with that rational number.

321 U The replacement for a must be less than the replacement for c.

322 C Point $\frac{a}{b}$ is to the right of point $\frac{c}{d}$.

322 D The number obtained from ad is greater than the number obtained from cb.

325 B Yes. You know that the product of two natural numbers is always a natural number. * Yes. Since neither b nor d is replaced by 0, you know that the product of b and d

is not 0, so the product of b and d is a counting number. * Yes. Since ac is a natural number and bd is a natural number other than zero, $\frac{ac}{bd}$ is a rational number.

325 C Yes. Every rational number can be indicated by a fraction. Therefore, you can think of $\frac{a}{b}$ as indicating x and $\frac{c}{d}$ as indicating y. You already know that the product $\frac{ac}{bd}$ is a rational number.

326 D Multiplication of rational numbers is a binary operation because it is a mapping of an ordered pair of rational numbers onto a single rational number.

326 T Yes. Think of $\frac{a}{b}$ and $\frac{c}{d}$ as any two rational numbers. If the universe for k and m is C, then $\frac{ka}{kb}$ represents each rational number that is equal to $\frac{a}{b}$, and $\frac{mc}{md}$ represents each rational number that is equal to $\frac{c}{d}$. You know that $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ by the definition of the product of two rational numbers. By the same property, you know that $\frac{ka}{kb} \times \frac{mc}{md} = \frac{(ka)(mc)}{(kb)(md)}$. Using the associative and commutative properties of natural numbers, along with the reduction property of rational numbers, you can show that $\frac{(ka)(mc)}{(kb)(md)} = \frac{ac}{bd}$.

326 U Yes * Yes * More than one pair of rational numbers can be mapped onto the same product.

329 B From the closure property of addition of natural numbers, you know that the sum of two natural numbers is always a natural number. * The universe for b is the set of counting numbers.

329 D Yes. If $\frac{a}{b}$ and $\frac{c}{b}$ are any two rational numbers, then their sum is $\frac{a+c}{b}$. You already know that, for each replacement of a, b, and c, $\frac{a+c}{b}$ is a rational number.

329 G Yes * $\frac{7}{14}$ is the same rational number as $\frac{7\times1}{7\times2}$. From the reduction property of rational numbers, you know that $\frac{7\times1}{7\times2}$ is the same rational number as $\frac{1}{2}$. * $\frac{21}{14}$ is the same rational number as $\frac{7\times3}{7\times2}$. From the reduction property of rational numbers, you know that $\frac{7\times3}{7\times2}$ is the same rational number as $\frac{3}{2}$. * Yes. $\frac{28}{14}$ is the same rational number as $\frac{7\times4}{7\times2}$. From the reduction property of rational numbers, you know that $\frac{7\times4}{7\times2}$ is the same rational number as $\frac{4}{2}$.

329 B Yes * Yes * From the reduction property of rational numbers, you know that $\frac{1}{2}$ is the same rational number as $\frac{1\times3}{2\times3}$.

331 B The well-defined property of addition of rational numbers * The well-defined property of addition of rational numbers.

332 E Yes * If the difference of a first and second number is the same as a third number, then the first number is the same as the sum of the second and third numbers.

333 G No * There is no rational number of arithmetic that can be added to $\frac{11}{9}$ to get a sum of $\frac{5}{9}$.

333 H No * No * If R_a were closed under subtraction, the difference of any two members of R_a would be a member of R_a. This is not true, since the difference of $\frac{5}{9}$ and $\frac{11}{9}$ is not a member of R_a.

333 J Yes. You know that $a - c$ is a natural number if a is greater than c. Therefore, $\frac{a-c}{b}$ is a member of R_a.

333 K Yes. If a is equal to c, then $a - c$ is equal to the natural number 0. Therefore, $\frac{a-c}{b}$ is the same as $\frac{0}{b}$, which is a member of R_a.

333 L No. If a is less than c, then $a - c$ is not a natural number, and $\frac{a-c}{b}$ is not a member of R_a.

334 A Yes. Because of the reduction property of rational numbers * Yes. Because of the reduction property of rational numbers * By using the definition of the difference of two rational numbers * Yes * For each replacement of a, b, and c, if a is greater than or equal to c, then $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$.

334 B Yes. If the rational numbers that form a difference are replaced by equal rational numbers, then the difference is still the same rational number. * Yes. $\frac{(2\times4)-(3\times1)}{3\times4} = \frac{5}{12}$; therefore, $\frac{2}{3} - \frac{1}{4} = \frac{5}{12}$.

338 I Yes. When you divide both the numerator and denominator of a fraction by their greatest common factor, you obtain a fraction whose numerator and denominator are relatively prime.

338 P 1. The numerator and denominator of a basic fraction are relatively prime and have no common factors other than 1.

340 R Think of the numbers 6 and 15. $6 = 2^1 \times 3^1$. $15 = 3^1 \times 5^1$. You know that the product of 6 and 15 is a multiple of both 6 and 15. The product of 6 and 15 is $2^1 \times 3^2 \times 5^1$, or 90. You also know how to find the least common multiple of 6 and 15. $2^1 \times 3^1 \times 5^1$, or 30, is the least common multiple of 6 and 15. Notice that the least common multiple of 6 and 15 has 3^1 as a factor, while the product of 6 and 15 has 3^2 as a factor. 3 is the greatest common factor of 6 and 15. If you divide $2^1 \times 3^2 \times 5^1$ (the product of 6 and 15) by 3, you obtain $2^1 \times 3^1 \times 5^1$, which is the least common multiple of the two numbers.

340 T Yes. Think of a and b as any two counting numbers. ab is a common multiple of a and b because both a and b are factors of ab. * No. Think of the numbers 4 and 8. The product $4 \cdot 8$ is a common multiple of both 4 and 8, but it is not the least common multiple. The least common multiple of 4 and 8 is 2^3, or 8.

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342 D Yes. Each member of set S is a rational number. * Yes. The members of set S go on and on. There is no last member.

342 H Yes. $\frac{7}{1}$ and $\frac{14}{2}$ are the same rational number. * Yes. $\frac{7}{1}$ and $\frac{35}{5}$ are the same rational number. * Yes. For each replacement of n, $\frac{7n}{n}$ and $\frac{7}{1}$ are the same rational number.

343 J $\frac{a}{1}$, or any rational number equal to $\frac{a}{1}$.

343 N Yes. The definition of the sum of two rational numbers * Yes * Yes. Since $\frac{a}{1}+\frac{b}{1}$ is the same rational number as $\frac{a+b}{1}$, and $\frac{a+b}{1}$ is the mate of $a+b$, then $\frac{a}{1}+\frac{b}{1}$ and $a+b$ are mates.

343 R Yes. The definition of the product of two rational numbers * Yes * Yes. Since $\frac{a}{1}\cdot\frac{b}{1}$ is the same rational number as $\frac{a\cdot b}{1}$, and $\frac{a\cdot b}{1}$ is the mate of $a\cdot b$, then $\frac{a}{1}\cdot\frac{b}{1}$ and $a\cdot b$ are mates.

345 B Yes. The reduction property of rational numbers * Yes. The rational number $\frac{3}{1}$ is the mate of the natural number 3. $\frac{3}{1}$ and $\frac{15}{5}$ are the same rational number. * The numerals 3 and $\frac{15}{5}$ name the same rational number.

349 B The transitive property of equality tells you that, if $x=y$ and $y=z$, then $x=z$. Since $y=z$ also means $z=y$, you also know that if $x=y$ and $z=y$, then $x=z$. You can obtain the statement expressed by sentence G from $x=y$ by replacing x by $\frac{1}{5}\times\frac{2}{1}$ and y by $\frac{2}{5}$. You can obtain the statement expressed by sentence H from $z=y$ by replacing z by $\frac{2}{1}\times\frac{1}{5}$ and y by $\frac{2}{5}$. Since these two statements are true, you know that the statement obtained by using the same replacements for x and z in $x=z$ is also true.

349 H The operation of multiplication of rational numbers has the commutative property.

349 J Yes * The definition of the product of two rational numbers.

349 K Yes * The definition of the product of two rational numbers.

349 L Yes * The commutative property of multiplication of natural numbers.

349 M Yes * The transitive property of equality.

349 N Yes * The transitive property of equality.

350 H The operation of addition of rational numbers has the commutative property.

350 O $\frac{3}{10}-\frac{7}{10}$ is not a rational number of arithmetic because 3 is less than 7.

350 Q No * Since you know of at least one set of replacements for which you do not obtain a true statement from $x-y=y-x$, you know that subtraction of rational numbers does not have the commutative property.

352 J The operation of multiplication of rational numbers has the associative property.

353 O A condition is a property only if a true statement is obtained from the condition for each replacement of the variables. Therefore, subtraction of rational numbers does not have the associative property.

354 B Yes * The associative property.

356 O For each replacement of the variables: $(y+z)x=x(y+z)$, because of the commutative property of multiplication; $x(y+z)=xy+yz$, because of the distributive property of multiplication over addition; $(y+z)x=xy+xz$, because of the transitive property of equality; $xy+xz=yx+zx$ because of the commutative property of multiplication; and $(y+z)x=yx+zx$ because of the transitive property of equality.

357 D No. A condition is a property only if a true statement is obtained for each replacement of the variables. * No.

358 H If the replacement for y is greater than or equal to the replacement for z, then $x(y-z)=xy-yz$ and multiplication distributes over subtraction.

358 F Yes. With respect to the operations of addition and multiplication and their properties, the rational numbers behave like the natural-number system.

359 J No. With respect to the operations of addition and multiplication and their properties, set M does not behave like the natural-number system.

360 F For each replacement of the variables, $\frac{0}{b}+\frac{c}{d}=\frac{0\cdot d+bc}{bd}$ and $\frac{0\cdot d+bc}{bd}=\frac{bc}{bd}$. Then, from the transitive property of equality, you know that, for each replacement of the variables, $\frac{0}{b}+\frac{c}{d}=\frac{bc}{bd}$.

360 H For each replacement of the variables, $\frac{0}{b}+\frac{c}{d}=\frac{bc}{bd}$. Also, for each replacement of the variables, $\frac{bc}{bd}=\frac{c}{d}$. Then, from the transitive property of equality, you know that, for each replacement of the variables, $\frac{0}{b}+\frac{c}{d}=\frac{c}{d}$.

361 K $x+0=x$. * Yes * From the commutative property of addition, you know that $x+0=x$ also means $0+x=x$. So, for each replacement of the variable, $0+x=x$.

361 L The conditions expressed by sentences K and L. $x-x=0$ also means $x=x+0$. $x=x+0$ is the same as $x+0=x$, and you know that $x+0=x$ is a property. $x-0=x$ also means $x=0+x$. $x=0+x$ is the same as $0+x=x$, and you know that $0+x=x$ is a property.

361 F Yes * $\frac{1}{1}\times\frac{c}{d}=\frac{1\times c}{1\times d}$ and $\frac{1\times c}{1\times d}=\frac{c}{d}$ for each replacement of the variables. Then, from the transitive property of equality, you know that, for each replacement of the variables, $\frac{1}{1}\times\frac{c}{d}=\frac{c}{d}$.

362 L Yes * From the definition of the product of two rational numbers, you

know that $\frac{0}{b} \times \frac{c}{d} = \frac{0 \cdot c}{bd}$. From the zero property of multiplication of natural numbers, you know that, for each replacement of c, $0 \cdot c = 0$. Therefore, $\frac{0}{b} \times \frac{c}{d} = \frac{0}{bd}$.

362 M Yes * For each replacement of b and d, $\frac{0}{b}$ and $\frac{0}{bd}$ are the same as 0. Therefore, the condition $\frac{0}{b} \cdot \frac{c}{d} = \frac{0}{bd}$ is the same condition as $0 \cdot \frac{c}{d} = 0$.

362 N Yes * From the commutative property of multiplication, you know that the condition $0 \cdot x = x$ is the same as the condition $x \cdot 0 = x$.

363 P No. The product of 0 and any rational number cannot be 1. * No. Zero cannot be the reciprocal of a number for the reason given above.

364 D Yes. The definition of the product of two rational numbers * Yes. The reduction property of rational numbers * Yes. The transitive property of equality.

364 G Yes * The well-defined property of multiplication of rational numbers.

364 K All of them * One * $\frac{5}{3}$, $\frac{10}{6}$, $\frac{15}{9}$, and $\frac{20}{12}$ are the same rational number.

364 L 1 * The reciprocal of 1 is $\frac{1}{1}$, and 1 and $\frac{1}{1}$ are the same rational number.

365 C No * The quotient of two natural numbers is not always a natural number.

367 L No * If c were replaced by 0, $\frac{0}{d}$ would not have a reciprocal, and $\frac{d}{0}$ would be meaningless.

367 M
$\frac{5}{6} \div \frac{1}{8} = x.$
$\frac{5}{6} = x \times \frac{1}{8}.$
$\frac{5}{6} \times \frac{8}{1} = (x \times \frac{1}{8})\frac{8}{1}.$
$\frac{5}{6} \times \frac{8}{1} = x(\frac{1}{8} \times \frac{8}{1}).$
$\frac{5}{6} \times \frac{8}{1} = x \times 1.$
$\frac{5}{6} \times \frac{8}{1} = x.$
$\frac{5}{6} \div \frac{1}{8} = \frac{5}{6} \times \frac{8}{1}.$

367 N
$\frac{a}{b} \div \frac{c}{d} = y.$
$\frac{a}{b} = y \cdot \frac{c}{d}.$
$\frac{a}{b} \times \frac{d}{c} = (y \cdot \frac{c}{d})\frac{d}{c}.$
$\frac{a}{b} \times \frac{d}{c} = y(\frac{c}{d} \cdot \frac{d}{c}).$
$\frac{a}{b} \times \frac{d}{c} = y \cdot 1.$
$\frac{a}{b} \times \frac{d}{c} = y.$
$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$

367 B For each replacement of the variables, $\frac{a}{b} \times \frac{d}{c}$ is a rational number because the product of any two rational numbers is a rational number. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ for each replacement of the variables. Therefore $\frac{a}{b} \div \frac{c}{d}$ must be a rational number for each replacement of the variables.

367 H $\frac{a}{b} \div 1 = \frac{a}{b} \times 1$, since the reciprocal of 1 is 1. From the identity-element property of rational numbers, you know that, for each replacement of a and b, $\frac{a}{b} \times 1$ is the same as $\frac{a}{b}$. Therefore, for each replacement of a and b, $\frac{a}{b} \div 1 = \frac{a}{b}$. * The quotient of any given rational number and 1 is the given rational number.

368 J Division of rational numbers does not have the commutative property. * The solution set of $\frac{7}{8} \div \frac{1}{3} = x$ is $\{\frac{21}{8}\}$. The solution set of $\frac{1}{3} \div \frac{7}{8} = y$ is $\{\frac{8}{21}\}$. Therefore, $\frac{7}{8} \div \frac{1}{3}$ is not equal to $\frac{1}{3} \div \frac{7}{8}$. This one example is sufficient to prove that division of rational numbers does not have the commutative property because a property is true for all members of the set.

368 K Division of rational numbers does not have the associative property. * The solution set of $(\frac{4}{5} \div \frac{2}{3}) \div \frac{1}{8} = m$ is $\{\frac{48}{5}\}$. The solution set of $\frac{4}{5} \div (\frac{2}{3} \div \frac{1}{8}) = n$ is $\{\frac{3}{20}\}$. Therefore, $(\frac{4}{5} \div \frac{2}{3}) \div \frac{1}{8}$ is not equal to $\frac{4}{5} \div (\frac{2}{3} \div \frac{1}{8})$. This one example is sufficient to prove that division of rational numbers does not have the associative property.

368 B $\frac{a}{1} \div \frac{b}{1} = \frac{a}{1} \times \frac{1}{b}$, from the quotient property of rational numbers. $\frac{a}{1} \times \frac{1}{b} = \frac{a \cdot 1}{1 \cdot b}$, from the definition of the product of two rational numbers. $\frac{a}{1} \div \frac{b}{1} = \frac{a \cdot 1}{b \cdot 1}$, from the transitive property of equality. $\frac{a \cdot 1}{1 \cdot b} = \frac{a}{b}$, from the identity-element property of multiplication of natural numbers. $\frac{a}{1} \div \frac{b}{1} = \frac{a}{b}$, from the transitive property of equality.

370 F The well-defined property of addition of rational numbers

371 Q 5, 6, and 4 * The digit 5 is associated with $\frac{1}{10^1}$, the digit 6 is associated with $\frac{1}{10^2}$, and the digit 4 is associated with $\frac{1}{10^3}$. Each of the numbers, $\frac{1}{10^1}$, $\frac{1}{10^2}$, and $\frac{1}{10^3}$ is less than 1.

374 G No * Yes * Because 106 hundredths is the same as the sum of 100 hundredths, 0 hundredths, and 6 hundredths. You know that 100 hundredths is the same as 1 one, and 0 hundredths is the same as 0 tenths. * Yes.

377 J $3 * 4 * 7 * 12.784$ is the same as $\frac{12784}{1000}$. .0509 is

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the same as $\frac{509}{10000}$. The product of $\frac{12784}{1000}$ and $\frac{509}{10000}$ if $\frac{6507056}{10000000}$, or .6507056.

377 W 72400 * 6000

377 X Fifth power of 10 * Fourth power of 10 * Third power of 10

378 D Yes * $\frac{ma}{nb} \cdot \frac{nd}{mc} = \frac{(ma)(nd)}{(nb)(mc)}$, from the definition of the product of two rational numbers. $\frac{(ma)(nd)}{(nb)(mc)} = \frac{m(an)d}{n(bm)c}$, from the associative property of multiplication of natural numbers. $\frac{ma}{nb} \cdot \frac{nd}{mc} = \frac{m(an)d}{n(bm)c}$, from the transitive property of equality. $\frac{m(an)d}{n(bm)c} = \frac{m(na)d}{n(mb)c}$, from the commutative property of multiplication of natural numbers. $\frac{ma}{nb} \cdot \frac{nd}{mc} = \frac{m(na)d}{n(mb)c}$, from the transitive property of equality. $\frac{m(na)d}{n(mb)c} = \frac{(mn)(ad)}{(nm)(bc)}$, from the associative property of multiplication of natural numbers. $\frac{ma}{nb} \cdot \frac{nd}{mc} = \frac{(mn)(ad)}{(nm)(bc)}$, from the transitive property of equality. $\frac{(mn)(ad)}{(nm)(bc)} = \frac{(mn)(ad)}{(mn)(bc)}$, from the commutative property of multiplication of natural numbers. $\frac{ma}{nb} \cdot \frac{nd}{mc} = \frac{(mn)(ad)}{(mn)(bc)}$, from the transitive property of equality. $\frac{(mn)(ad)}{(mn)(bc)} = \frac{ad}{bc}$, from the reduction property of rational numbers. $\frac{ma}{nb} \cdot \frac{nd}{mc} = \frac{ad}{bc}$, from the transitive property of equality.

381 W $248.36 \div 10 = 24.836$. * $248.36 \div 100 = 2.4836$. * $248.36 \div 1000 = .24836$. * $248.36 \div 10000 = .024836$. * $248.36 \div 100000 = .0024836$.

384 G Yes * For the second step, you divide 40 by 11 to help find the second digit in the decimal for the quotient. For the fourth step, you also divide 40 by 11 to help find the fourth digit in the decimal for the quotient.

384 K When you use decimals to divide 7 by 11, the fourth remainder is 7, not 0. This means that the quotient of 7 and 11, or $\frac{7}{11}$, is greater than .6363.

385 C Yes. This decimal would name the same rational number as the decimal .25. Therefore, this decimal would name $\frac{1}{4}$.

385 E First, find the basic fraction that indicates the rational number. If the denominator of this fraction has no prime factors other than 2 or 5, then the rational number can be expressed by a terminating decimal.

386 G You know that $.\overline{5}$ can be expressed as $.5\overline{5}$, and you know that .55 is less than $.5\overline{5}$. Therefore, .55 must be less than $.\overline{5}$.

386 H If the decimals do not have the same number of digits to the left of the decimal point, then the decimal with more digits to the left of the decimal point expresses the greater number. If the decimals have the same number of digits to the left of the decimal points, you can compare corresponding digits in order, beginning with those on the left. The first two corresponding digits that are different tell you which decimal expresses the greater number.

389 K Yes * $\frac{5}{8} < \frac{3}{4}$ because $5 \cdot 4 < 3 \cdot 8$. $\frac{3}{4} < 1$, or $\frac{3}{4} < \frac{1}{1}$, because $3 \cdot 1 < 1 \cdot 4$. Therefore, $\frac{3}{4}$ is between $\frac{5}{8}$ and 1.

389 R Find the sum of $\frac{4}{5}$ and $\frac{9}{7}$ and divide this sum by 2. * $\frac{73}{70}$

390 V From the density property of the set of rational numbers, you know that there is a rational number between x and y. Let a be the variable for this rational number. From the density property, you also know that there is a rational number between x and a and another between a and y. Let the variables for these numbers be b and c. You also know that there is a rational number between x and b, another between b and a, another between a and c, and another between c and y. In this way, you can go on and on finding more numbers between x and y. Each of these numbers that you find will be between the given numbers.

390 F If a rational number immediately followed a given rational number, then there would not be a rational number between them. This cannot be true because of the density property which tells you that there is a rational number between any two rational numbers.

390 H No. There is no rational number that immediately follows zero. * No. Because you cannot name the first three members of the set.

390 J No * For any number you name, there is another rational number between the one named and $\frac{1}{3}$ that is a member of $\{x \mid x < \frac{1}{3}\}$.

391 E $\frac{3}{4}$ * You find the sum of $\frac{1}{2}$ and $\frac{1}{4}$.

392 J $\{\frac{1}{6}\}$ * You should have found the difference of $\frac{5}{6}$ and $\frac{2}{3}$.

392 O {.31} * You should have found the difference of .49 and .18.

392 V $\{\frac{1}{2}\}$ * You should have found the difference of $\frac{3}{5}$ and $\frac{1}{10}$.

393 G Yes * The solution set of $x + \frac{3}{8} = \frac{5}{8}$ is the same as the solution set of $x = \frac{1}{4}$. From this, you know that the solution set of $x + \frac{3}{8} > \frac{5}{8}$ is $\{x \mid x > \frac{1}{4}\}$.

395 E Yes * The intersection of two sets is a set that contains only those members that are in both sets. In the same way, the solution set of a compound condition involving "and" contains only those numbers that satisfy both of the simple conditions.

395 I The portion of the graph of the universe that has a bar above it as well as below it indicates the graph of $\{x \mid x > \frac{1}{3} \wedge x < \frac{3}{2}\}$. * Yes.

395 M Yes * The solution set of $x > 4$ is the same as the solution set of $x - 1 > 3$. Since the solution set of $x > 4$ is a subset of the solution set of $2 + x > 4$, $\{x \mid x > 4\}$ is the solution set of $2 + x > 4 \wedge x - 1 > 3$.

397 H {2.59} * $2.59

398 O The solution set of $1\frac{1}{4} + x = 2\frac{3}{4}$ is the same as the solution set of $x = 1\frac{1}{2}$. From this, you know that the solution set of $1\frac{1}{4} + x < 2\frac{3}{4}$ is $\{x \mid x < 1\frac{1}{2}\}$.

398 Q No * 0 is not a member of the universe.

398 E The solution set of $s + 2.75 = 14.00$ is the same as the solution set of $s = 11.25$. From this, you know that the solution set of $s + 2.75 < 14.00$ is $\{s \mid s < 11.25\}$.

398 F No * Not all numbers that are less than 11.25 are greater than 9.10.

398 I 10.256 is greater than 9.10 and less than 11.25. * 11.0012 is also greater than 9.10 and less than 11.25.

399 L Only numbers that can be expressed in hundredths are members of the set tabulated in D5.

399 R $\{m \mid m > 2\frac{1}{4} \wedge m < 3\frac{1}{4}\}$ * Ralph used between $2\frac{1}{4}$ and $3\frac{1}{4}$ gal. of paint.

399 C Yes * $x < 3\frac{1}{3}$ has the same solution set as $2\frac{1}{3} + x < 5\frac{2}{3}$.

403 X $\frac{1}{10}$ * Find the product of $\frac{1}{10}$ and 15.

404 Z Yes * Find the product of $\frac{b}{a}$ and $\frac{c}{a}$.

406 L The first component of $(1\frac{1}{2}, 4\frac{1}{2})$ is the replacement for x, and x refers to the average weight per package.

407 D No * You do not buy $\frac{1}{8}$ of a pencil.

407 E No * Mark does not have enough money to buy 4 of these pencils.

407 F You can usually buy a part of a pound or a part of a foot of anything that is sold by the pound or by the foot.

407 J No * There is no coin that has a value of $\frac{1}{3}$ cent.

407 O To the nearer hundredth, 285.83 is the solution of $1715/6 \sim s/1$.

409 F Each rate pair whose second component is 100 represents a comparison of the amount of money put in a savings account for each $100 of income.

417 K The numeral $\frac{1}{2}$ names the same rational number as the numeral .5.

419 B $.375(100) = 37\frac{1}{2}(1)$. * Divide each component of $37\frac{1}{2}/100$ by 100.

420 P $\frac{1}{3}/1$ * Divide each component of the rate pair $33\frac{1}{3}/100$ by 100.

420 S To find the number that satisfies $\frac{1}{3}y = 132$, you should multiply 132 by the reciprocal of $\frac{1}{3}$, which is 3.

421 C The number of dollars that Mr. Harris used to buy the car for each 1 dollar of his savings.

421 J $\frac{2}{3}/1 \sim 48/x$. * {72}

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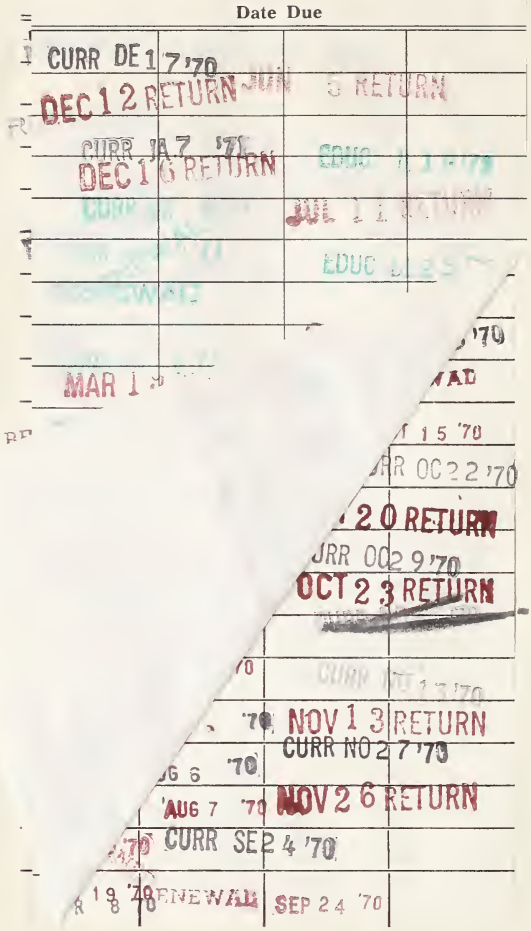
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